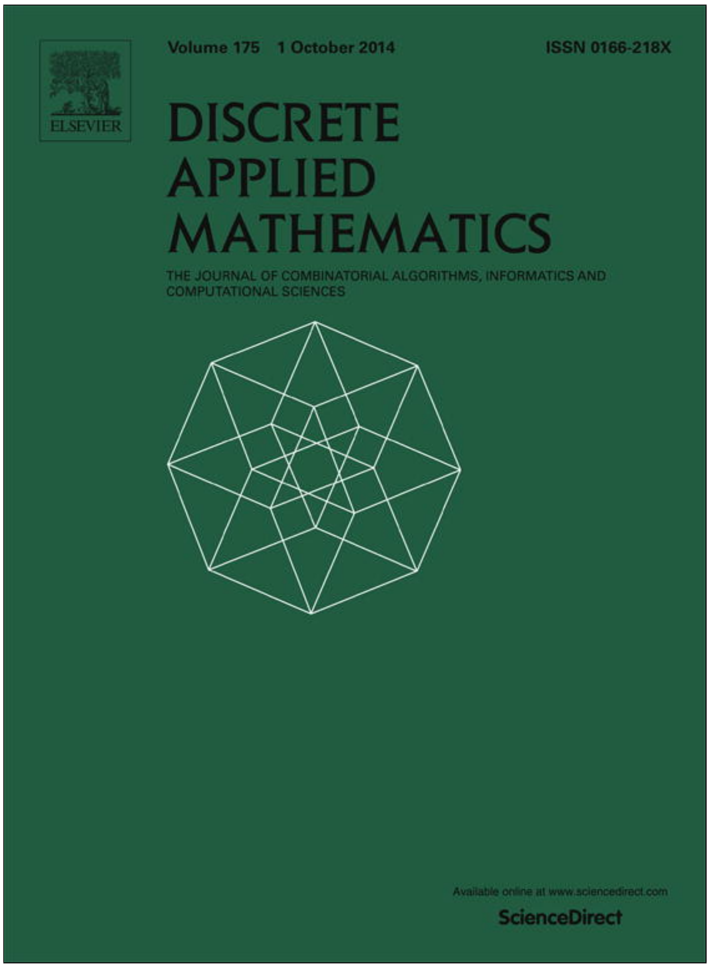
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Trees with maximum p-reinforcement number[☆]

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ABSTRACT

Let $G = (V, E)$ be a graph and p a positive integer. The p-domination number $\gamma_p(G)$ is the minimum cardinality of a set $D \subseteq V$ with $|N_G(x) \cap D| \geq p$ for all $x \in V \setminus D$. The p-reinforcement number $r_p(G)$ is the smallest number of edges whose addition to G results in a graph G' with $\gamma_p(G') < \gamma_p(G)$. It is showed by Lu et al. (2013) that $r_p(T) \leq p + 1$ for any tree T and $p \geq 2$. This paper characterizes all trees attaining this upper bound when $p \geq 3$.

1. Introduction

For graph-theoretical terminology and notation not defined here we follow [19]. Let $G = (V, E) = (V(G), E(G))$ be a simple graph and $x \in V$. The *neighborhood* and *degree* of x are $N_G(x) = \{y \in V : xy \in E\}$ and $d_G(x) = |N_G(x)|$, respectively. If $d_G(x) = 1$, then x is called a *leaf* and its unique neighbor is called a *stem*. The set of leaves of G is denoted by $L(G)$. Let $p \geq 1$ be an integer and $X \subseteq V$ with $x \in X$. A vertex $y \in N_G(x)$ is called a *p-private neighbor* of x with respect to X if $y \in V - X$ and $|N_G(y) \cap X| = p$. We use $N_p(x, X, G)$ to denote the set of p-private neighbors of x with respect to X in G.

For $X \subseteq V$, the subgraph induced by X (resp. $V - X$) is denoted by $G[X]$ (resp. $G - X$). The complement G^c of G is the simple graph with vertex-set V and edge-set $E(G^c) = \{xy : xy \notin E\}$. For $B \subseteq E(G^c)$, $G + B$ denotes the graph obtained from G by adding B. To simplify notation, for $x \in V$ and subgraph $H \subseteq G$, we write $G - x$ and $G - H$ for $G - \{x\}$ and $G - V(H)$, respectively.

Let $p \geq 1$ be an integer and $X \subseteq V$. For $Y \subseteq V$, *$X p$-dominates* Y in G if for each $y \in Y$, either $y \in X$ or $|N_G(y) \cap X| \geq p$. We write $X \succ_p Y$ if $X p$-dominates Y, and write $X \not\succ_p Y$ otherwise. In particular, if $X \succ_p V$ then X is called a *p-dominating set*, abbreviated DS_p, of G. The *p-domination number* $\gamma_p(G)$ is the minimum cardinality of a DS_p of G. A DS_p with cardinality $\gamma_p(G)$ is called a *γ_p-set* of G. The *p-reinforcement number* $r_p(G)$ is the smallest number of edges in G^c that have to be added to G in order to reduce $\gamma_p(G)$, that is

$$r_p(G) = \min\{|B| : B \subseteq E(G^c) \text{ with } \gamma_p(G + B) < \gamma_p(G)\}.$$

By convention,

$$r_p(G) = 0 \quad \text{if } \gamma_p(G) \leq p. \tag{1.1}$$

Clearly, γ_1 and r_1 are the well-known domination γ and reinforcement r, respectively.

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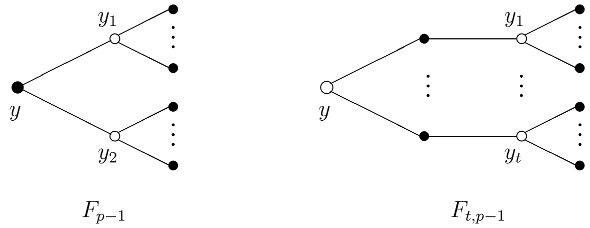

Fig. 1. Trees F_{p-1} and $F_{t,p-1}$, where $t \geq p$ and each y_i has $p-1$ leaves.

The concept of p-domination was introduced by Fink and Jacobson [10] in 1985 and has been well studied for recent decade (see, for example, [2–4,7–9,11]). Chellali et al. [5] gave an excellent survey on this topic. The p-reinforcement number, introduced by Lu, Hu and Xu [17], is a parameter for measuring vulnerability of p-domination, is also a natural extension of the classical reinforcement number which was introduced by Kok and Mynhardt [15] and studied by a number of authors (see, for example, [6,12–14,20]). Motivated by the work of these authors, Lu, Hu and Xu [17] studied p-reinforcement, found a method to determine r_p in terms of γ_p and showed that the decision problem on r_p is NP-hard and established some upper bounds.

Surprisingly, for a tree T of order n, the known upper bounds for $r_p(T)$ are of distinct forms according to $p = 1$ and $p \geq 2$. For $p = 1$, Blair et al. [1] gave a sharp upper bound $r_1(T) \leq \frac{n}{2}$. For $p \geq 2$, however, there is an upper bound for $r_p(T)$ which is independent of n.

Theorem 1.1 (*Lu, Hu and Xu [17]*). *$r_p(T) \leq p + 1$ for any tree T and $p \geq 2$.*

In this paper we characterize all extremal trees in Theorem 1.1 for $p \geq 3$ by a recursive construction. The rest of this paper is organized as follows. The main result of this paper is stated in Section 2. To prove the main result, we propose two needed parameters η_p and μ_p in Section 3 and use them to establish some structural properties of a tree T with $r_p(T) = p+1$ for $p \geq 3$ in Section 4. In Section 5 we complete the proof of the main result. A conclusion is in Section 6.

2. Main result

Throughout this paper, we always suppose that $p \geq 3$ is an integer. In this section we will give a constructive characterization of trees with p-reinforcement number $p + 1$. First, we state two known results.

Lemma 2.1. *Every DS_p of a graph contains all vertices of degree less than p.*

Lemma 2.2 (*Lu et al. [16]*). *Let $p \geq 2$ be an integer and D be a DS_p of a tree T. Then D is the unique γ_p-set of T if and only if for each $x \in D$ with $d_T(x) \geq p$, $|N_G(x) \cap D| \leq p - 2$ or $|N_p(x, D, T)| \geq 2$.*

Let $t \geq p$ be an integer. The *spider* S_t is a tree obtained from a star $K_{1,t}$ by attaching one leaf at each leaf of $K_{1,t}$. Two important trees F_{p-1} and $F_{t,p-1}$ in our construction are shown in Fig. 1, where F_{p-1} (resp. $F_{t,p-1}$) is obtained from a star $K_{1,2}$ (resp. a spider S_t) by attaching $p-1$ leaves at each leaf of $K_{1,2}$ (resp. S_t).

In Fig. 1, we call y the *center* of F_{p-1} (resp. $F_{t,p-1}$). It is obvious that the set of black vertices in F_{p-1} (resp. $F_{t,p-1}$ for $t \geq p$) is the unique γ_p-set of F_{p-1} (resp. $F_{t,p-1}$). For a star $K_{1,m}$ ($m \geq 2$), the unique stem is also called the *center* of $K_{1,m}$.

