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# The $p$-domination number of complete multipartite graphs

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Let $G = (V, E)$ be a graph and $p$ be a positive integer. A subset $S \subseteq V$ is called a $p$-dominating set of $G$ if every vertex not in $S$ has at least $p$ neighbors in $S$. The $p$-domination number is the minimum cardinality of a $p$-dominating set in $G$. This paper establishes an exact formula of the $p$-domination number of all complete multipartite graphs for arbitrary positive integer $p$.

*Keywords*: Combinatorics; $p$-domination set; $p$-domination number; complete multipartite graph.

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## 1. Introduction

For notation and graph-theoretical terminology not defined here, we refer the reader to [3]. Let $G = (V, E)$ be a finite simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *neighborhood* and *degree* of a vertex $v \in V$ are $N_G(v) = \{u \in V : uv \in E\}$ and $d_G(v) = |N_G(v)|$, respectively. A *dominating set* of $G$ is a subset $S \subseteq V$ such that every vertex of $V - S$ has at least one neighbor in $S$. The *domination number* $\gamma(G)$ is the minimum cardinality of all dominating sets in $G$. The domination is a classical concept in graph theory. The early literature on the domination with related topics is, in detail, surveyed in the two outstanding books by Haynes, Hedetniemi and Slater [10, 11].

Fink and Jacobson [8, 9] generalized the concept of dominating set. Let $p$ be a positive integer. A subset $D \subseteq V$ is a *p-dominating set* of $G$ if $|N_G(v) \cap D| \geq p$ for each $v \in V - D$. The *p-domination number* $\gamma_p(G)$ is the minimum cardinality of all $p$-dominating sets in $G$. A $p$-dominating set $D$ with $|D| = \gamma_p(G)$ is called a $\gamma_p$-set of $G$ (for short, $\gamma_p(G)$-set). For $S, T \subseteq V$, $S$ $p$-dominate $T$ in $G$ if $|N_G(v) \cap S| \geq p$ for each $v \in T - S$. Clearly, the 1-dominating set is the well-known dominating set in a graph $G$, and so $\gamma_1(G) = \gamma(G)$. By the definition of $p$-dominating set, the following observation is obvious.

**Observation 1.** Every $p$-dominating set contains all the vertices with degree at most $p - 1$.

The determination of the $p$-domination number for graphs seems to be a difficult problem. In 1989, Jacobson and Peters [12] showed that the problem is NP-complete in general graphs. In 1994, Bean, Henning and Swart [1] proved the problem remains NP-complete in bipartite or chordal graphs. These results show that the following study is of important significance.

- Find the lower and upper bounds of $\gamma_p$ with difference as small as possible.
- Determine exact values of $\gamma_p$ for some graphs, specially well-known networks.

Many works focused on the bounds of $\gamma_p$ for general graphs or some special classes of graphs (see, for example, [2, 4, 6, 7, 13]). Very recently, Chellali *et al.* [5] have given an excellent survey on this topics. Until now, however, no research has been done on calculating the exact values of $\gamma_p$ even for some particular graphs except [14]. In [14], the author obtained the exact 2-domination number of the toroidal grid graphs $C_m \Box C_n$ in some cases.

In this paper, we give an exact formula of $\gamma_p$ for arbitrary positive integer $p$ and the complete $t$-partite graph $K_{n_1,n_2,\ldots,n_t}$.

Throughout this paper, the graph $G$ always denotes a complete $t$-partite graph $K_{n_1,n_2,\ldots,n_t}$ with $t$-partition $\{V_1, V_2, \ldots, V_t\}$, $N_t = \{1, 2, \ldots, t\}$ and

$$f(I) = \sum_{i \in I} n_i \quad \text{for } I \subseteq N_t.$$

Note that if $t = 1$ or $f(N_t) \leq p$ then $\gamma_p(G) = |V(G)|$ by Observation 1. Thus, we always assume $t \geq 2$ and $f(N_t) > p$.

