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# Fault-tolerance of $(n, k)$-star networks ☆

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ABSTRACT

This paper considers a refined measure $\kappa_s^{(h)}$ for the fault-tolerance of a network and, for the generalized star network $S_{n,k}$, determines $\kappa_s^{(h)}(S_{n,k}) = n + h(k-2) - 1$ for $2 \leqslant k \leqslant n-1$ and $0 \leqslant h \leqslant n-k$, which implies that at least $n + h(k-2) - 1$ vertices of $S_{n,k}$ have to be removed to get a disconnected graph without vertices of degree less than $h$. This work generalizes some known results. When the $(n, k)$-star graph is used to model the topological structure of a large-scale parallel processing system, this result can provide a more accurate measure for the fault tolerance of the system.

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## 1. Introduction

It is well known that interconnection networks play an important role in parallel computing/communication systems. An interconnection network can be modeled by a graph in which vertices correspond to processors and edges correspond to communication links.

The *connectivity* $\kappa(G)$ of a graph $G$ is defined as the minimum number of vertices whose deletion disconnects $G$. As an important measure for the fault-tolerance of a network, the larger connectivity $\kappa$ is, the more reliable the network is. However, the definition of $\kappa$ is implicitly assumed that any subset of system components is equally likely to be faulty simultaneously, which may not be true in real applications, thus connectivity $\kappa$ underestimate the reliability of a network. To compensate such shortcoming, Harary [12] introduced the concept of the conditional connectivity by appending some requirements on the resulting graph. In this trend, Esfahanian [11] proposed the concept of the restricted connectivity, Latifi et al. [16] generalized it to the restricted $h$-connectivity which can measure fault tolerance of an interconnection network more accurately than the classical connectivity $\kappa$. The concepts stated here are slightly different from theirs.

For a given nonnegative integer $h$, a subset $S$ of vertices of a connected graph $G$ is called an *h-super vertex-cut*, or *h-cut* for short, if $G - S$ is disconnected and has the minimum degree at least $h$. The *h-super connectivity* of $G$, denoted by $\kappa_s^{(h)}(G)$, is defined as the minimum cardinality over all $h$-cuts of $G$. Since a complete graph $K_n$ is nonseparable, $\kappa_s^{(h)}(K_n)$ does not exist for any $h$ with $0 \leqslant h \leqslant n-1$. Furthermore, if $G$ is not a complete graph then $\kappa_s^{(0)}(G) = \kappa(G)$; for $h \geqslant 1$, if $\kappa_s^{(h)}(G)$ exists, then $\kappa_s^{(h-1)}(G) \leqslant \kappa_s^{(h)}(G)$. For any graph $G$ and integer $h$, determining $\kappa_s^{(h)}(G)$ is quite difficult. In fact, the existence of $\kappa_s^{(h)}(G)$ is an open problem so far when $h \geqslant 1$. Only a little knowledge of results have been known on $\kappa_s^{(h)}$ for particular classes of graphs and small $h$'s.

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As a topological structure of interconnection networks, the star graph $S_n$, proposed by Akers and Krishnamurthy [1], is an attractive alternative to the hypercube as an interconnection network, and has superior degree and diameter compared to the comparable hypercube as well as it is highly hierarchical and symmetrical [9]. However, the number of vertices of an $n$-dimensional star is $n!$, there is a large gap between $n!$ and $(n+1)!$ if $S_n$ is extended to $S_{n+1}$. To achieve scalability, Chiang and Chen [7] generalized the star graph $S_n$ to the $(n,k)$-star graph $S_{n,k}$, which preserves many ideal properties of the star graph [8]. Since then the $(n,k)$-star graph has received considerable attention in the literature [2,3,5,6,4,10,14,15,19,18,22,24–27].