For two disjoint graphs G and H, let $G \oplus_{xy} H$ denote the graph with vertex-set $V(G) \cup V(H)$ and edge-set $E(G) \cup \{xy\} \cup E(H)$, where $x \in V(G)$ and $y \in V(H)$.

Definition 2.3. Let G be a tree with a unique γ_p-set X. A new tree T is constructed from G by the following operation $\mathscr{O}$.

$$\mathscr{O}: \ T = G \oplus_{xy} H,$$

where $H \in \{K_{1,p-1}, K_{1,p}, F_{p-1}, F_{t,p-1}\}$, y is the center of H and x must fulfil the following conditions:

(1) $x \in X$ if $H = K_{1,p-1}$.
(2) $x \notin X$ if $H = K_{1,p}$.
(3) $x \in X$ and $|N_p(x, X, G)| \geq \min\{p + 1, |N_G(x) \cap X| + 2\}$ if $H = F_{p-1}$.
(4) x is an arbitrary vertex in G if $H = F_{t,p-1}$.

Note that the tree $H \in \{K_{1,p}, F_{p-1}, F_{t,p-1}\}$ with $t \geq p$ has a unique γ_p-set, denoted by $\mathscr{U}_H$. By Definition 2.3, the following observation follows almost immediately from Lemmas 2.1 and 2.2.

Observation 2.4. *Let $p \geq 3$ and $t \geq p$ be two integers and G a tree different to $K_{1,p-1}$ with a unique γ_p-set X. Then the tree $T = G \oplus_{xy} H$ obtained from G by operation $\mathscr{O}$ has a unique γ_p-set*

$$X \cup \begin{cases} L(H) & \text{if } H = K_{1,p-1}; \\ \mathscr{U}_H & \text{if } H \in \{K_{1,p}, F_{p-1}, F_{t,p-1}\}. \end{cases}$$

Since the star $K_{1,p}$ has a unique γ_p-set, by Observation 2.4, we can define a family $\mathscr{T}_p$ of trees as follows.

$\mathscr{T}_p = \{T : T$ is obtained from the star $K_{1,p}$ via a finite series of operation $\mathscr{O}\}$.

It must be pointed out that $K_{1,p} \notin \mathscr{T}_p$. We now are ready to establish our main result whose proof is postponed to Section 5.

Theorem 2.5. *For an integer $p \geq 3$ and a tree T, $r_p(T) = p + 1$ if and only if $T \in \mathscr{T}_p$.*

3. Notations and lemmas

The notations η_p and μ_p introduced by Lu, Hu and Xu [17] play important roles in the study of p-reinforcement. In this section, we present their definitions and fundamental results.

Let $G = (V, E)$ be a graph and $X \subseteq V$. For each vertex $x \in V$, define

$$\eta_p(x, X, G) = \begin{cases} p - |N_G(x) \cap X| & \text{if } X \not\succ_p x; \\ 0 & \text{if } X \succ_p x. \end{cases} \quad (3.1)$$

If $|X| \geq p$, then there is a subset $B_x \subseteq E(G^c)$ with $|B_x| = \eta_p(x, X, G)$ such that $X \succ_p x$ in $G+B_x$, and so X is a DS_p of $G+(\cup_{x\in V} B_x)$, which implies that $r_p(G) \leq |\cup_{x\in V} B_x| = \sum_{x\in V} \eta_p(x, X, G)$ by the definition of r_p. Motivated by this inequality, Lu, Hu and Xu [17] define

$$\eta_p(S, X, G) = \sum_{x\in S} \eta_p(x, X, G) \quad \text{for } S \subseteq V, \quad (3.2)$$

and prove the following two lemmas.

Lemma 3.1 (*Lu, Hu and Xu [17]*)**.** *Let p be an integer and G a graph. If $\gamma_p(G) > p$, then*

$$r_p(G) = \min\{\eta_p(V(G), X, G) : X \subseteq V(G) \text{ with } |X| < \gamma_p(G)\}.$$

Let G be a graph and $X \subseteq V(G)$. If $|X| < \gamma_p(G)$ and $\eta_p(V(G), X, G) = r_p(G)$, then X is called an *η_p-set* of G.

Lemma 3.2 (*Lu, Hu and Xu [17]*)**.** *Let p be an integer and G be a graph. If X is an η_p-set of G, then $|X| = \gamma_p(G) - 1$.*

The following observation is trivial by (3.1) and (3.2).

Observation 3.3. *Let G be a graph and $S, X \subseteq V(G)$. Then*

(1) *$\eta_p(S, X, G) \geq \eta_p(S_1, X, G)$ for any $S_1 \subseteq S$.*
(2) *$\eta_p(S, X, G) \leq \eta_p(S, X_1, G)$ for any $X_1 \subseteq X$.*
(3) *$\eta_p(S, X, G) \geq \eta_p(S, X, H)$ for any supergraph H of G.*

By the definitions of η_p and $\oplus$, the following lemma follows from Observation 3.3 and Lemmas 3.2 and 3.1 immediately.

Lemma 3.4. *Let G_i be a graph with $x_i \in V(G_i)$ for $i = 1, 2$ and $H = G_1 \oplus_{x_1x_2} G_2$.*

(1) *For any $X_i \subseteq V(G_i)$ $(i = 1, 2)$,*

$$\eta_p(V(G_1), X_1, G_1) - \eta_p(V(G_1), X_1 \cup X_2, H) = \begin{cases} 1 & \text{if } X_1 \not\succ_p x_1 \text{ and } X_2 \ni x_2; \\ 0 & \text{otherwise.} \end{cases}$$

(2) *If $\gamma_p(G_1) > p$ and $\gamma_p(H) \geq \gamma_p(G_1) + \gamma_p(G_2)$, then $r_p(H) \leq r_p(G_1)$.*

Now we present the parameter μ_p. Let $G = (V, E)$ be a graph and $X \subseteq V$. For $x \in X$, define

$$\mu_p(x, X, G) = |N_p(x, X, G)| + \max\{0, p - |N_G(x) \cap X|\}. \quad (3.3)$$

Lemma 3.5 (*Lu, Hu and Xu [17]*)**.** *For a graph G,*

$$r_p(G) \leq \min\{\mu_p(x, X, G) : X \text{ is a } \gamma_p\text{-set of } G \text{ and } x \in X\}.$$

4. Properties for a tree T with $r_p(T) = p + 1$

In this section, we use the parameters η_p and μ_p to establish some lemmas of a tree T with $r_p(T) = p + 1$, which will be applied in the proof of Theorem 2.5.

Lemma 4.1. *Let $p \geq 3$ and T a tree with $r_p(T) = p + 1$. If D is a γ_p-set of T, then*

(1) *$N_p(x, D, T) \neq \emptyset$ for each $x \in D$.*
(2) *D is the unique γ_p-set of T.*

Proof. Let x be any vertex in D. Then

$$\begin{aligned}|N_p(x, D, T)| &= \mu_p(x, D, T) - \max\{0, p - |N_T(x) \cap D|\} \quad \text{(by (3.3))}\\ &\geq \mu_p(x, D, T) - p\\ &\geq r_p(T) - p \quad \text{(by Lemma 3.5)}\\ &= 1\end{aligned}$$

and so the conclusion (1) holds.

We now prove the conclusion (2). Since D is a γ_p-set of T and $r_p(T) = p + 1 > 0$, $|D| = \gamma_p(T) \geq p + 1$ by (1.1). If $d_T(x) < p$ for any $x \in D$, then D is the unique γ_p-set by Lemma 2.1, and so the conclusion follows. Assume now that there is some $x \in D$ such that $d_T(x) \geq p$. By Lemma 3.5 and (3.3),

$$p + 1 = r_p(T) \leq \mu_p(x, D, T) = |N_p(x, D, T)| + \max\{0, p - |N_T(x) \cap D|\},$$

that is,

$$|N_p(x, D, T)| \geq p + 1 - \max\{0, p - |N_T(x) \cap D|\}.$$

If $|N_T(x) \cap D| \geq p - 1$, then $\max\{0, p - |N_T(x) \cap D|\} \leq 1$, and so $|N_p(x, D, T)| \geq p \geq 3$. This fact implies that D satisfies the second condition in Lemma 2.2, from which D is the unique γ_p-set of T. The lemma follows. ■

Let $p \geq 3$ and T a tree with $r_p(T) = p + 1$. Through this paper, we use $\mathscr{U}_T$ to represent the unique γ_p-set of T. For any $xy \in E(T)$, let T_y denote the component of $T - x$ containing y.