## 2. Optimal $\gamma_p$-Sets of $G$

For any $D \subseteq V(G)$, define

$$D_i = V_i \cap D \quad \text{for each } i \in N_t \quad \text{and} \quad I_D = \{i \in N_t : |D_i| = |V_i|\}.$$

**Lemma 2.** *If $t \geq 2$ and $f(N_t) > p$, then*

$$\gamma_p(G) \leq \min\{f(I) : I \subseteq N_t \text{ with } f(I) \geq p\}$$

*with equality if $G$ has a $\gamma_p$-set $D$ with $f(I_D) \geq p$.*

**Proof.** Let $I \subseteq N_t$ with $f(I) \geq p$ and $S = \bigcup_{i \in I} V_i$. Then

$$|S| = \sum_{i \in I} |V_i| = \sum_{i \in I} n_i = f(I) \geq p.$$

Since $G$ is a complete $t$-partite graph, for any $v \in V(G) - S$, we have $S \subseteq N_G(v)$ and so $|N_G(v) \cap S| = |S| \geq p$. This implies that $S$ is a $p$-dominating set of $G$, and so

$$\gamma_p(G) \leq \min\{f(I) : I \subseteq N_t \text{ with } f(I) \geq p\}.$$

On the other hand, let $D$ be a $\gamma_p(G)$-set with $f(I_D) \geq p$. Then

$$\begin{aligned}\gamma_p(G) = |D| &\geq \sum_{i \in I_D} |V_i| = \sum_{i \in I_D} n_i = f(I_D) \\ &\geq \min\{f(I) : I \subseteq N_t \text{ with } f(I) \geq p\}.\end{aligned}$$

The lemma follows. □

**Lemma 3.** *If $t \geq 2$ and $f(N_t) > p$, then $|I_D| \leq t - 2$ for any $\gamma_p(G)$-set $D$ with $f(I_D) < p$.*

**Proof.** Clearly $|I_D| \leq t - 1$ by $f(N_t) > p > f(I_D)$. If $|I_D| = t - 1$, then there is a unique index $i_0 \in N_t$ such that $N_t - I_D = \{i_0\}$. By the definition of $I_D$, $V(G) - V_{i_0} \subseteq D$ and there exists a vertex $x$ in $V_{i_0}$ but not in $D$. Since $D$ is a $\gamma_p(G)$-set and $f(I_D) < p$, we can deduce a contradiction as follows:

$$p \leq |N_G(x) \cap D| = |V(G) - V_{i_0}| = \sum_{i \in I_D} n_i = f(I_D) < p.$$

Hence $|I_D| \leq t - 2$. □

For a $\gamma_p(G)$-set $D$ with $|I_D| < t$, $|D| = f(I_D) + \sum_{i \in N_t - I_D} |D_i|$. By Lemma 3, the value of $||D_i| - \frac{|D| - f(I_D)}{t - |I_D|}|$ is well-defined for any $i \in N_t - I_D$ if $t \geq 2$ and $f(N_t) > p$. Define

$$\mu(D) = \sum_{i \in N - I_D} \left| |D_i| - \frac{|D| - f(I_D)}{t - |I_D|} \right|.$$

**Definition 2.1.** A $\gamma_p(G)$-set $D$ is called to be *optimal* if the following conditions hold: (1) $f(I_D) < p$; (2) $|I_D| \geq |I_S|$ for any $\gamma_p(G)$-set $S$; (3) $\mu(D) \leq \mu(S)$ for any $\gamma_p(G)$-set $S$ with $I_D = I_S$.

By the definition, if each $\gamma_p(G)$-set $D$ has $f(I_D) < p$, then there must be at least one optimal $\gamma_p$-set in $G$. To obtain the upper bound of $\gamma_p(G)$, by Lemma 2, we only need to consider the case that every $\gamma_p(G)$-set $D$ satisfies $f(I_D) < p$. We investigate properties of optimal $\gamma_p$-sets starting with the following critical lemma.

**Lemma 4.** $||D_i| - |D_j|| \leq 1$ *for any optimal $\gamma_p(G)$-set $D$ and $i, j \in N_t - I_D$.*

**Proof.** By Lemma 3, $t - |I_D| \geq 2$ and so $N_t - I_D \neq \emptyset$. Let

$$|D_s| = \max\{|D_i| : i \in N_t - I_D\} \quad \text{and} \quad |D_w| = \min\{|D_i| : i \in N_t - I_D\}.$$

Suppose, to be contrary, that $|D_s| - |D_w| \geq 2$. Clearly, $|D_s| \geq 2$. Since $w \in N_t - I_D$, $D_w \subsetneq V_w$. Hence there are $x \in D_s$ and $y \in V_w - D_w$. Let

$$D^* = (D - \{x\}) \cup \{y\}.$$

Then

$$I_{D^*} = \begin{cases} I_D & \text{if } |D_w| < |V_w| - 1; \\ I_D \cup \{w\} & \text{if } |D_w| = |V_w| - 1. \end{cases} \tag{2.1}$$