This paper is concerned about $\kappa_s^{(h)}$ for the $(n,k)$-star graph $S_{n,k}$. For $k=n-1$, $S_{n,n-1}$ is isomorphic to a star graph $S_n$, Hu and Yang [13], Nie et al. [20] and Rouskovet al. [21], independently, determined $\kappa_s^{(1)}(S_n)=2n-4$ for $n\geqslant 3$. Wan and Zhang [23] showed $\kappa_s^{(2)}(S_n)=6n-18$ for $n\geqslant 4$. Yang et al. [26] proved that if $2\leqslant k\leqslant n-2$ then $\kappa_s^{(1)}(S_{n,k})=n+k-3$ for $n\geqslant 3$ and $\kappa_s^{(2)}(S_{n,k})=n+2k-5$ for $n\geqslant 4$.

We, in this paper, will generalize these results by proving that $\kappa_s^{(h)}(S_{n,k})=n+h(k-2)-1$ for $2\leqslant k\leqslant n-1$ and $0\leqslant h\leqslant n-k$.

The main proof of this result is in Section 3. In Section 2, we recall the structure of $S_{n,k}$ and some lemmas used in our proofs. Conclusions and some remarks are in Section 4.

## 2. Definitions and lemmas

For a given integer $n$ with $n\geqslant 2$, let $I_n=\{1,2,\ldots,n\}$, $I'_n=\{2,\ldots,n\}$. For an integer $k$ with $1\leqslant k\leqslant n-1$, let $P(n,k)=\{p_1p_2\ldots p_k:\ p_i\in I_n,\ p_i\neq p_j,\ 1\leqslant i\neq j\leqslant k\}$, the set of $k$-permutations on $I_n$. Clearly, $|P(n,k)|=\frac{n!}{(n-k)!}$.

**Definition 2.1.** (Chiang et al. [7]) The $(n,k)$-star graph $S_{n,k}$ is a graph with vertex-set $P(n,k)$. The adjacency is defined as follows: a vertex $p=p_1p_2\ldots p_i\ldots p_k$ is adjacent to a vertex

(a) $p_ip_2\ldots p_{i-1}p_1p_{i+1}\ldots p_k$, where $i\in I'_k$ (swap $p_1$ with $p_i$).
(b) $p'_1p_2p_3\ldots p_k$, where $p'_1\in I_n\setminus\{p_i:\ i\in I_k\}$ (replace $p_1$ by $p'_1$).

The vertices of type $(a)$ are referred to as *swap-neighbors* of the vertex $p$ and the edges between them are referred to as *swap-edges* or *i-edges*. The vertices of type $(b)$ are referred to as *unswap-neighbors* of the vertex $p$ and the edges between them are referred to as *unswap-edges*. Clearly, every vertex in $S_{n,k}$ has $k-1$ swap-neighbors and $n-k$ unswap-neighbors. Usually, if $p=p_1p_2\ldots p_k$ is a vertex in $S_{n,k}$, we call $p_i$ the $i$th *bit* of $p$ for each $i\in I_k$.

It has been known that the $(n,k)$-star graph $S_{n,k}$ is a vertex transitive graph with order $\frac{n!}{(n-k)!}$ and regular degree $n-1$ (see Chiang et al. [7]). In addition, $S_{n,n-1}$ is isomorphic to the star graph $S_n$, and $S_{n,1}$ is isomorphic to the complete graph $K_n$. Fig. 1 shows the $(4,2)$-star $S_{4,2}$ and the $(4,3)$-star $S_{4,3}$.

**Lemma 2.2.** *For any* $\alpha=p_2p_3\ldots p_k\in P(n,k-1)$ $(k\geqslant 2)$, *let* $V_\alpha=\{p_1\alpha:\ p_1\in I_n\setminus\{p_i:\ i\in I'_k\}\}$. *Then the subgraph of* $S_{n,k}$ *induced by* $V_\alpha$ *is a complete graph of order* $n-k+1$, *denoted by* $K^{\alpha}_{n-k+1}$.