Lemma 4.2. *Let $p \geq 3$ and T a tree with $r_p(T) = p + 1$. For any $x \in \mathscr{U}_T$ and $y \in N_T(x)$,*

(1) *If $y \notin N_p(x, \mathscr{U}_T, T)$, then $r_p(T_y) = p + 1$ and $\mathscr{U}_{T_y} = \mathscr{U}_T \cap V(T_y)$.*
(2) *If $y \in N_p(x, \mathscr{U}_T, T)$, then*
   (a) *either T_y is a star $K_{1,p-1}$ with center y or $r_p(T_y) = 1$ and $\mathscr{U}_T \cap V(T_y)$ is an η_p-set of T_y, and*
   (b) *$\eta_p(V(T_y), X, T_y) \geq p - 1$ for $X \subseteq V(T_y)$ with $y \in X$ and $|X| \leq |\mathscr{U}_T \cap V(T_y)|$.*

Proof. Let $Y = \mathscr{U}_T \cap V(T_y)$ and $Z = \{z \in V(T_y) \setminus Y : |N_T(z) \cap \mathscr{U}_T| = p\}$. Note that $x \in \mathscr{U}_T$ and $\mathscr{U}_T$ is the unique γ_p-set of T. Since $p \geq 3$, $Y \neq \emptyset$ and so $Z \neq \emptyset$ by Lemma 4.1(1). For any $z \in Z$, since $\mathscr{U}_T \succ_p z$ and $x \in \mathscr{U}_T$,

$$|N_{T_y}(z) \cap \mathscr{U}_T| = |N_T(z) \cap \mathscr{U}_T - \{x\}| \geq p - 1,$$

with equality if and only if $z = y$. Hence either T_y is a star $K_{1,p-1}$ with center y or $|V(T_y)| > p$. In the former case, the conclusion (b) in (2) is trivial by (3.1) and (3.2). Thus, to prove the lemma, we only need to consider the case of $|V(T_y)| > p$.

We claim that $\gamma_p(T_y) > p$. Suppose, to be contrary, that $\gamma_p(T_y) \leq p$. Furthermore, $\gamma_p(T_y) = p$ since $|V(T_y)| > p$. Note that $p \geq 3$ and p vertices in a tree have at most one common neighbor. Since T_y is a tree, $T_y = K_{1,p}$. Let z be the center of T_y. Since $p \geq 3$ and $\mathscr{U}_T$ is a DS_p of T, it follows from Lemmas 2.1 and 4.1(1) that $L(T_y) = Y$ and $z \in Z$. If $z = y$, then $|N_T(z) \cap \mathscr{U}_T| = |L(T_y) \cup \{x\}| = p + 1$, which contradicts that $z \in Z$. If $z \neq y$, then $y \in Y \subseteq \mathscr{U}_T$ and $N_T(y) = \{x, z\}$, furthermore, $N_T(y) \cap \mathscr{U}_T = \{x\}$ and $N_p(y, \mathscr{U}_T, T) = \{z\}$. By (3.3),

$$\mu_p(y, \mathscr{U}_T, T) = |N_p(y, \mathscr{U}_T, T)| + \max\{0, p - |N_T(y) \cap \mathscr{U}_T|\} = 1 + (p - 1) = p,$$

from which and Lemma 3.5 we obtain that $r_p(T) \leq \mu_p(y, \mathscr{U}_T, T) = p$, a contradiction. The claim holds.

Firstly, we prove (1). Let $T - T_y = T_x$. Then $T = T_x \oplus_{xy} T_y$. Since $x \in \mathscr{U}_T$ and $y \notin N_p(x, \mathscr{U}_T, T)$, $\mathscr{U}_T \cap V(T_x) \succ_p V(T_x)$ and $Y \succ_p V(T_y)$. It follows that $\gamma_p(T) = |\mathscr{U}_T| = |\mathscr{U}_T \cap V(T_x)| + |Y| \geq \gamma_p(T_x) + \gamma_p(T_y)$, furthermore, $\gamma_p(T) = \gamma_p(T_x) + \gamma_p(T_y)$ since the union of a γ-set of T_x and a γ_p-set of T_y is a DS_p of T. So Y is a γ_p-set of T_y. By Theorem 1.1 and Lemma 3.4(2), $p + 1 \geq r_p(T_y) \geq r_p(T) = p + 1$, and so $\mathscr{U}_{T_y} = Y$ by Lemma 4.1(2).

Secondly, we prove the conclusion (a) of (2). Since $y \in N_p(x, \mathscr{U}_T, T)$ and $\mathscr{U}_T \succ_p V(T)$, $|N_{T_y}(y) \cap Y| = |N_T(y) \cap \mathscr{U}_T - \{x\}| = p - 1$ and $Y \succ_p V(T_y) - \{y\}$. Thus, by (3.2) and (3.1),

$$\eta_p(V(T_y), Y, T_y) = \eta_p(y, Y, T_y) + \sum_{z \in V(T_y) - \{y\}} \eta_p(z, Y, T_y) = 1. \tag{4.1}$$

We claim that $|Y| < \gamma_p(T_y)$. Assume, to the contrary, that $|Y| \geq \gamma_p(T_y)$. Let Y' be a γ_p-set of T_y. Since $x \in \mathscr{U}_T$ and $\mathscr{U}_T$ is a unique γ_p-set of T, $\mathscr{U}_T - Y \succ_p V(T) - V(T_y)$. So $(\mathscr{U}_T - Y) \cup Y' \succ_p V(T)$ and

$$|(\mathscr{U}_T - Y) \cup Y'| = (|\mathscr{U}_T| - |Y|) + |Y'| = \gamma_p(T) - |Y| + \gamma_p(T_y) \leq \gamma_p(T).$$

This fact means that $(\mathscr{U}_T - Y) \cup Y'$ is a γ_p-set of T different from $\mathscr{U}_T$, a contradiction. The claim holds. Therefore, by Lemma 3.1 and (4.1),

$$r_p(T_y) \leq \eta_p(V(T_y), Y, T_y) = 1.$$

Note that $r_p(T_y) \geq 1$ by (1.1) since $\gamma_p(T_y) > p$. Thus, $r_p(T_y) = \eta_p(V(T_y), Y, T_y) = 1$ and Y $(= \mathscr{U}_T \cap V(T_y))$ is an η_p-set of T_y.

Finally, we prove the conclusion (b) of (2). Let $X \subseteq V(T_y)$ such that $y \in X$ and $|X| \le |Y|$. It suffices to show that $\eta_p(V(T_y), X, T_y) \ge p-1$. For any $u \in N_{T_y}(y)$, let T_u be the component of $T-y$ containing u. Since $y \in X - \mathscr{U}_T$ and $|X| \le |Y|$,

$$\sum_{u \in N_{T_y}(y)} |X \cap V(T_u)| = |X - \{y\}| \le |Y| - 1 = \sum_{u \in N_{T_y}(y)} |\mathscr{U}_T \cap V(T_u)| - 1,$$

which implies that there is some $u \in N_{T_y}(y)$ such that $|X \cap V(T_u)| < |\mathscr{U}_T \cap V(T_u)|$. Let

$$S = (\mathscr{U}_T - V(T_u)) \cup (X \cap V(T_u)).$$

Then $|S| < |\mathscr{U}_T| = \gamma_p(T)$. Since $|N_T(y) \cap S| \ge |N_T(y) \cap (\mathscr{U}_T - V(T_u))| \ge |N_T(y) \cap \mathscr{U}_T| - 1 = p-1$, by (3.1),

$$\eta_p(y, S, T) \le 1. \tag{4.2}$$

Since $\mathscr{U}_T - V(T_u) \succ_p V(T - T_u - y)$, by (3.1) and (3.2),

$$\eta_p(V(T), S, T) = \eta_p(V(T_u), S, T) + \eta_p(y, S, T). \tag{4.3}$$

It follows from Lemma 3.1 that

$$\begin{aligned} p+1 = r_p(T) &\le \eta_p(V(T), S, T) \\ \text{(by (4.3))} &= \eta_p(V(T_u), S, T) + \eta_p(y, S, T) \\ \text{(by Observation 3.3(2) and (4.2))} &\le \eta_p(V(T_u), S \cap V(T_u), T) + 1 \\ \text{(by Observation 3.3(3))} &\le \eta_p(V(T_u), S \cap V(T_u), T_y) + 1, \end{aligned}$$

that is, $\eta_p(V(T_u), S \cap V(T_u), T_y) \ge p$. Note that $T_y = (T_y - T_u) \oplus_{yu} T_u$ and $X \cap V(T_u) = S \cap V(T_u)$. Therefore,

$$\begin{aligned} \eta_p(V(T_y), X, T_y) &\ge \eta_p(V(T_u), X, T_y) \quad \text{(by Observation 3.3(1))} \\ &\ge \eta_p(V(T_u), S \cap V(T_u), T_y) - 1 \quad \text{(by (3.1) and (3.2))} \\ &\ge p-1 \end{aligned}$$

as required. The lemma follows. ■

Remark 4.3. With a similar argument, both Lemmas 4.1 and 4.2 are also true for $p = 2$.