Thus $I_D \subseteq I_{D^*}$. We first claim that $D^*$ is a $\gamma_p(G)$-set. In fact, it is easy to see that $D^*$ can $p$-dominate $V(G) - V_w$. By the choice of $s \in N_t - I_D$, $V_s - D_s \neq \emptyset$. Since $D - D_s$ can $p$-dominate $V_s - D_s$, we have $|D| - |D_s| \geq p$. It follows that, for any vertex $z \in V_w - D^*$,

$$|N_G(z) \cap D^*| = |D^*| - |D_w^*| = |D| - (|D_w| + 1) \geq |D| - |D_s| + 1 \geq p + 1,$$

which means that $D^*$ can $p$-dominate $z$ and, hence, $D^*$ is a $\gamma_p(G)$-set.

By the second condition of the optimality of $D$, we have $|I_D| \geq |I_{D^*}|$. Thus $I_D = I_D^*$ by $I_D \subseteq I_{D^*}$. Combined with $|D| = |D^*| = \gamma_p(G)$, we can obtain that

$$\frac{|D| - f(I_D)}{t - |I_D|} = \frac{|D^*| - f(I_{D^*})}{t - |I_{D^*}|}.$$

For convenience, we use the notation $\lambda$ to represent them.

We now show $\mu(D^*) - \mu(D) < 0$. Since $|D| = \gamma_p(G) = f(I_D) + \sum_{i \in N_t - I_D} |D_i|$,

$$\lambda = \frac{1}{t - |I_D|} \sum_{i \in N_t - I_D} |D_i|.$$

By $|D_s| - |D_w| \geq 2$ and the choices of $s$ and $w$, we have that

$$|D_w| + 1 \leq |D_s| - 1 \quad \text{and} \quad |D_w| < \lambda < |D_s|.$$

It follows that

$$\begin{aligned} \mu(D^*) - \mu(D) &= \sum_{i \in N_t - I_{D^*}} |(|D_i^*| - \lambda)| - \sum_{i \in N_t - I_D} |(|D_i| - \lambda)| \\ &= |(|D_w^*| - \lambda)| + |(|D_s^*| - \lambda)| - (\lambda - |D_w|) - (|D_s| - \lambda) \\ &= |(|D_w| + 1 - \lambda)| + |(|D_s| - 1 - \lambda)| - (|D_s| - |D_w|) \\ &= \begin{cases} 2(|D_w| - \lambda) & \text{if } \lambda < |D_w| + 1, \\ -2 & \text{if } |D_w| + 1 \leq \lambda \leq |D_s| - 1, \\ 2(\lambda - |D_s|) & \text{if } \lambda > |D_s| - 1 \end{cases} \\ &< 0. \end{aligned}$$

This contradicts with the third condition of the optimality of $D$, and so $|D_s| - |D_w| \leq 1$.

The lemma follows. □

For an optimal $\gamma_p(G)$-set $D$, $t - |I_D| \geq 2$ by Lemma 3, and so $N_t - I_D \neq \emptyset$. Thus we denote

$$k = \max\{|D_i| : i \in N_t - I_D\} \quad \text{and} \quad \ell = \min\{|D_i| : i \in N_t - I_D\}. \tag{2.2}$$

If $k \neq \ell$, then $k = \ell + 1$ by Lemma 4. Define

$$A = \begin{cases} \{i \in N_t - I_D : |D_i| = \ell + 1\} & \text{if } k = \ell + 1; \\ \emptyset & \text{if } k = \ell, \end{cases} \tag{2.3}$$
$$B = \{i \in N_t - I_D : |D_i| = \ell\}.$$

Then $\{A, B\}$ is a partition of $N_t - I_D$ and $B \neq \emptyset$.

**Lemma 5.** $|A| = 0$ *or* $2 \leq |A| \leq t - |I_D| - 1$ *for any optimal* $\gamma_p(G)$*-set* $D$.

**Proof.** Since $\{A, B\}$ is a partition of $N_t - I_D$ and $B \neq \emptyset$, it is obvious that $|A| \leq t - |I_D| - 1$. We now show $|A| \neq 1$. Assume to the contrary that $|A| = 1$.

