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RESEARCH ARTICLE

Transitivity of varietal hypercube networks

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Abstract The varietal hypercube VQ_n is a variant of the hypercube Q_n and has better properties than Q_n with the same number of edges and vertices. This paper proves that VQ_n is vertex-transitive. This property shows that when VQ_n is used to model an interconnection network, it is high symmetrical and obviously superior to other variants of the hypercube such as the crossed cube.

Keywords Combinatorics, graph, transitivity, varietal hypercube network
MSC 05C60, 68R10

1 Introduction

We follow [8] for graph-theoretical terminology and notation not defined here. A graph $G = (V, E)$ always means a simple undirected graph, where $V = V(G)$ is the vertex set and $E = E(G)$ is the edge set of G. It is well known that interconnection networks play an important role in parallel computing/communication systems. An interconnection network can be modeled by a graph $G = (V, E)$, where V is the set of processors and E is the set of communication links in the network.

The hypercube network Q_n has proved to be one of the most popular interconnection networks since it has a simple structure and has many nice properties. The varietal hypercubes were proposed by Cheng and Chuang [2] in 1994 as an attractive alternative to Q_n when they were used to model the interconnection network of a large-scale parallel processing system.

The n-dimensional *varietal hypercube* VQ_n is the labeled graph defined recursively as follows. VQ_1 is the complete graph of two vertices labeled with 0 and 1, respectively. Assume that VQ_{n-1} has been constructed. Let VQ_n^0 (resp. VQ_n^1) be a labeled graph obtained from VQ_{n-1} by inserting a zero (resp. 1) in front of each vertex-labeling in VQ_{n-1}. For $n > 1$, VQ_n is obtained

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by joining vertices in VQ_n^0 and VQ_n^1, according to the following rule: a vertex $X_n = 0x_{n-1}x_{n-2}x_{n-3}\cdots x_2x_1$ in VQ_n^0 and a vertex $Y_n = 1y_{n-1}y_{n-2}y_{n-3}\cdots y_2y_1$ in VQ_n^1 are adjacent in VQ_n if and only if

(i) $x_{n-1}x_{n-2}x_{n-3}\cdots x_2x_1 = y_{n-1}y_{n-2}y_{n-3}\cdots y_2y_1$ if $n \neq 3k$, or

(ii) $x_{n-3}\cdots x_2x_1 = y_{n-3}\cdots y_2y_1$ and $(x_{n-1}x_{n-2}, y_{n-1}y_{n-2}) \in I$ if $n = 3k$, where $I = \{(00,00),(01,01),(10,11),(11,10)\}$.

Figure 1 shows the examples of varietal hypercubes VQ_n for $n = 1, 2, 3$, and 4.

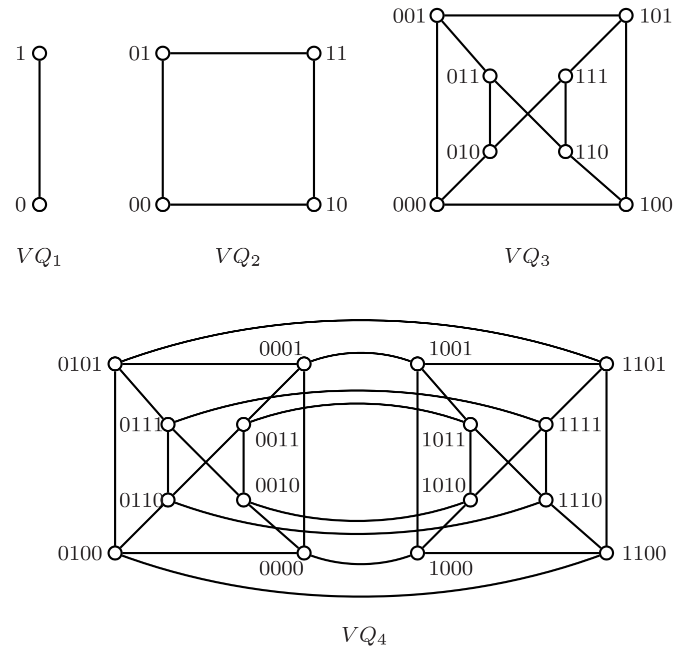

Fig. 1 Varietal hypercubes VQ_1, VQ_2, VQ_3, and VQ_4

Like Q_n, VQ_n is an n-regular graph with 2^n vertices and $n2^{n-1}$ edges, and has many properties similar or superior to Q_n. For example, the connectivity and restricted connectivity of VQ_n and Q_n are the same (see Wang and Xu [7]), while, all the diameter and the average distance, fault-diameter and wide-diameter of VQ_n are smaller than that of the hypercube (see Cheng and Chuang [2], Jiang et al. [4]). Very recently, Cao et al. [1] have shown that VQ_n has better pancyclicity and panconnectivity than Q_n.