Lemma 4.4. *Let $p \ge 3$, T a tree with $r_p(T) = p+1$, and $x \in \mathscr{U}_T$ such that $\mu_p(x, \mathscr{U}_T, T) \ge p+2$. For any $X \subseteq V(T-x)$ with $|X| < \gamma_p(T)$, $\eta_p(V(T), X, T) \ge p+2$.*

Proof. Let X be a counterexample to the lemma with $|X \cap \mathscr{U}_T|$ as large as possible. Since $|X| < \gamma_p(T)$, Lemma 3.1 implies that $\eta_p(V(T), X, T) \ge r_p(T) = p+1$, furthermore, $\eta_p(V(T), X, T) = p+1 = r_p(T)$ since X is a counterexample to the lemma. Thus X is an η_p-set of T and $|X| = \gamma_p(T) - 1$ by Lemma 3.2.

Let $N_p(x, \mathscr{U}_T, T) = \{x_1, \ldots, x_t\}$ and $N_T(x) = \{x_1, \ldots, x_t, x_{t+1}, \ldots, x_d\}$, where $d = d_T(x)$. For each i, let T_i be the component of $T - x$ containing x_i. Since $x \in \mathscr{U}_T - X$ and $|X| = \gamma_p(T) - 1$,

$$\sum_{i=1}^{d} |X \cap V(T_i)| = |X| = \gamma_p(T) - 1 = |\mathscr{U}_T| - 1 = \sum_{i=1}^{d} |\mathscr{U}_T \cap V(T_i)| \tag{4.4}$$

and, by (3.2) and Lemma 3.4(1),

$$p+1 = \eta_p(V(T), X, T) = \eta_p(x, X, T) + \sum_{i=1}^{d} \eta_p(V(T_i), X \cap V(T_i), T_i). \tag{4.5}$$

Claim 1. *For $t+1 \le i \le d$, $X \cap V(T_i) = \mathscr{U}_T \cap V(T_i)$ if $|X \cap V(T_i)| = |\mathscr{U}_T \cap V(T_i)|$.*

Proof. Suppose, to be contrary, that $X \cap V(T_i) \ne \mathscr{U}_T \cap V(T_i)$. Let $X' = (\mathscr{U}_T \cap V(T_i)) \cup (X - V(T_i))$. Then $x \notin X'$, $|X'| = |X| < \gamma_p(T)$ and $|X' \cap \mathscr{U}_T| > |X \cap \mathscr{U}_T|$.

Since $x_i \notin N_p(x, \mathscr{U}_T, T)$, $\mathscr{U}_T \cap V(T_i)$ is the unique γ_p-set of T_i by Lemma 4.2(1), and so $X \cap V(T_i) \not\succ_p V(T_i)$ but $X' \succ_p V(T_i)$ in T. Thus,

$$\eta_p(V(T_i), X \cap V(T_i), T) \ge 1, \tag{4.6}$$

$$\eta_p(V(T_i), X', T) = 0. \tag{4.7}$$

Note that $T = T_i \oplus_{x_i x} (T - T_i)$ and $X' \cap V(T - T_i) = X' - V(T_i) = X - V(T_i)$. By (3.2),

$$\begin{aligned}
&\eta_p(V(T), X', T) = \eta_p(V(T_i), X', T) + \eta_p(V(T - T_i), X', T)\\
&\text{(by (4.7) amd Lemma 3.4(1))} \;\le\; 0 + \eta_p(V(T - T_i), X' - V(T_i), T - T_i)\\
&\qquad = \eta_p(V(T - T_i), X - V(T_i), T - T_i)\\
&\text{(by Lemma 3.4(1))} \le \eta_p(V(T - T_i), X, T) + 1\\
&\text{(by (4.6))} \le \eta_p(V(T - T_i), X, T) + \eta_p(V(T_i), X \cap V(T_i), T)\\
&\text{(since } x \notin X) \;=\; \eta_p(V(T - T_i), X, T) + \eta_p(V(T_i), X, T)\\
&\qquad = \eta_p(V(T), X, T)\\
&\qquad = p + 1,
\end{aligned}$$

which means that X' is another counterexample to Lemma 4.4 with $|X' \cap \mathscr{U}_T| > |X \cap \mathscr{U}_T|$, a contradiction to the choice of X. □

Claim 2. *$|X \cap V(T_i)| = |\mathscr{U}_T \cap V(T_i)|$ for $1 \le i \le d$.*

Proof. Suppose not, (4.4) implies that there is some i such that $|X \cap V(T_i)| < |\mathscr{U}_T \cap V(T_i)|$. Let

$$X' = (\mathscr{U}_T - V(T_i)) \cup (X \cap V(T_i)).$$

Then $|X'| < |\mathscr{U}_T| = \gamma_p(T)$. Since $\mathscr{U}_T \succ_p V(T)$ and $x \in \mathscr{U}_T - V(T_i) \subseteq X'$, $X' \succ_p V(T - T_i)$ in T and so $\eta_p(V(T - T_i), X', T) = 0$. Therefore,

$$\begin{aligned}
\eta_p(V(T_i), X \cap V(T_i), T_i) &\ge \eta_p(V(T_i), X', T) \quad \text{(by Lemma 3.4(1))}\\
&= \eta_p(V(T), X', T)\\
&\ge r_p(T) = p + 1, \quad \text{(by Lemma 3.1)}
\end{aligned}$$

from which and (4.5), it follows that

$$\eta_p(V(T), X', T) = p + 1, \tag{4.8}$$

$$\eta_p(V(T_j), X \cap V(T_j), T_j) = 0 \quad \text{for } j \ne i, \tag{4.9}$$

$$\eta_p(x, X, T) = 0. \tag{4.10}$$

Note that X' is an η_p-set of T by (4.8) since $r_p(T) = p + 1$ and $|X'| < \gamma_p(T)$. By Lemma 3.2, $|X'| = \gamma_p(T) - 1 = |\mathscr{U}_T| - 1$ and so

$$|X \cap V(T_i)| = |X'| - |\mathscr{U}_T - V(T_i)| = |\mathscr{U}_T| - 1 - |\mathscr{U}_T - V(T_i)| = |\mathscr{U}_T \cap V(T_i)| - 1. \tag{4.11}$$

On the other hand, (4.9) implies that, for $j \ne i$, $X \cap V(T_j) \succ_p V(T_j)$ in T_j and it follows from Lemmas 4.2 and 3.2 that

$$|X \cap V(T_j)| \ge \gamma_p(T_j) = \begin{cases} |\mathscr{U}_T \cap V(T_j)| + 1 & \text{if } 1 \le j \le t;\\ |\mathscr{U}_T \cap V(T_j)| & \text{if } t + 1 \le j \le d. \end{cases} \tag{4.12}$$

(4.4), (4.11) and (4.12) together imply that $|\{1, \ldots, t\} - \{i\}| \le 1$, that is, $t \le 2$. Furthermore, from the hypothesis $\mu_p(x, \mathscr{U}_T, T) \ge p + 2$ and (3.3), we obtain that

$$t = |N_p(x, \mathscr{U}_T, T)| = \mu_p(x, \mathscr{U}_T, T) - \max\{0, p - |N_T(x) \cap \mathscr{U}_T|\} = 2$$

and $|N_T(x) \cap \mathscr{U}_T| = 0$. It follows from (4.4), (4.11) and (4.12) that

$$|X \cap V(T_j)| = |\mathscr{U}_T \cap V(T_j)|, \quad \text{for } j \ge t + 1 = 3.$$

By Claim 1, $X \cap V(T_j) = \mathscr{U}_T \cap V(T_j)$ for $3 \le j \le d$ and so

$$\begin{aligned}
|N_T(x) \cap X| &= \sum_{j=1}^{2} |N_T(x) \cap (X \cap V(T_j))| + \sum_{j=3}^{d} |N_T(x) \cap (\mathscr{U}_T \cap V(T_j))|\\
&\le (1 + 1) + |N_T(x) \cap \mathscr{U}_T| = 2 < p,
\end{aligned}$$

which means that $\eta_p(x, X, T) \ge 1$ by (3.1) since $x \notin X$, a contradiction to (4.10). The claim follows. □

We now continue to prove the lemma. Let $I = \{i \mid 1 \le i \le t \text{ and } x_i \in X\}$. Since $p \ge 3$, it follows from Claim 2 and the conclusion (b) in Lemma 4.2(2) that

$$\sum_{i \in I} \eta_p(V(T_i), X \cap V(T_i), T_i) \ge \sum_{i \in I} (p - 1) = |I|(p - 1). \tag{4.13}$$

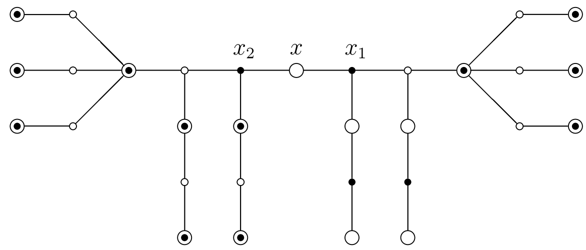

Fig. 2. A tree T with $r_2(T) = 3$.