**Proof.** For any two vertices $p_1\alpha$ and $p'_1\alpha$ in $V_\alpha$ with $p_1\neq p'_1$, by the condition $(b)$ of Definition 2.1, $p_1\alpha$ and $p'_1\alpha$ are linked in $S_{n,k}$ by an unswap-edge. Thus, the subgraph of $S_{n,k}$ induced by $V_\alpha$ is a complete graph $K_{n-k+1}$. $\square$

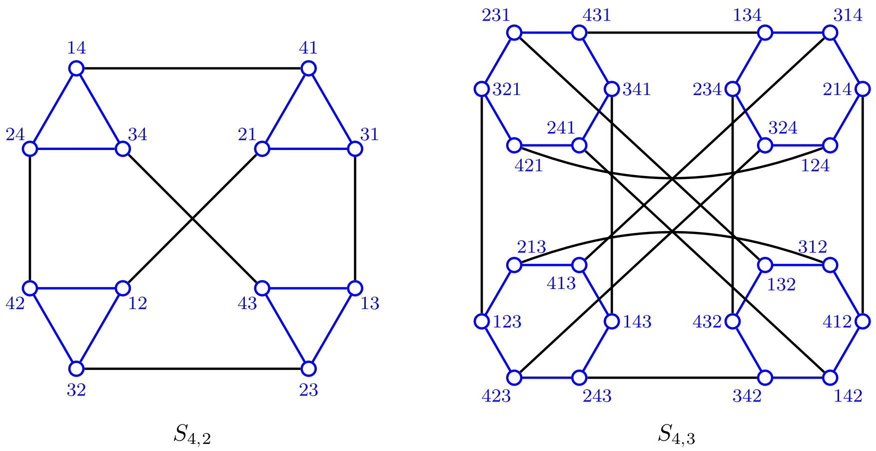


**Fig. 1.** The (4,2)-star $S_{4,2}$ and the (4,3)-star $S_{4,3}$

By Lemma 2.2, the vertex-set $P(n,k)$ of $S_{n,k}$ can be decomposed into $|P(n,k-1)|$ subsets, each of which induces a complete graph $K_{n-k+1}$. It is clear that, if $uv$ is an edge between two different complete subgraphs $K^{\alpha}_{n-k+1}$ and $K^{\beta}_{n-k+1}$ $(\alpha \neq \beta)$, then $\alpha$ and $\beta$ differ in only one bit and $uv$ is a swap-edge. Thus, we have the following conclusion.

**Lemma 2.3.** *The vertex-set of $S_{n,k}$ can be partitioned into $|P(n,k-1)|$ subsets, each of which induces a complete graph of order $n-k+1$. Furthermore, there is at most one swap-edge between any two complete graphs.*

Let $S^{t:i}_{n-1,k-1}$ denote a subgraph of $S_{n,k}$ induced by all vertices with $i$ in the $t$th bit for $t \in I'_k$. The following lemma is a slight modification of the result of Chiang and Chen [7].

**Lemma 2.4.** *For a fixed integer $t$ with $2 \leqslant t \leqslant k$, $S_{n,k}$ can be partitioned into $n$ subgraphs $S^{t:i}_{n-1,k-1}$, which is isomorphic to $S_{n-1,k-1}$, for each $i \in I_n$. Moreover, there are $\frac{(n-2)!}{(n-k)!}$ independent swap-edges between $S^{t:i}_{n-1,k-1}$ and $S^{t:j}_{n-1,k-1}$ for any $i,j \in I_n$ with $i \neq j$.*

**Lemma 2.5.** (Chen et al. [3]) *In $S_{n,k}$, a cycle has length at least 6 if it contains a swap-edge.*

**Lemma 2.6.** (Chiang et al. [7]) $\kappa(S_{n,k}) = n-1$.

## 3. Main results

In this section, we present our main results, that is, we determine the $h$-super connectivity of the $(n,k)$-star graph $S_{n,k}$. Since $S_{n,1} \cong K_n$, we only consider the case of $k \geqslant 2$ in the following discussion.

**Lemma 3.1.** $\kappa^{(h)}_s(S_{n,k}) \leqslant n + h(k-2) - 1$ *for* $2 \leqslant k \leqslant n-1$ *and* $0 \leqslant h \leqslant n-k$.