An *automorphism* of a graph G is a permutation σ on $V(G)$ satisfying the adjacency-preserving condition

$$xy \in E(G) \Longleftrightarrow \sigma(x)\sigma(y) \in E(G).$$

Under the operation of composition, the set of all automorphisms of G forms a group, denoted by $\mathrm{Aut}(G)$ and referred to as the *automorphism group* of G.

A graph G is *vertex-transitive* if for any given two vertices x and y in G, there is some $\sigma \in \mathrm{Aut}(G)$ such that $y = \sigma(x)$. A graph G is called to be *edge-transitive* if for any two given edges xy and uv of G, there is some $\sigma \in \mathrm{Aut}(G)$ such that $uv = \sigma(x)\sigma(y)$.

It has known that Q_n is vertex-transitive and edge-transitive, the folded hypercube FQ_n is vertex-transitive (see Ma and Xu [6]), the crossed hypercube CQ_n is not vertex-transitive for $n \geqslant 5$ (see Kulasinghe and Bettayeb [5]). However, transitivity of some variants of the hypercube has not been investigated (see Xu [9]). In this paper, we consider transitivity of VQ_n. Choose three vertices $X = 0101$, $Y = 1101$, and $Z = 0001$ in VQ_4 (see Fig. 1), the edge XZ is contained in a cycle of length 5, but the edge XY is not. This fact shows that there is no $\sigma \in \mathrm{Aut}(VQ_4)$ such that $XZ = \sigma(X)\sigma(Y)$. However, we can show that VQ_n is vertex-transitive. This property shows that when VQ_n is used to model an interconnection network, it is high symmetrical and obviously superior to other variants of the hypercube, such as the crossed cube CQ_n.

2 Main results

An edge X_nY_n in VQ_n, where

$$X_n = x_n x_{n-1} \cdots x_2 x_1, \quad Y_n = y_n y_{n-1} \cdots y_2 y_1,$$

is called the *i-transversal edge* if

$$x_n \cdots x_{i-1} = y_n \cdots y_{i-1}, \quad x_i \neq y_i.$$

For convenience, we express VQ_n as $VQ_n^0 \odot VQ_n^1$, where

$$VQ_n^0 \cong VQ_n^1 \cong VQ_{n-1}.$$

Then edges between VQ_n^0 and VQ_n^1 are n-transversal edges. We call the edges of Type 2 *crossing edges* when

$$(x_{n-1}x_{n-2}, y_{n-1}y_{n-2}) \in \{(10, 11), (11, 10)\},$$

and call the other edges *normal edges*. For a given position integer n, let $I_n = \{1, 2, \ldots, n\}$, and let

$$V_n = \{x_n \cdots x_2 x_1 \mid x_i \in \{0, 1\},\ i \in I_n\}.$$

Clearly, $V(VQ_n) = V_n$. For a given $X_n = x_n \cdots x_2 x_1 \in V_n$, let $X_i = x_i \cdots x_2 x_1$. For $b \in \{0, 1\}$, let $\overline{b} = \{0, 1\} \setminus \{b\}$. By definitions, we immediately obtain the following simple observation.

Observation 1 Let $VQ_n = VQ_n^0 \odot VQ_n^1$, and let X_nY_n be an n-transversal edge in VQ_n, where $X_n \in VQ_n^0$ and $Y_n \in VQ_n^1$. For $n \geqslant 3$, if

$$X_n = 0ab x_{n-3} \cdots x_1,$$

then

$$Y_n = 1a'b'X_{n-3},$$

where $ab = a'b'$ if X_nY_n is a normal edge, and $(ab, a'b') = (1b, 1\overline{b})$ if X_nY_n is a crossing edge.