Let $A = \{i_1\}$. Then $|D_{i_1}| = \ell + 1 \geq 1$ and $V_{i_1} - D_{i_1} \neq \emptyset$ since $i_1 \in N_t - I_D$. Since $D - D_{i_1}$ $p$-dominates $V_{i_1} - D_{i_1}$, we have $|D| - |D_{i_1}| \geq p$. Take any vertex $x \in D_{i_1}$ and let

$$D' = D - \{x\}.$$

Consider any vertex $y$ in $V(G) - D'$. If $y \in V_{i_1}$, then

$$|N_G(y) \cap D'| = |D'| - |D'_{i_1}| = (|D| - 1) - (|D_{i_1}| - 1)| = |D| - |D_{i_1}| \geq p.$$

If $y \notin V_{i_1}$, then there exists some $j \in B$ such that $y \in V_j$. Noting $|D_j| = |D_{i_1}| - 1$, we have that

$$|N_G(y) \cap D'| = |D'| - |D'_j| = (|D| - 1) - |D_j| = |D| - |D_{i_1}| \geq p.$$

Hence $D'$ is a $p$-dominating set of $G$ with $|D'| = |D| - 1 = \gamma_p(G) - 1$, a contradiction. The lemma follows. □

**Lemma 6.** $\gamma_p(G) \geq p + \ell + \delta_A$ *for any optimal* $\gamma_p(G)$*-set* $D$*, where* $\ell$ *and* $A$ *are defined in* (2.2) *and* (2.3)*, respectively,* $\delta_A$ *is the characteristic function on* $A$*, i.e.,* $\delta_A = 0$ *if* $|A| = 0$ *and* $\delta_A = 1$ *otherwise.*

**Proof.** Note that $N_t - I_D \neq \emptyset$ and $V_i - D_i \neq \emptyset$ for $i \in N_t - I_D$. To $p$-dominate $V_i - D_i$, $|D - D_i| = |D| - |D_i| \geq p$ for $i \in N_t - I_D$.

If $|A| = 0$, then $\delta_A = 0$ and $N_t - I_D = B$. For any $i \in B$, $|D_i| = \ell$ by (2.3), and so $\gamma_p(G) = |D| \geq p + |D_i| = p + \ell = p + \ell + \delta_A$.

If $|A| \neq 0$, then $\delta_A = 1$. For $i \in A$, $|D_i| = \ell + 1$ by (2.3). Thus $\gamma_p(G) = |D| \geq p + |D_i| = p + \ell + 1 = p + \ell + \delta_A$.

The lemma follows. □

**Lemma 7.** $\lceil \frac{p - f(I_D)}{t - |I_D| - 1} \rceil \leq n_i$ *for any optimal* $\gamma_p(G)$*-set* $D$ *and* $i \in N_t - I_D$.

**Proof.** Let $N - I_D = A \cup B$ as defined in (2.3). Then $|D_i| = \ell + 1$ for $i \in A$ and $|D_j| = \ell$ for $j \in B$. Note that $n_i = |V_i| \geq |D_i| + 1 \geq \ell + 1$ for any $i \in N_t - I_D$. It follows that

$$\begin{aligned} |D| &= f(I_D) + \sum_{i \in A} |D_i| + \sum_{i \in B} |D_i| \\ &= f(I_D) + |A|(\ell + 1) + (t - |I_D| - |A|)\ell \\ &= f(I_D) + (t - |I_D| - 1)\ell + \ell + |A|, \end{aligned}$$

from which we have

$$\begin{aligned} \left\lceil \frac{p - f(I_D)}{t - |I_D| - 1} \right\rceil &= \ell + \left\lceil \frac{|A| - \delta_A}{t - |I_D| - 1} - \frac{|D| - (p + \ell + \delta_A)}{t - |I_D| - 1} \right\rceil \\ &\leq \ell + \left\lceil \frac{|A| - \delta_A}{t - |I_D| - 1} \right\rceil \quad \text{(by Lemma 6)} \\ &\leq \ell + \delta_A \quad \text{(by Lemma 5)} \\ &\leq \ell + 1 \\ &\leq n_i \quad \text{for any } i \in N_t - I_D \end{aligned}$$

as desired, and so the lemma follows. □

## 3. Main Results

In this section, we will give an exact formula of $\gamma_p$ for a complete $t$-partite graph $G = K_{n_1, n_2, \ldots, n_t}$. By Lemma 2, if $G$ contains a $\gamma_p$-set $D$ with $f(I_D) \geq p$, then