For $i \in \{1, \ldots, t\} \setminus I$, Claim 2 and the conclusion (a) in Lemma 4.2(2) together imply that $|X \cap V(T_i)| = |\mathscr{U}_T \cap V(T_i)| < \gamma_p(T_i)$, and so $X \cap V(T_i) \not\succ_p V(T_i)$. Thus,

$$\sum_{i\in\{1,\ldots,t\}\setminus I} \eta_p(V(T_i), X \cap V(T_i), T_i) \geq \sum_{i\in\{1,\ldots,t\}\setminus I} 1 = t - |I| = |N_p(x, \mathscr{U}_T, T)| - |I|. \tag{4.14}$$

On the other hand, Claims 1 and 2 together imply that $X \cap V(T_i) = \mathscr{U}_T \cap V(T_i)$ for $t + 1 \leq t \leq d$. Note that $x_i \notin \mathscr{U}_T$ for $1 \leq i \leq t$. Thus

$$\begin{aligned} |N_T(x) \cap X| &= \sum_{i=1}^{d} |N_T(x) \cap (X \cap V(T_i))| \\ &= |I| + \sum_{i=t+1}^{d} |N_T(x) \cap (\mathscr{U}_T \cap V(T_i))| = |N_T(x) \cap \mathscr{U}_T| + |I|. \end{aligned} \tag{4.15}$$

Since $x \notin X$, $\eta_p(x, X, T) = \max\{0, p - |N_T(x) \cap X|\}$ by (3.1). Therefore,

$$\begin{aligned} p + 1 &= \eta_p(x, X, T) + \sum_{i=1}^{d} \eta_p(V(T_i), X \cap V(T_i), T_i) \quad (\text{by } (4.5)) \\ &\geq \max\{0, p - |N_T(x) \cap X|\} + |N_p(x, \mathscr{U}_T, T)| + |I|(p-2) \quad (\text{by } (4.13)\text{–}(4.14)) \\ &\geq \max\{0, p - |N_T(x) \cap \mathscr{U}_T|\} - |I| + |N_p(x, \mathscr{U}_T, T)| + |I|(p-2) \quad (\text{by } (4.15)) \\ &\geq \mu_p(x, \mathscr{U}_T, T) \quad (\text{by } (3.3),\ \text{since } p \geq 3) \\ &\geq p + 2, \end{aligned}$$

a contradiction. The lemma follows. ■

Remark 4.5. Lemma 4.4 is not true for $p = 2$.

Consider the tree T shown in Fig. 2, in which $\mathscr{U}_T$ consists of all large circles in T, $\gamma_2(T) = |\mathscr{U}_T| = 17$, $r_2(T) = 3$, $x \in \mathscr{U}_T$ and $\mu_2(x, \mathscr{U}_T, T) = 4$ by (3.3). Let X be the set of black vertices in T. Then $|X| = 16 < \gamma_2(T)$, however, $\eta_2(V(T), X, T) = 3$ by (3.1) and (3.2).

5. Proof of Theorem 2.5

In this section, we will complete the proof of Theorem 2.5. For the convenience, let $H_1 = K_{1,p-1}$, $H_2 = K_{1,p}$, $H_3 = F_{p-1}$ and $H_4 = F_{t,p-1}$ with $t \geq p$. Let $\mathscr{O}_i$ denote the operation $\mathscr{O}$ if $H = H_i$ for $i \in \{1, 2, 3, 4\}$ in Definition 2.3.

Let $p \geq 3$ and T a tree obtained from a star $K_{1,p}$ by $\mathscr{O}_i$ for some $i \in \{1, 2, 3, 4\}$. By the condition of $\mathscr{O}_i$, $i \neq 3$ and

$$T = \begin{cases} F_{p-1} & \text{if } i = 1; \\ S_{p,p} & \text{if } i = 2; \\ F_{t+1,p-1} \text{ or } K_{1,p} \oplus_{xy} F_{t,p-1} & \text{if } i = 4, \end{cases}$$

where $t \geq p$, x is the center of $K_{1,p}$, and $S_{p,p}$ is a tree obtained from a complete graph K_2 by attaching p leaves at each vertex of K_2. By calculating η_p in (3.1) and (3.2), $r_p(T) = p + 1$ by Lemma 3.1.

The sufficiency of Theorem 2.5 follows from the above fact and the following lemma by the definition of $\mathscr{T}_p$.

Lemma 5.1. *Let $p \geq 3$ be an integer and G a tree with $r_p(G) = p + 1$. If T is obtained from G by operation $\mathscr{O}_i$ for $i = 1, 2, 3, 4$, then $r_p(T) = p + 1$.*

Proof. Since T is obtained from G by operation $\mathscr{O}_i$ for some $i \in \{1, 2, 3, 4\}$,

$$T = G \oplus_{xy} H_i,$$

where y is the center of H_i and x satisfies the conditions in Definition 2.3. Note that $r_p(G) = p + 1$ and G has the unique γ_p-set $\mathscr{U}_G$ by Lemma 4.1(2). By Observation 2.4,

$$\gamma_p(T) = |\mathscr{U}_G| + \begin{cases} p-1 & \text{if } i = 1; \\ \gamma_p(H_i) & \text{if } i \in \{2, 3, 4\}. \end{cases} \tag{5.1}$$

To complete the proof of the lemma, it suffices to show that $r_p(T) \geq p + 1$ by Theorem 1.1. Suppose, to be contrary, that $r_p(T) \leq p$. Let S be an η_p-set of T such that

(1) $|S \cap V(G)|$ is as small as possible,
(2) subject to (1), $|S \cap \mathscr{U}_G|$ is as large as possible.

Then $\eta_p(V(T), S, T) = r_p(T) \leq p$ and $|S| = \gamma_p(T) - 1$ by Lemma 3.2. We will deduce a contradiction by distinguishing the following two cases.

Case 1. $|S \cap V(G)| \geq |\mathscr{U}_G|$.

We claim that $S \cap V(G) = \mathscr{U}_G$. Suppose, to be contrary, that $S \cap V(G) \neq \mathscr{U}_G$. Let

$$S' = \mathscr{U}_G \cup \begin{cases} S \cap V(H_i) & \text{if } |S \cap V(G)| = |\mathscr{U}_G|; \\ (S \cap V(H_i)) \cup \{y\} & \text{if } |S \cap V(G)| > |\mathscr{U}_G|. \end{cases}$$

Then $|S'| \leq |S| = \gamma_p(T) - 1$ and either $|S' \cap V(G)| < |S \cap V(G)|$ or $|S' \cap V(G)| = |S \cap V(G)|$ and $|S' \cap \mathscr{U}_G| > |S \cap \mathscr{U}_G|$. This contradicts the choice of S if S' is an η_p-set of T. Thus, to prove the claim, it suffices to show that S' is an η_p-set of T. Since S is an η_p-set of T, by the definition of η_p-set,

$$\eta_p(V(T), S', T) \geq \eta_p(V(T), S, T). \tag{5.2}$$

Note that $\eta_p(V(G), S', T) = 0$ since $S' \cap V(G) = \mathscr{U}_G \succ_p V(G)$. If $|S \cap V(G)| > |\mathscr{U}_G|$, then $S' \cap V(H_i) = S \cap V(H_i) \cup \{y\}$ and so $\eta_p(V(H_i), S', T) \leq \eta_p(V(H_i), S, T)$ by (3.1) and (3.2). Therefore, by (3.2),

$$\begin{aligned} \eta_p(V(T), S', T) &= \eta_p(V(G), S', T) + \eta_p(V(H_i), S', T) \\ &\leq \eta_p(V(G), S, T) + \eta_p(V(H_i), S, T) = \eta_p(V(T), S, T). \end{aligned} \tag{5.3}$$

If $|S \cap V(G)| = |\mathscr{U}_G|$, then $S \cap V(G) \not\succ_p V(G)$ since $S \cap V(T) \neq \mathscr{U}_G$ and $\mathscr{U}_G$ is the unique γ_p-set of G. Note that $S \cap V(H_i) = S' \cap V(H_i)$ and $T = G \oplus_{xy} H_i$. Let $\delta = 0$ if $y \in S$, and $\delta = 1$ if $y \notin S$. By (3.1) and (3.2),

$$\begin{aligned} \eta_p(V(T), S, T) &= \eta_p(V(G), S, T) + \eta_p(V(H_i), S, T) \\ &\geq \delta + (\eta_p(V(H_i), S', T) - \delta) \\ &= \eta_p(V(G), S', T) + \eta_p(V(H_i), S', T) \\ &= \eta_p(V(T), S', T). \end{aligned} \tag{5.4}$$

Since S is an η_p-set of T, S' is also an η_p-set of T by (5.2)–(5.4). The claim holds.