**Proof.** By our hypothesis of $h \leqslant n-k$, for any $\alpha \in P(n,k-1)$, we can choose a subset $X \subseteq V(K^{\alpha}_{n-k+1})$ such that $|X| = h+1$. Then the subgraph of $K^{\alpha}_{n-k+1}$ induced by $X$ is a complete graph $K_{h+1}$. Let $S$ be the neighbor-set of $X$ in $S_{n,k} - X$. Clearly, $V(K^{\alpha}_{n-k+1} - X) \subseteq S$, that is, $X$ has exactly $n-k+1-|X|$ unswap-neighbors in $V(K^{\alpha}_{n-k+1} - X) \cap S$. Since $S_{n,k}$ is $(n-1)$-regular, every vertex of $X$ has exactly $(k-1)$ swap-neighbors are not in $K^{\alpha}_{n-k+1}$. Moreover, by Lemma 2.3, every vertex outside of $K^{\alpha}_{n-k+1}$ has at most one swap-neighbor in $K^{\alpha}_{n-k+1}$, thus any two swap-neighbors of $X$ are different from each other. It follows that

$$|S| = n-k+1-|X| + |X|(k-1) = n + h(k-2) - 1. \tag{3.1}$$

Since $|S| + |X| = n + h(k-1) < \frac{n!}{(n-k)!}$ for $k \geqslant 2$, there exists some vertex not adjacent to $X$ in $S_{n,k}$, and so $S$ is a vertex-cut of $S_{n,k}$. We now need to show that $S$ is an $h$-cut of $S_{n,k}$. We show that every vertex of $S_{n,k} - (X \cup S)$ has degree at least $h$. Let $u$ be a vertex in $S_{n,k} - (X \cup S)$. If $u$ has a neighbor $v$ in $S \cap V(K^{\alpha}_{n-k+1})$, then $u$ is a swap-neighbor of $v$ since all the unswap-neighbors of $v$ are in $V(K^{\alpha}_{n-k+1})$. If $u$ has a neighbor $v$ in $S \setminus V(K^{\alpha}_{n-k+1})$, then $v$ has a swap-neighbor in $V(K^{\alpha}_{n-k+1})$. Moreover, if $u$ has two neighbor $v$, $v'$ in $S$, then three vertices $u$, $v$ and $v'$ are contained in a cycle that has length at most 5 and contains at least one swap-edge, which contradicts with Lemma 2.5. Thus, $u$ has at most one neighbor in $S$. In other words, $u$ has at least $n-2$ neighbors in $S_{n,k} - S$. Since $n-2 \geqslant n-k \geqslant h$ for $k \geqslant 2$, $u$ has degree at least $h$ in $S_{n,k} - S$. By the arbitrariness of $u \in S_{n,k} - (X \cup S)$, $S$ is an $h$-cut of $S_{n,k}$, and so

$$\kappa^{(h)}_s(S_{n,k}) \leqslant |S| = n + h(k-2) - 1$$

as required. The Lemma follows. □

**Corollary 3.2.** $\kappa^{(h)}_s(S_{n,2}) = n-1$ *for* $0 \leqslant h \leqslant n-2$.

**Proof.** It is easy to know $\kappa^{(h)}_s(S_{n,2}) \leqslant n-1$ by Lemma 3.1. And on the other hand, by Lemma 2.6, $\kappa^{(h)}_s(S_{n,2}) \geqslant \kappa(S_{n,2}) = n-1$. Thus the conclusion holds. □

To state and prove our main results, we need some notations. Let $S$ be an $h$-cut of $S_{n,k}$ and $X$ be the vertex-set of a connected component of $S_{n,k} - S$. For a fixed $t \in I'_k$ and any $i \in I_n$, let

$$\begin{aligned} &Y = V(S_{n,k} - S - X), \quad S_i = S \cap V(S^{t:i}_{n-1,k-1}), \\ &X_i = X \cap V(S^{t:i}_{n-1,k-1}), \quad Y_i = Y \cap V(S^{t:i}_{n-1,k-1}), \end{aligned} \tag{3.2}$$

and let

$$J_X = \{i \in I_n : X_i \neq \emptyset\}, \; J_Y = \{i \in I_n : Y_i \neq \emptyset\}, \; J_0 = J_X \cap J_Y. \tag{3.3}$$