Lemma 1 *Define a mapping*

$$\begin{aligned} \sigma_1\colon\ & V_n \to V_n, \\ & X_n \mapsto \overline{x}_n X_{n-1}. \end{aligned} \tag{1}$$

Then $\sigma_1 \in \mathrm{Aut}(VQ_n)$.

Proof Clearly, σ_1 is a permutation on V_n. We only need to show that σ_1 preserves the adjacency of vertices in VQ_n. To the end, let $X_nY_n \in E(VQ_n)$, where

$$X_n = x_n x_{n-1} \cdots x_2 x_1, \quad Y_n = y_n y_{n-1} \cdots y_2 y_1.$$

Without loss of generality, assume $x_n = 0$. Then $\sigma_1(X_n) = 1X_{n-1}$ in VQ_n^1.

If $y_n = 0$, then $X_nY_n \in E(VQ_n^0)$, and so $X_{n-1}Y_{n-1} \in E(VQ_{n-1})$. Since $\sigma_1(Y_n) = 1Y_{n-1}$ in VQ_n^1, $\sigma_1(X_n)\sigma_1(Y_n)$ is an edge in VQ_n^1, and so in VQ_n.

We now assume $y_n = 1$. Then Y_n is in VQ_n^1 and $\sigma_1(Y_n)$ is in VQ_n^0. Thus, X_nY_n is an n-transversal edge.

If X_nY_n is a normal edge, then $X_{n-1} = Y_{n-1}$, and clearly,

$$\sigma_1(X_n)\sigma_1(Y_n) = Y_nX_n \in E(VQ_n).$$

If X_nY_n is a crossing edge, then

$$(x_{n-1}x_{n-2}, y_{n-1}y_{n-2}) = (1x_{n-2}, 1\overline{x}_{n-2}).$$

That is,

$$X = 01x_{n-2}X_{n-3}, \quad Y = 11\overline{x}_{n-2}X_{n-3},$$

and so

$$\begin{aligned} \sigma_1(X_n) &= 11x_{n-2}X_{n-3}, \\ \sigma_1(Y_n) &= 01\overline{x}_{n-2}X_{n-3}, \end{aligned}$$

which shows that $\sigma_1(X_n)\sigma_1(Y_n)$ is a crossing edge in VQ_n.

The lemma follows. □

Lemma 2 *For a given* $\phi \in \mathrm{Aut}(VQ_{n-3})$, *define a mapping* φ_i *from* V_{n-1} *to* V_{n-1} *for each* $i = 0, 1, 2, 3$ *subjected to*

$$\begin{aligned} \varphi_0(X_{n-1}) &= x_{n-1}x_{n-2}\phi(X_{n-3}), \\ \varphi_1(X_{n-1}) &= x_{n-1}\overline{x}_{n-2}\phi(X_{n-3}), \\ \varphi_2(X_{n-1}) &= \overline{x}_{n-1}x_{n-2}\phi(X_{n-3}), \\ \varphi_3(X_{n-1}) &= \overline{x}_{n-1}\overline{x}_{n-2}\phi(X_{n-3}). \end{aligned} \tag{2}$$

If $n = 3k$*, then* $\varphi_i \in \mathrm{Aut}(Q_{n-1})$ *for each* $i = 0, 1, 2, 3$.

Proof We first show $\varphi_1, \varphi_2 \in \mathrm{Aut}(Q_{n-1})$. To the end, we only need to show that both φ_1 and φ_2 preserve the adjacency of vertices in VQ_{n-1}. Let $X_{n-1}Y_{n-1} \in E(VQ_{n-1})$, where

$$X_{n-1} = x_{n-1} \cdots x_2 x_1, \quad Y_{n-1} = y_{n-1} \cdots y_2 y_1.$$

We want to show that

$$\varphi_i(X_{n-1})\varphi_i(Y_{n-1}) \in E(VQ_{n-1})$$

for each $i = 1, 2$.