By the above claim, $|\mathscr{U}_G| + |S \cap V(H_i)| = |S| = \gamma_p(T) - 1$ and so, by (5.1),

$$|S \cap V(H_i)| = \begin{cases} p-2 & \text{if } i = 1; \\ \gamma_p(H_i) - 1 & \text{if } i \in \{2, 3, 4\}. \end{cases}$$

Note that $H_i \in \{K_{1,p-1}, K_{1,p}, F_{p-1}, F_{t,p-1}\}$ with $t \geq p \geq 3$. By calculating directly η_p by (3.1) and (3.2),

$$\eta_p(V(H_i), S \cap V(H_i), H_i) \geq \begin{cases} p+2 & \text{if } i \neq 2 \text{ and } S \cap V(H_i) \not\succ_p y; \\ p+1 & \text{otherwise.} \end{cases} \tag{5.5}$$

Note that $T = G \oplus_{xy} H_i$. Since $S \cap V(G) = \mathscr{U}_G$, $\eta_p(V(G), S, T) = 0$ and, by Definition 2.3, $i \neq 2$ if $x \in S$. It follows from (3.2), Lemma 3.4 and (5.5) that

$$\begin{aligned} \eta_p(V(T), S, T) &= \eta_p(V(H_i), S, T) + \eta_p(V(G), S, T) \\ &= \begin{cases} \eta_p(V(H_i), S \cap V(H_i), H_i) - 1 & \text{if } x \in S \text{ and } S \cap V(H_i) \not\succ_p y; \\ \eta_p(V(H_i), S \cap V(H_i), H_i) & \text{otherwise.} \end{cases} \\ &\geq p + 1. \end{aligned}$$

Since S is an η_p-set of T, $r_p(T) = \eta_p(V(T), S, T) \geq p + 1$ by Lemma 3.1, which contradicts the assumption $r_p(T) \leq p$.

Case 2. $|S \cap V(G)| < |\mathscr{U}_G|$.

Note that $T = G \oplus_{xy} H_i$ with $r_p(T) \le p$ and S is an η_p-set of T. Since $|S \cap V(G)| < |\mathscr{U}_G|$, Lemma 3.1 implies that $\eta_p(V(G), S \cap V(G), G) \ge r_p(G) = p + 1$, and hence

$$\begin{aligned} p &\ge r_p(T) = \eta_p(V(T), S, T) \\ \text{(by (3.2))} &= \eta_p(V(G), S, T) + \eta_p(V(H_i), S, T) \\ \text{(by Lemma 3.4(1))} &\ge \eta_p(V(G), S \cap V(G), G) - \begin{cases} 1 & \text{if } S \cap V(G) \not\succ_p x \text{ and } y \in S; \\ 0 & \text{otherwise.} \end{cases} \\ &\ge \begin{cases} p & \text{if } S \cap V(G) \not\succ_p x \text{ and } y \in S; \\ p+1 & \text{otherwise,} \end{cases} \end{aligned}$$

from which we obtain that $S \cap V(G) \not\succ_p x$ (and so $x \notin S$), $y \in S$,

$$\eta_p(V(G), S \cap V(G), G) = p + 1, \quad \text{and} \tag{5.6}$$

$$\eta_p(V(H_i), S, T) = 0. \tag{5.7}$$

By (5.6), $\eta_p(V(G), S \cap V(G), G) = r_p(G)$ and so $S \cap V(G)$ is an η_p-set of G. Note that $|S| = \gamma_p(T) - 1$. By Lemma 3.2 and (5.1), $|S \cap V(G)| = |\mathscr{U}_G| - 1$ and

$$|S \cap V(H_i)| = \begin{cases} p-1 & \text{if } i = 1; \\ \gamma_p(H_i) & \text{if } i \in \{2, 3, 4\}. \end{cases} \tag{5.8}$$

Note that $H_i \in \{K_{1,p-1}, K_{1,p}, F_{p-1}, F_{t,p-1}\}$ $(t \ge p \ge 3)$ with center y and $y \in S \cap V(H_i)$. Since $x \notin S$, $S \cap V(H_i) \succ_p V(H_i)$ by (5.7), which and (5.8) together imply that $H_i = F_{p-1}$. By the definition of $\mathscr{O}$ in Definition 2.3,

$$|N_p(x, \mathscr{U}_G, G)| \ge \min\{p + 1, |N_G(x) \cap \mathscr{U}_G| + 2\}.$$

Thus either $\mu_p(x, \mathscr{U}_G, G) \ge p + 2$ by (3.3) or $|N_p(x, \mathscr{U}_G, G)| = p + 1$ and $|N_G(x) \cap \mathscr{U}_G| \ge p$.

In the former case, it follows from Lemma 4.4 that $\eta_p(V(G), S \cap V(G), G) \ge p + 2$ since $x \notin S$ and $S \cap V(G) = |\mathscr{U}_G| - 1$, which contradicts (5.6).

In the latter case, since $S \cap V(G) \not\succ_p x$ and $|N_G(x) \cap \mathscr{U}_G| \ge p$, there is a vertex $z \in N_G(x)$ such that $z \in \mathscr{U}_G$ but $z \notin S$. Let G_z denote the component of $G - x$ containing z. Note that $z \notin N_p(x, \mathscr{U}_G, G)$ and $\eta_p(x, S \cap V(G), G) \ge 1$ since $x \in \mathscr{U}_G$ and $S \cap V(G) \not\succ_p x$. If $|S \cap V(G_z)| < |\mathscr{U}_G \cap V(G_z)|$, then $\eta_p(V(G_z), S \cap V(G_z), G_z) \ge r_p(G_z) = p + 1$ by Lemmas 3.1 and 4.2(1). Since $x \notin S$, Lemma 3.4(1) implies that $\eta_p(V(G_z), S \cap V(G), G) = \eta_p(V(G_z), S \cap V(G_z), G_z)$. Hence, by (3.2),

$$\begin{aligned} \eta_p(V(G), S \cap V(G), G) &\ge \eta_p(x, S \cap V(G), G) + \eta_p(V(G_z), S \cap V(G), G) \\ &\ge 1 + \eta_p(V(G_z), S \cap V(G_z), G_z) \\ &\ge p + 2, \end{aligned}$$

which contradicts (5.6). If $|S \cap V(G_z)| \ge |\mathscr{U}_G \cap V(G_z)|$, then let

$$S' = (S - V(G_z)) \cup (\mathscr{U}_G \cap V(G_z)).$$

Obviously, $|S' \cap V(G)| \le |S \cap V(G)|$ and $|S' \cap \mathscr{U}_G| > |S \cap \mathscr{U}_G|$ since $z \in \mathscr{U}_G - S$. To end the proof, it suffices to prove that S' is an η_p-set of T (this is a contradiction to the choice of S). Note that $T = (T - G_z) \oplus_{xz} G_z$. Since $S' - V(G_z) = S - V(G_z)$ and $z \in S' - S$, it follows from (3.1) and (3.2) that

$$\eta_p(V(T - G_z), S', T) \le \eta_p(V(T - G_z), S, T).$$

Since $z \notin N_p(x, \mathscr{U}_G, G)$, Lemma 4.2(1) implies that $\mathscr{U}_G \cap V(G_z)$ is the unique γ_p-set of G_z and so $S' \succ_p V(G_z)$ in T. By (3.2),

$$\begin{aligned} \eta_p(V(T), S', T) &= \eta_p(V(T - G_z), S', T) + \eta_p(V(G_z), S', T) \\ &\le \eta_p(V(T - G_z), S, T) + 0 \\ &\le \eta_p(V(T - G_z), S, T) + \eta_p(V(G_z), S, T) = \eta_p(V(T), S, T). \end{aligned}$$

On the other hand, since $|S'| = |S - V(G_z)| + |\mathscr{U}_G \cap V(G_z)| \le |S - V(G_z)| + |S \cap V(G_z)| = |S|$ and S is an η_p-set of T, $\eta_p(V(T), S', T) \ge \eta_p(V(T), S, T)$ by Lemma 3.1. Hence,

$$\eta_p(V(T), S', T) = \eta_p(V(T), S, T),$$

which means that S' is also an η_p-set of T. The lemma holds. ■

It remains to establish the necessity of Theorem 2.5. We do so by proving Lemma 5.2.

Let T be a tree with the unique γ_p-set $\mathscr{U}_T$. Define

$$M_p(T) = \{x \in V(T) \mid \text{ there is some } y \in \mathscr{U}_T \text{ such that } x \in N_p(y, \mathscr{U}_T, T)\}$$

and $m_p(T) = |M_p(T)|$. Since each vertex in $M_p(T)$ is the common p-private neighbor of exactly p vertices of $\mathcal{U}_T$ with respect to $\mathcal{U}_T$,

$$m_p(T) = |M_p(T)| = \frac{1}{p} \sum_{x \in \mathcal{U}_T} |N_p(x, \mathcal{U}_T, T)|. \tag{5.9}$$

Let T be a tree rooted at r and $y \in V(T)$. We use $C(y)$ and $D(y)$ to denote the sets of children and descendants, respectively, of y, and define $D[y] = D(y) \cup \{y\}$. For $x \in V(T)$, the notation $d_T(x, y)$ represents the distance between x and y in T. Define

$$\ell(y) = \max\{d_T(x, y) \mid x \in D[y]\}. \tag{5.10}$$

Lemma 5.2. *Let $p \geq 3$ be an integer and T a tree. If $r_p(T) = p + 1$, then $T \in \mathcal{T}_p$.*

Proof. Induction on $m_p(T)$. Since $p \geq 3$ and $r_p(T) = p + 1$, Lemma 4.1 implies that T has the unique γ_p-set $\mathcal{U}_T$ and $|N_p(v, \mathcal{U}_T, T)| \geq 1$ for $v \in \mathcal{U}_T$. Note that $|\mathcal{U}_T| = \gamma_p(T) > p$ by (1.1) since $r_p(T) = p + 1 > 0$. By (5.9),

$$m_p(T) = \frac{1}{p} \sum_{v \in \mathcal{U}_T} |N_p(v, \mathcal{U}_T, T)| \geq \frac{1}{p} \sum_{v \in \mathcal{U}_T} 1 = \frac{1}{p} |\mathcal{U}_T| > 1.$$

Furthermore, $m_p(T) \geq 2$ since $m_p(T)$ is an integer.