$$\gamma_p(G) = \min\{f(I) : I \subseteq N_t \text{ with } f(I) \geq p\}.$$

Thus, we only need to consider the case of $f(I_D) < p$ for any $\gamma_p(G)$-set $D$. In this case, $G$ must have optimal $\gamma_p(G)$-sets. Moreover, for any optimal $\gamma_p(G)$-set $D$, $|I_D| \leq t - 2$ by Lemma 3, and $\lceil \frac{p - f(I_D)}{t - |I_D| - 1} \rceil \leq n_i$ for any $i \in N_t - I_D$ by Lemma 7. Thus, the following family $\mathscr{I}_p$ of the subsets of $N_t$ is well-defined.

$$\mathscr{I}_p = \left\{ I \subset N_t : |I| \leq t - 2, f(I) < p \text{ and } \left\lceil \frac{p - f(I)}{t - |I| - 1} \right\rceil \leq n_i \right. \\ \left. \text{for each } i \in N_t - I \right\}.$$

Some examples of $\mathscr{I}_p$ for $G = K_{2,2,10,17}$ can be found in Table 1.

Table 1. Examples of $s_1$, $\mathscr{I}_p$, $s_2$ and $\gamma_p(G)$ for $G = K_{2,2,10,17}$, where $N_4 = \{1, 2, 3, 4\}$.

| $p$ | $s_1$ | $\mathscr{I}_p$ | $s_2$ | $\gamma_p(G)$ |
|---|---|---|---|---|
| 1 | 2 | $\{\emptyset\}$ | 1 | $s_1 = p + s_2 = 2$ |
| 2 | 2 | $\{\emptyset\}$ | 1 | $s_1 = 2$ |
| 3 | 4 | $\{\emptyset, \{1\}, \{2\}\}$ | 1 | $s_1 = p + s_2 = 4$ |
| 4 | 4 | $\{\emptyset, \{1\}, \{2\}\}$ | 1 | $s_1 = 4$ |
| 5 | 10 | $\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ | 1 | $p + s_2 = 6$ |
| 6 | 10 | $\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ | 2 | $p + s_2 = 8$ |
| 7 | 10 | $\{\{1, 2\}\}$ | 3 | $s_1 = p + s_2 = 10$ |
| 9 | 10 | $\{\{1, 2\}\}$ | 5 | $s_1 = 10$ |
| 11 | 12 | $\{\{1\}, \{2\}, \{3\}, \{1, 2\}\}$ | 1 | $s_1 = p + s_2 = 12$ |
| 13 | 14 | $\{\{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$ | 1 | $s_1 = p + s_2 = 14$ |
| 14 | 17 | $\{\{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$ | 2 | $p + s_2 = 16$ |
| 15 | 17 | $\emptyset$ | $\infty$ | $s_1 = 17$ |

Let

$$s_1 = \min\{f(I) : I \subseteq N_t \text{ with } f(I) \geq p\} \quad \text{and}$$

$$s_2 = \begin{cases} \min\left\{ \left\lceil \dfrac{p - f(I)}{t - |I| - 1} \right\rceil : I \in \mathscr{I}_p \right\} & \text{if } \mathscr{I}_p \neq \emptyset; \\ \infty & \text{if } \mathscr{I}_p = \emptyset. \end{cases}$$

**Lemma 8.** *Let $G = K_{n_1,n_2,\ldots,n_t}$ with $t \geq 2$ and $f(N_t) > p$. Then $\gamma_p(G) \leq p + s_2$.*

**Proof.** If $\mathscr{I}_p = \emptyset$, then $s_2 = \infty$ and so $\gamma_p(G) < p + s_2$. Assume that $\mathscr{I}_p \neq \emptyset$ below. Let $I \in \mathscr{I}_p$ (without loss of generality, say $I = \{1, \ldots, k\}$) with

$$k \leq t - 2, \quad f(I) < p \quad \text{and} \quad s_2 = \left\lceil \frac{p - f(I)}{t - k - 1} \right\rceil \leq n_i \quad \text{for each } i \in \{k + 1, \ldots, t\}.$$