Without loss of generality, assume $x_{n-1}x_{n-2} = 0b$, where $b \in \{0, 1\}$. Then

$$\varphi_1(X_{n-1}) = 0\bar{b}\phi(X_{n-3})$$

is in VQ^0_{n-1} and

$$\varphi_2(X_{n-1}) = 1b\phi(X_{n-3})$$

is in VQ^1_{n-1}. Note $n - 1 \neq 3k$ since $n = 3k$. There are two cases according to $y_{n-1} = 0$ or 1.

Case 1 $y_{n-1} = 0$.

Then $X_{n-1}Y_{n-1} \in E(VQ^0_{n-1})$.

(i) If $y_{n-2} = b$, then

$$X_{n-1}Y_{n-1} \in E(VQ^{0b}_{n-1}),$$

where VQ^{0b}_{n-1} denotes a subgraph of VQ_{n-1} obtained from VQ_{n-3} by inserting two digits $0b$ in front of each vertex-labeling in VQ_{n-3}, which is isomorphic to VQ_{n-3}. Thus,

$$X_{n-3}Y_{n-3} \in E(VQ_{n-3}).$$

Since $\phi \in \mathrm{Aut}(VQ_{n-3})$, we have

$$\phi(X_{n-3})\phi(Y_{n-3}) \in E(VQ_{n-3}).$$

Thus, two vertices $0b\phi(X_{n-3})$ and $0b\phi(Y_{n-3})$ are adjacent in VQ^{0b}_{n-1}, and so two vertices

$$\varphi_1(X_{n-1}) = 0\bar{b}\phi(X_{n-3}), \quad \varphi_1(Y_{n-1}) = 0\bar{b}\phi(Y_{n-3})$$
$$(\text{resp.} \quad \varphi_2(X_{n-1}) = 1b\phi(X_{n-3}), \quad \varphi_2(Y_{n-1}) = 1b\phi(Y_{n-3}))$$

are adjacent in $VQ^{0\bar{b}}_{n-1}$ (resp. VQ^{1b}_{n-1}), that is,

$$\varphi_i(X_{n-1})\varphi_i(Y_{n-1}) \in E(VQ_{n-1})$$

for each $i = 1, 2$.

(ii) If $y_{n-2} = \overline{b}$, then $(bX_{n-3})(\overline{b}Y_{n-3})$ is an $(n-2)$-transversal normal edge in VQ^0_{n-1} since $n-2 \neq 3k$. Thus, $X_{n-3} = Y_{n-3}$, and so $\phi(X_{n-3}) = \phi(Y_{n-3})$. Since

$$\varphi_1(X_{n-1}) = 0\overline{b}\phi(X_{n-3}), \quad \varphi_1(Y_{n-1}) = 0b\phi(X_{n-3})$$

$$(\text{resp.} \quad \varphi_2(X_{n-1}) = 1b\phi(X_{n-3}), \quad \varphi_2(Y_{n-1}) = 1\overline{b}\phi(X_{n-3})),$$

we know that $\varphi_1(X_{n-1})\varphi_1(Y_{n-1})$ (resp. $\varphi_2(X_{n-1})\varphi_2(Y_{n-1})$) is also an $(n-2)$-transversal normal edge in VQ^0_{n-1} (resp. VQ^1_{n-1}).

Case 2 $y_{n-1} = 1$.

Then $X_{n-1}Y_{n-1}$ is an $(n-1)$-transversal normal edge in VQ_{n-1} since $n-1 \neq 3k$, and so $X_{n-2} = Y_{n-2}$. Since $\phi(X_{n-3}) = \phi(Y_{n-3})$, we have

$$\varphi_1(X_{n-1}) = 0\overline{b}\phi(X_{n-3}), \quad \varphi_1(Y_{n-1}) = 1\overline{b}\phi(X_{n-3})$$

$$(\text{resp.} \quad \varphi_2(X_{n-1}) = 1b\phi(X_{n-3}), \quad \varphi_2(Y_{n-1}) = 0b\phi(X_{n-3})),$$

which implies that $\varphi_i(X_{n-1})\varphi_i(Y_{n-1})$ is an $(n-1)$-transversal edge in VQ_{n-1} for each $i = 1, 2$.