If $m_p(T) = 2$, then let $M_p(T) = \{x, y\}$. Since $|N_p(v, \mathcal{U}_T, T)| \geq 1$ for $v \in \mathcal{U}_T$, $\mathcal{U}_T = (N_T(x) \cap \mathcal{U}_T) \cup (N_T(y) \cap \mathcal{U}_T)$. Suppose that T has a vertex z not in $\mathcal{U}_T \cup \{x, y\}$. Since $p \geq 3$ and $\mathcal{U}_T \succ_p z$, z has two neighbors in either $N_T(x) \cap \mathcal{U}_T$ or $N_T(y) \cap \mathcal{U}_T$, which means that T contains a cycle with length 4. This contradicts that T is a tree. So

$$V(T) = \mathcal{U}_T \cup \{x, y\} = (N_T(x) \cap \mathcal{U}_T) \cup (N_T(y) \cap \mathcal{U}_T) \cup \{x, y\}.$$

Note that x and y have at most one common neighbor in T. Since $|N_T(x) \cap \mathcal{U}_T| = p$ and $|N_T(y) \cap \mathcal{U}_T| = p$,

$$T = F_{p-1} \quad \text{or} \quad S_{p,p}.$$

Thus T is obtained from $K_{1,p}$ by $\mathcal{O}_1$ if $T = F_{p-1}$, otherwise by $\mathcal{O}_2$, and so $T \in \mathcal{T}_p$. This establishes the base case.

Let $m_p(T) \geq 3$. Assume that, for any tree T' with $r_p(T') = p + 1$, if $m_p(T') < m_p(T)$ then $T' \in \mathcal{T}_p$.

We root T at a leaf r. Since $m_p(T) \geq 3$, T is not a star and so $\ell(r) \geq 3$ by (5.10). By (5.10), $0 \leq \ell(y) \leq \ell(r)$ for $y \in V(T)$. Let

$$V_i = \{y \in V(T) \mid \ell(y) = i\} \quad \text{for } 0 \leq i \leq \ell(r). \tag{5.11}$$

Then $\{V_0, V_1, \ldots, V_{\ell(r)}\}$ is a partition of $V(T)$, and satisfies the following properties.

I. $V_0 = L(T) - \{r\} \subseteq \mathcal{U}_T$. It is trivial by (5.11) and Lemma 2.1.
II. For $y \in V_1$, (i) $y \in M_p(T)$; (ii) $d_T(y) = p$ or $p + 1$; (iii) $T \in \mathcal{T}_p$ if $d_T(y) = p + 1$.

Proof of II. By (5.11), $D(y) \subseteq V_0$ and y has exactly one neighbor (i.e., the father of y) not in V_0. By I and Lemma 4.1(1), $y \in M_p(T)$ and so $d_T(y) = p$ or $p + 1$. Both (i) and (ii) hold.

We now prove (iii). Let $T' = T - D[y]$ and x be the father of y. Since $D(y) \subseteq V_0 \subseteq \mathcal{U}_T$ by I and $d_T(y) = p + 1$, $x \notin \mathcal{U}_T$ and $T[D[y]] = K_{1,p}$ with center y. So $T = T' \oplus_{xy} K_{1,p}$. Since $\mathcal{U}_T$ is a DS_p of T containing no $\{x, y\}$, $\mathcal{U}_T \cap V(T') \succ_p V(T')$ and $D(y) \succ_p D[y]$. Thus

$$\gamma_p(T) = |\mathcal{U}_T| = |\mathcal{U}_T \cap V(T')| + |D(y)| \geq \gamma_p(T') + \gamma_p(K_{1,p}).$$

Furthermore, $\gamma_p(T) = \gamma_p(T') + \gamma_p(K_{1,p})$ since the union between a γ_p-set of T' and a γ_p-set of $K_{1,p}$ is also a DS_p of T, which implies that $\mathcal{U}_T \cap V(T')$ is a γ_p-set of T'. Since $M_p(T) \geq 3$, T' contains at least two vertices not in $\mathcal{U}_T$ and, to p-dominate them, $|\mathcal{U}_T \cap V(T')| \geq p + 1$. By Theorem 1.1 and Lemma 3.4(2),

$$p + 1 \geq r_p(T') \geq r_p(T) = p + 1,$$

which implies that $r_p(T') = p + 1$. So $\mathcal{U}_{T'} = \mathcal{U}_T \cap V(T')$ by Lemma 4.1(2), and then $m_p(T') = m_p(T) - 1$ since $y \in M_p(T)$ by (i). Applying the induction on T', $T' \in \mathcal{T}_p$. Since $T = T' \oplus_{xy} K_{1,p}$ and $y \notin \mathcal{U}_T \cap V(T')$, T is obtained from T' by $\mathcal{O}_2$, and so $T \in \mathcal{T}_p$. (iii) follows. □

To the end, assume, by II, that

$$v \in M_p(T) \quad \text{and} \quad d_T(v) = p, \quad \text{for each } v \in V_1. \tag{5.12}$$

Then the father of each vertex in V_1 belongs to $\mathcal{U}_T$, and so

$$V_2 \subseteq \mathcal{U}_T. \tag{5.13}$$

Let $x \in V_3$ and $P = xwvu$ be a path in $T[D[x]]$ such that $d_T(w)$ is as large as possible. By I, (5.12) and (5.13), $u \in V_0 \subseteq \mathcal{U}_T$, $v \in V_1 \subseteq M_p(T)$ is a stem of T with $d_T(x) = p$, and $w \in V_2 \subseteq \mathcal{U}_T$. By Lemma 3.5,

$$\mu_p(w, \mathcal{U}_T, T) \geq r_p(T) = p + 1.$$

Case 1. $\mu_p(w, \mathscr{U}_T, T) \geq p+2$.

Let $T' = T - D[v]$. Since v is a stem of T and $|D(v)| = d_T(v) - 1 = p-1$, $T[D[v]] = K_{1,p-1}$ with center v and so $T = T' \oplus_{wv} K_{1,p-1}$. Since $m_p(T) \geq 3$, T' contains at least two p-private neighbors with respect to $\mathscr{U}_T$ and hence $|V(T')| \geq p+2$, which implies that $\gamma_p(T') \geq p+1$ since p vertices of the tree T' have at least one common neighbor in T'.

We claim that $r_p(T') = p+1$. It suffices to prove $r_p(T') \geq p+1$ by Theorem 1.1. Since $w \in V_2 \subseteq \mathscr{U}_T$, $\mathscr{U}_T \cap V(T') \succ_p T'$, and so $\gamma_p(T) = |\mathscr{U}_T| = |\mathscr{U}_T \cap V(T')| + (p-1) \geq \gamma_p(T') + p - 1$. Let X' be an η_p-set of T' and $X = X' \cup D(v)$. By Lemma 3.2,

$$|X| = |X'| + |D(v)| = (\gamma_p(T') - 1) + (p-1) < \gamma_p(T),$$

and then $\eta_p(V(T), X, T) \geq r_p(T) = p+1$ by Lemma 3.1. Since $v \notin X$, Lemma 3.4(1) implies that $\eta_p(V(T'), X', T') = \eta_p(V(T'), X, T)$. If $w \in X'$, then $\eta_p(V(K_{1,p-1}), X, T) = 0$ and

$$\begin{aligned} r_p(T') &= \eta_p(V(T'), X', T') = \eta_p(V(T'), X, T) \\ &= \eta_p(V(T), X, T) - \eta_p(V(K_{1,p-1}), X, T) \geq p+1. \end{aligned}$$

If $w \notin X'$, then $\eta_p(V(K_{1,p-1}), X, T) = 1$ by (3.1) and (3.2), and $\eta_p(V(T), X, T) \geq p+2$ by Lemma 4.4 since $r_p(T) = p+1$ and $\mu_p(w, \mathscr{U}_T, T) \geq p+2$. Thus

$$\begin{aligned} r_p(T') &= \eta_p(V(T'), X', T') = \eta_p(V(T'), X, T) \\ &= \eta_p(V(T), X, T) - \eta_p(V(K_{1,p-1}), X, T) \geq p+1. \end{aligned}$$

The claim follows.