Since $t - k - 1 \geq 1$ and $p - f(I) > 0$, there are two integers $q$ and $r$ with $q \geq 0$ and $0 \leq r \leq t - k - 2$ such that

$$p - f(I) = q(t - k - 1) + r.$$

Then for each $i \in \{k + 1, \ldots, t\}$,

$$n_i \geq s_2 = \begin{cases} q + 1 & \text{if } r \neq 0; \\ q & \text{if } r = 0. \end{cases} \tag{3.1}$$

Thus, we can choose $D \subseteq V(G)$ such that

$$D = (V_1 \cup \cdots \cup V_k) \cup (V'_{k+1} \cup \cdots \cup V'_{k+r}) \cup (V'_{k+r+1} \cup \cdots \cup V'_{t-1}) \cup V'_t,$$

where, for each $i \in \{k + 1, \ldots, t\}$, $V'_i$ is a subset of $V_i$ satisfying

$$|V'_i| = \begin{cases} q + 1 & \text{if } k + 1 \leq i \leq k + r, \\ q & \text{if } k + r + 1 \leq i \leq t - 1, \\ s_2 & \text{if } i = t. \end{cases} \tag{3.2}$$

Thus,

$$\begin{aligned}|D| &= \sum_{i=1}^{k}|V_i| + \sum_{i=k+1}^{k+r}|V_i'| + \sum_{j=k+r+1}^{t-1}|V_j'| + |V_t'| \\ &= (n_1+\cdots+n_k) + r(q+1) + (t-k-r-1)q + s_2 \\ &= (f(I) + q(t-k-1) + r) + s_2 \\ &= p + s_2.\end{aligned}$$

To complete the proof, we only need to show that $D$ is a $p$-dominating set of $G$. To this aim, let $v$ be any vertex in $V(G) - D$. By the choice of $D$, there is some $i_0 \in \{k+1, \ldots, t\}$ such that $v \in V_{i_0} - V'_{i_0}$. Since $G$ is a complete $t$-partite graph,

$$|N_G(v) \cap D| = |D| - |V'_{i_0}| = p + s_2 - |V'_{i_0}|.$$

By (3.1) and (3.2), we have

$$\begin{aligned} s_2 - |V'_{i_0}| &= \begin{cases} 1 & \text{if } r \neq 0 \text{ and } k+r+1 \leq i_0 \leq t-1, \\ 0 & \text{otherwise} \end{cases} \\ &\geq 0. \end{aligned}$$

It follows that $|N_G(v) \cap D| = p + s_2 - |V'_{i_0}| \geq p$, which implies that $D$ can $p$-dominate $v$. Hence $D$ is a $p$-dominating set of $G$. The lemma follows. □

We now state our main result as follows.

**Theorem 9.** *For any integer $p \geq 1$ and a complete $t$-partite graph $G = K_{n_1,n_2,\ldots,n_t}$ with $t \geq 2$ and $f(N_t) > p$,*

$$\gamma_p(G) = \min\{s_1, p + s_2\}.$$

**Proof.** From Lemmas 2 and 8, we can obtain that $\gamma_p(G) \leq \min\{s_1, p+s_2\}$, and if $G$ has a $\gamma_p$-set $D$ with $f(I_D) \geq p$ then $\gamma_p(G) = s_1 \geq \min\{s_1, p+s_2\}$.

In the following, assume that every $\gamma_p(G)$-set $D$ satisfies $f(I_D) < p$. Let $D$ be an optimal $\gamma_p(G)$-set. To the end, we only need to show $\gamma_p(G) \geq p + s_2$.

Since $|I_D| \leq t-2$ by Lemma 3 and $\lceil \frac{p-f(I_D)}{t-|I_D|-1} \rceil \leq n_i$ for any $i \in N_t - I_D$ by Lemma 7, we have $I_D \in \mathscr{I}_p$, and so $\lceil \frac{p-f(I_D)}{t-|I_D|-1} \rceil \geq s_2$. From the proof of Lemma 7, we know that $\ell + \delta_A \geq \lceil \frac{p-f(I_D)}{t-|I_D|-1} \rceil$. Hence, by Lemma 6,

$$\gamma_p(G) \geq p + \ell + \delta_A \geq p + \left\lceil \frac{p - f(I_D)}{t - |I_D| - 1} \right\rceil \geq p + s_2.$$

The theorem follows. □

Some illustrations of $s_1$, $s_2$ and $\gamma_p(G) = \min\{s_1, p+s_2\}$ for the complete 4-partite graph $G = K_{2,2,10,17}$ are shown in Table 1.

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