Thus, $\varphi_i \in \mathrm{Aut}(Q_{n-1})$ for each $i = 1, 2$. Since $\varphi_3 = \varphi_1\varphi_2$ and $\varphi_0 = \varphi_3^2$, we have $\varphi_3, \varphi_0 \in \mathrm{Aut}(Q_{n-1})$ immediately. The lemma follows. □

Lemma 3 *For a given* $\varphi \in \mathrm{Aut}(VQ_{n-1})$*, define a mapping*

$$\begin{aligned} \sigma_0 \colon\ & V_n \to V_n, \\ & X_n \mapsto x_n\varphi(X_{n-1}). \end{aligned} \tag{3}$$

When $n \neq 3k$*,* $\sigma_0 \in \mathrm{Aut}(VQ_n)$ *for any* $\varphi \in \mathrm{Aut}(VQ_{n-1})$*. When* $n = 3k$*, if*

$$\begin{aligned} &\varphi = \varphi_0 \quad \text{or} \quad \varphi_1 \quad \text{or} \\ &\varphi = \varphi_2 \quad \text{when } x_n = b \quad \text{and} \quad \varphi = \varphi_3 \quad \text{when } x_n = \overline{b}, \quad b \in \{0,1\}, \end{aligned} \tag{4}$$

where φ_i *is defined in* (2) *for each* $i = 0, 1, 2, 3$*, then* $\sigma_0 \in \mathrm{Aut}(VQ_n)$*.*

Proof It is easy to see that σ_0 is a permutation on V_n. We show $\sigma_0 \in \mathrm{Aut}(VQ_n)$. To the end, we only need to prove that σ_0 preserves the adjacency of vertices in VQ_n. Let

$$VQ_n = VQ^0_n \odot VQ^1_n,$$

and let

$$X_n = x_nx_{n-1}\cdots x_2x_1, \quad Y_n = y_ny_{n-1}\cdots y_2y_1$$

be any two adjacent vertices in VQ_n. Without loss of generality, let $x_n = 0$. Then X_n is in VQ^0_n. There are two cases according to $y_n = 0$ or 1.

Case 1 $y_n = 0$.

Since both X_n and Y_n are in VQ^0_n, for any $\varphi \in \mathrm{Aut}(VQ_{n-1})$, by (3), we have

$$\sigma_0(X_n) = 0\varphi(X_{n-1}), \quad \sigma_0(Y_n) = 0\varphi(Y_{n-1}).$$

Since $X_nY_n \in E(VQ_n^0)$, we have

$$X_{n-1}Y_{n-1} \in E(VQ_{n-1}),$$

and so

$$\varphi(X_{n-1})\varphi(Y_{n-1}) \in E(VQ_{n-1}),$$

and hence,

$$\varphi(X_n)\varphi(Y_n) \in E(VQ_n^0) \subset E(VQ_n).$$

Case 2 $y_n = 1$.

In this case, Y_n is in VQ_n^1 and X_nY_n is an n-transversal edge in VQ_n, which is either a normal edge or a crossing edge.

(i) X_nY_n is a crossing edge.

In this subcase, $n = 3k$, $X_{n-3} = Y_{n-3}$, and

$$X_n = 01x_{n-2}X_{n-3}, \quad Y_n = 11\overline{x}_{n-2}X_{n-3}$$

by Observation 1.

Since

$$X_{n-1} = 1x_{n-2}X_{n-3}, \quad Y_{n-1} = 1\overline{x}_{n-2}X_{n-3},$$

we know that $X_{n-1}Y_{n-1}$ is an $(n-2)$-dimensional normal edge in VQ_{n-1}^1 since $n-1 \neq 3k$. Thus, for any $\varphi \in \mathrm{Aut}(VQ_{n-1})$, $\varphi(X_{n-1})\varphi(Y_{n-1})$ is an $(n-2)$-dimensional normal edge in VQ_{n-1}^1. Without loss of generality, let

$$\varphi(X_{n-1}) = 10U_{n-3}.$$

Then

$$\varphi(Y_{n-1}) = 11U_{n-3},$$

and so

$$\sigma_0(X_n) = x_n\varphi(X_{n-1}) = 010U_{n-3},$$
$$\sigma_0(Y_n) = y_n\varphi(Y_{n-1}) = 111U_{n-3}.$$

Thus, $\sigma_0(X_n)\sigma_0(Y_n)$ is an n-dimensional crossing edge in VQ_n.

(ii) X_nY_n is a normal edge.