Since $r_p(T') = p+1$, T' has the unique γ_p-set $\mathscr{U}_{T'}$ of T' by Lemma 4.1(1). We now show $\mathscr{U}_{T'} = \mathscr{U}_T \cap V(T')$. Suppose, to the contrary, that $\mathscr{U}_{T'} \neq \mathscr{U}_T \cap V(T')$. Note that $\mathscr{U}_T \cap V(T') \succ_p V(T')$ since $w \in \mathscr{U}_T$. Then $|\mathscr{U}_T \cap V(T')| \geq |\mathscr{U}_{T'}| + 1$. Since $\mathscr{U}_{T'} \cup D[v] \succ_p V(T)$ and $|\mathscr{U}_{T'} \cup D[v]| = |\mathscr{U}_{T'}| + p \leq |\mathscr{U}_T \cap V(T')| + (p-1) = \gamma_p(T)$, $\mathscr{U}_{T'} \cup D[v]$ is a γ_p-set of T different to $\mathscr{U}_T$. This contradicts that $\mathscr{U}_T$ is the unique γ_p-set of T. Hence $\mathscr{U}_{T'} = \mathscr{U}_T \cap V(T')$.

Since $\mathscr{U}_{T'} = \mathscr{U}_T \cap V(T')$ and $v \in N_p(w, \mathscr{U}_T, T)$, $m_p(T') = m_p(T) - 1$. Applying the induction on T', $T' \in \mathscr{T}_p$. Since $w \in \mathscr{U}_T \cap V(T') = \mathscr{U}_{T'}$, T is obtained from T' by $\mathscr{O}_1$, and hence $T \in \mathscr{T}_p$.

Case 2. $\mu_p(w, \mathscr{U}_T, T) = p+1$.

By the definition of μ_p in (3.3),

$$|N_p(w, \mathscr{U}_T, T)| + \max\{0, p - |N_T(w) \cap \mathscr{U}_T|\} = p+1. \tag{5.14}$$

Since $w \in V_2 \subseteq \mathscr{U}_T$ by (5.13), w is not a stem of T and so $C(w) \subseteq V_1 \subseteq M_p(T)$ by (5.11) and II(i). Therefore,

$$C(w) \subseteq N_p(w, \mathscr{U}_T, T), \tag{5.15}$$

and, for $v' \in C(w)$, the component of $T - w$ containing v' is a star $K_{1,p-1}$ with center v'.

Case 2.1 $x \in \mathscr{U}_T$.

Let $T' = T - D[w]$. Since $x \in \mathscr{U}_T$, $N_p(w, \mathscr{U}_T, T) = C(w)$ by (5.15) and $N_T(w) \cap \mathscr{U}_T = \{x\}$. Thus $|C(w)| = 2$ by (5.14). So $T[D[w]] = F_{p-1}$ with center w and $T = T' \oplus_{xw} F_{p-1}$.

Since $x \notin N_p(w, \mathscr{U}_T, T)$, $r_p(T') = p+1$ and $\mathscr{U}_{T'} = \mathscr{U}_T \cap V(T')$ by Lemma 4.2(1). Thus $m_p(T') = m_p(T) - |C(w)| < m_p(T)$. Applying the induction on T', $T' \in \mathscr{T}_p$. Hence if x satisfies the condition of $\mathscr{O}_3$, that is,

$$|N_p(x, \mathscr{U}_{T'}, T')| \geq \min\{p+1, |N_{T'}(x) \cap \mathscr{U}_{T'}| + 2\}, \tag{5.16}$$

then T is obtained from T' by $\mathscr{O}_3$ and $T \in \mathscr{T}_p$.

We now show (5.16). Since $x \in \mathscr{U}_T$ and $r_p(T) = p+1$, (3.3) and Lemma 3.5 together imply that $|N_p(x, \mathscr{U}_T, T)| + \max\{0, p - |N_T(x) \cap \mathscr{U}_T|\} = \mu_p(x, \mathscr{U}_T, T) \geq r_p(T) = p+1$, that is,

$$|N_p(x, \mathscr{U}_T, T)| \geq \min\{p+1, |N_T(x) \cap \mathscr{U}_T| + 1\}. \tag{5.17}$$

Since $w \in N_T(x) \cap \mathscr{U}_T$ and $\mathscr{U}_{T'} = \mathscr{U}_T \cap V(T')$, $|N_p(x, \mathscr{U}_{T'}, T')| = |N_p(x, \mathscr{U}_T, T)|$ and $|N_{T'}(x) \cap \mathscr{U}_{T'}| = |N_T(x) \cap \mathscr{U}_T| - 1$. Therefore, (5.16) follows from (5.17).

Case 2.2 $x \notin \mathscr{U}_T$.

Let $T' = T - D[x]$ and $T_0 = T[D[x]]$. Then $T = T' \oplus_{yx} T_0$, where y is the father of x.

We claim that $T_0 = F_{t,p-1}$ with center x, where $t = |C(x)| \geq p$. Note that $N_T(w) = C(w) \cup \{x\}$. Since $x \notin \mathscr{U}_T$ and $C(w) \subseteq N_p(w, \mathscr{U}_T, T)$ by (5.15), $N_T(w) \cap \mathscr{U}_T = \emptyset$ and so $|N_p(w, \mathscr{U}_T, T)| = 1$ by (5.14). Therefore, $C(w) = N_p(w, \mathscr{U}_T, T) = \{v\}$ and

$$x \notin \mathscr{U}_T \cup N_p(w, \mathscr{U}_T, T). \tag{5.18}$$

By (5.18), $|N_T(x) \cap \mathscr{U}_T| \geq p+1$ and so $t = |C(x)| = |N_T(x)| - 1 \geq p$. Let $w' \in C(x)$. By the choice of $P_4 = xwvu$, $d_T(w') \leq d_T(w) = 2 < p$. By Lemmas 2.1 and 4.1(1), $w' \in \mathscr{U}_T$ and $N_p(w', \mathscr{U}_T, T) \neq \emptyset$. It follows that $d_T(w') = 2$ since $x \in N_T(w')$ is not a p-private neighbor with respect to $\mathscr{U}_T$. Let $N_p(w', \mathscr{U}_T, T) = \{v'\}$. Then $v' \in V_1$ and $d_T(v') = p$ by (5.12). By the arbitrariness of w', $T_0 = T[D[x]] = F_{t,p-1}$ with center x. The claim holds.

Since $\mathscr{U}_T \cap V(T') \succ_p V(T')$ and $\mathscr{U}_T \cap V(T_0) \succ_p V(T_0)$ by (5.18), $\gamma_p(T) = |\mathscr{U}_T| = |\mathscr{U}_T \cap V(T')| + |\mathscr{U}_T \cap V(T_0)| \geq \gamma_p(T') + \gamma_p(T_0)$. Furthermore, $\gamma_p(T) = \gamma_p(T') + \gamma_p(T_0)$ because the union between a γ_p-set of T' and a γ_p-set of T_0 is a DS_p of T. So $\mathscr{U}_T \cap V(T')$ (resp., $\mathscr{U}_T \cap V(T_0)$) is a γ_p-set of T' (resp., T_0).

Note that $N_p(z, \mathscr{U}_T, T) \neq \emptyset$ for any $z \in \mathscr{U}_T$ by Lemma 4.1(1). (5.18) implies that T' has at least one p-private neighbor with respect to $\mathscr{U}_T$, and so $\gamma_p(T') = |\mathscr{U}_T \cap V(T')| \geq p$.

If $\gamma_p(T') = p$, then $T' = K_{1,p}$. Thus T is obtained from $K_{1,p}$ by $\mathscr{O}_4$ and $T \in \mathscr{T}_p$.

If $\gamma_p(T') \geq p + 1$, then $r_p(T') \geq r_p(T) = p + 1$ by Lemma 3.4(2), furthermore, $r_p(T') = p + 1$ by Theorem 1.1. Since $\mathscr{U}_T \cap V(T')$ is a γ_p-set of T', it follows from Lemma 4.1(2) that $\mathscr{U}_{T'} = \mathscr{U}_T \cap V(T')$, and hence $m_p(T') = m_p(T) - m_p(T_0) < m_p(T)$. Applying the induction on T', $T' \in \mathscr{T}_p$. Thus T is obtained from T' by $\mathscr{O}_4$ and $T \in \mathscr{T}_p$. ■

6. Conclusion

We characterize all trees with p-reinforcement number $p + 1$ for $p \geq 3$ by a recursive construction. Our proof strongly depends on Lemma 4.4. However, Lemma 4.4 is not true for $p = 2$ (see Remark 4.5). When $p = 2$, Theorem 1.1 implies that $r_2(T) \leq 3$ for any tree T. Very recently, Lu, Song and Yang [18] have presented a sufficient and necessary condition for a tree to have the 2-reinforcement number 3.

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