In this subcase, $X_{n-1} = Y_{n-1}$, and so

$$\varphi(X_{n-1}) = \varphi(Y_{n-1})$$

for any $\varphi \in \mathrm{Aut}(VQ_{n-1})$. By (3), we have

$$\sigma_0(X_n) = 0\varphi(X_{n-1}), \quad \sigma_0(Y_n) = 1\varphi(Y_{n-1}).$$

If $n \neq 3k$, then $\sigma_0(X_n)\sigma_0(Y_n) \in E(VQ_n)$ is a normal edge. Assume now $n = 3k$. Then

$$\{x_{n-1}x_{n-2}, y_{n-1}y_{n-2}) \in \{(00, 00), (01, 01)\}.$$

If $\varphi = \varphi_0$ or φ_1, then

$$\begin{cases} \sigma_0(X_{n-1}) = 0\varphi_0(X_{n-1}) = 0x_{n-1}x_{n-2}\phi(X_{n-3}), \\ \sigma_0(Y_{n-1}) = 1\varphi_0(Y_{n-1}) = 1x_{n-1}x_{n-2}\phi(X_{n-3}) \end{cases}$$

or

$$\begin{cases} \sigma_0(X_n) = 0\varphi_1(X_{n-1}) = 0x_{n-1}\overline{x}_{n-2}\phi(X_{n-3}), \\ \sigma_0(Y_n) \ = 1\varphi_1(Y_{n-1}) = 1x_{n-1}\overline{x}_{n-2}\phi(X_{n-3}). \end{cases}$$

Thus, $\sigma_0(X_{n-1})\sigma_0(Y_{n-1})$ is an n-dimensional normal edge in VQ_n.

If $\varphi = \varphi_2$ when $x_n = 1$ and $\varphi = \varphi_3$ when $x_n = 0$, then

$$\begin{cases} \sigma_0(X_n) = 0\varphi_3(X_{n-1}) = 0\overline{x}_{n-1}\overline{x}_{n-2}\phi(X_{n-3}), \\ \sigma_0(Y_n) = 1\varphi_2(Y_{n-1}) = 1\overline{x}_{n-1}x_{n-2}\phi(X_{n-3}). \end{cases}$$

If $\varphi = \varphi_2$ when $x_n = 0$ and $\varphi = \varphi_3$ when $x_n = 1$, then

$$\begin{cases} \sigma_0(X_n) = 0\varphi_2(X_{n-1}) = 0\overline{x}_{n-1}x_{n-2}\phi(X_{n-3}), \\ \sigma_0(Y_n) \ = 1\varphi_3(Y_{n-1}) = 1\overline{x}_{n-1}\overline{x}_{n-2}\phi(X_{n-3}). \end{cases}$$

Since

$$\{\overline{x}_{n-1}\overline{x}_{n-2}, \overline{x}_{n-1}x_{n-2}) \in \{(11, 10), (10, 11)\},$$

we know that $\sigma_0(X_n)$ and $\sigma_0(Y_n)$ are linked by an n-dimensional crossing edge in VQ_n.

Thus, we have proved that σ_0 preserves the adjacency of vertices in VQ_n, and so $\sigma_0 \in \mathrm{Aut}(VQ_n)$. The lemma follows. □

Theorem 1 *VQ_n is vertex-transitive for any $n \geqslant 1$.*

Proof We proceed by induction on $n \geqslant 1$. The conclusion is true for each $n = 1, 2$, clearly. Since VQ_3 is isomorphic to a Cayley graph $C(\mathbb{Z}_8, \{1, 4, 7\})$ (see Huang and Xu [3]), the conclusion is also true for $n = 3$.

Assume the induction hypothesis for any positive integer fewer than n with $n \geqslant 4$, that is, for any i with $1 \leqslant i \leqslant n-1$, VQ_i is vertex-transitive.

Let $VQ_n = VQ_n^0 \odot VQ_n^1$ be an n-dimensional varietal hypercube with $n \geqslant 4$. To prove that VQ_n is vertex-transitive, we need to prove that for any two vertices X_n and Y_n in VQ_n, there is some $\sigma \in \mathrm{Aut}(VQ_n)$ such that $\sigma(X_n) = Y_n$. To the end, let

$$X_n = x_n x_{n-1} \cdots x_2 x_1, \quad Y_n = y_n y_{n-1} \cdots y_2 y_1$$

be any two vertices in VQ_n. By the induction hypothesis, there is some $\varphi \in \mathrm{Aut}(VQ_{n-1})$ such that $\varphi(X_{n-1}) = Y_{n-1}$.

We now construct a $\sigma \in \mathrm{Aut}(VQ_n)$ with $\sigma(X_n) = Y_n$. Without loss of generality, let $x_n = 0$. Then X_n is in VQ_n^0. There are two cases.

Case 1 $y_n = 0$.

In this case, both X_n and Y_n are in VQ_n^0. Assume first $n \neq 3k$. By the induction hypothesis, there is some $\varphi \in \mathrm{Aut}(VQ_{n-1})$ such that $\varphi(X_{n-1}) = Y_{n-1}$. Let $\sigma = \sigma_0$, where σ_0 is defined in (3). By Lemma 3, $\sigma \in \mathrm{Aut}(VQ_n)$, and

$$\sigma(X_n) = 0\varphi(X_{n-1}) = 0Y_{n-1} = Y_n.$$

Assume $n = 3k$ below. Since Y_{n-1} is one of the following four forms:

$$\begin{aligned} Y_{n-1} &= x_{n-1}x_{n-2}Y_{n-3}, \\ Y_{n-1} &= x_{n-1}\overline{x}_{n-2}Y_{n-3}, \\ Y_{n-1} &= \overline{x}_{n-1}x_{n-2}Y_{n-3}, \\ Y_{n-1} &= \overline{x}_{n-1}\overline{x}_{n-2}Y_{n-3}, \end{aligned}$$

by the induction hypothesis, there is some $\phi \in \mathrm{Aut}(VQ_{n-3})$ such that

$$\phi(X_{n-3}) = Y_{n-3}.$$

Let $\sigma = \sigma_0$, where $\sigma_0 \in \mathrm{Aut}(VQ_n)$ is defined in (3). Then

$$\sigma_0(X_n) = \begin{cases} x_n\varphi_0(X_{n-1}) = 0x_{n-1}x_{n-2}\phi(X_{n-3}) = Y_n, & Y_{n-1} = x_{n-1}x_{n-2}Y_{n-3}, \\ x_n\varphi_1(X_{n-1}) = 0x_{n-1}\overline{x}_{n-2}\phi(X_{n-3}) = Y_n, & Y_{n-1} = x_{n-1}\overline{x}_{n-2}Y_{n-3}, \\ 0\varphi_2(X_{n-1}) = 0\overline{x}_{n-1}x_{n-2}\phi(X_{n-3}) = Y_n, & Y_{n-1} = \overline{x}_{n-1}x_{n-2}Y_{n-3}, \\ 0\varphi_3(X_{n-1}) = 0\overline{x}_{n-1}\overline{x}_{n-2}\phi(X_{n-3}) = Y_n, & Y_{n-1} = \overline{x}_{n-1}\overline{x}_{n-2}Y_{n-3}. \end{cases}$$

Case 2 $y_n = 1$.

In this case, Y_n is in VQ_n^1. Consider σ_1 and σ_0, defined in (1) and (3), respectively. When $X_{n-1} = Y_{n-1}$, let $\sigma = \sigma_0$. Then $\sigma \in \mathrm{Aut}(VQ_n)$, and

$$\sigma(X_n) = \sigma_0(X_n) = 1X_{n-1} = Y_n.$$

When $X_{n-1} \neq Y_{n-1}$, let $\sigma = \sigma_1\sigma_0$. Then $\sigma = \sigma_1\sigma_0 \in \mathrm{Aut}(VQ_n)$, and

$$\sigma(X_n) = \sigma_1\sigma_0(X_n) = \sigma_1(x_n\varphi(X_{n-1})) = \sigma_1(0Y_{n-1}) = 1Y_{n-1} = Y_n.$$

Thus, we have proved that for any two vertices X_n and Y_n in VQ_n, there is a $\sigma \in \mathrm{Aut}(VQ_n)$ such that $\sigma(X_n) = Y_n$, and so VQ_n is vertex-transitive. The theorem follows. □

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