**Lemma 3.3.** *Let $S$ be a minimum $h$-cut of $S_{n,k}$. If $3 \leqslant k \leqslant n-1$ and $1 \leqslant h \leqslant n-k$ then, for any $t \in I'_k$,*

(a) *$S_i$ is an $(h-1)$-cut of $S^{t:i}_{n-1,k-1}$ for any $i \in J_0$,*
(b) *$\kappa_s^{(h)}(S_{n,k}) \geqslant |J_0|\ \kappa_s^{(h-1)}(S_{n-1,k-1})$,*
(c) *$J_X \cup J_Y = I_n$.*

**Proof**

(a) By the definition of $J_0, S_i$ is a vertex-cut of $S^{t:i}_{n-1,k-1}$ for any $i \in J_0$. For any vertex $x$ in $S^{t:i}_{n-1,k-1} - S_i$, since $x$ has degree at least $h$ in $S_{n,k} - S$ and has exactly one neighbor outsider $S^{t:i}_{n-1,k-1}, x$ has degree at least $h-1$ in $S^{t:i}_{n,k} - S_i$. This fact shows that $S_i$ is an $(h-1)$-cut of $S^{t:i}_{n-1,k-1}$ for any $i \in J_0$.
(b) By the assertion (a), we have $|S_i| \geqslant \kappa_s^{(h-1)}(S_{n-1,k-1})$, and so

$$\kappa_s^{(h)}(S_{n,k}) = |S| \geqslant \sum_{i\in J_0}|S_i| \geqslant |J_0|\kappa_s^{(h-1)}(S_{n-1,k-1}).$$

(c) If $J_X \cup J_Y \neq I_n$, that is, $I_n \setminus (J_X \cup J_Y) \neq \ = \emptyset$, then there exists an $i_0 \in I_n$ such that $V(S^{t:i_0}_{n-1,k-1}) = S_{i_0}$. Thus, we have

$$\begin{aligned}\kappa_s^{(h)}(S_{n,k}) &= |S| \geqslant |S_{i_0}| = \tfrac{(n-1)!}{(n-k)!}\\ &\geqslant (n-1)(n-2)\\ &> n+(n-3)(n-3)-1\\ &\geqslant n+h(k-2)-1,\end{aligned}$$

which contradicts to Lemma 3.1. Thus, $J_X \cup J_Y = I_n$. The Lemma follows. □

**Theorem 3.4.** *$\kappa_s^{(h)}(S_{n,k}) = n+h(k-2)-1$ for $2 \leqslant k \leqslant n-1$ and $0 \leqslant h \leqslant n-k$.*

**Proof.** By Corollary 3.2, the conclusion holds for $k=2$. And by Lemma 3.1, we only need to prove that, for $3 \leqslant k \leqslant n-1$ and $0 \leqslant h \leqslant n-k$,

$$\kappa_s^{(h)}(S_{n,k}) \geqslant n+h(k-2)-1. \tag{3.4}$$

For fixed $k$ and $n$, we prove the inequality (3.4) by induction on $h(\geqslant 0)$. Since $\kappa_s^{(0)}(S_{n,k}) = \kappa(S_{n,k}) = n-1$, the inequality (3.4) is true for $h=0$. Assume the induction hypothesis for $h-1$ with $h \geqslant 1$. We have

$$\kappa_s^{(h-1)}(S_{n-1,k-1}) \geqslant n+(h-1)(k-3)-2. \tag{3.5}$$

Let $S$ be a minimum $h$-cut of $S_{n,k}$ and $X$ be the vertex-set of a minimum connected component of $S_{n,k} - S$. Use notations defined in (3.2) and (3.3). Choose $t \in I'_k$ such that $|J_X|$ is as large as possible. For each $i \in I_n$, we write $S^i_{n-1,k-1}$ for $S^{t:i}_{n-1,k-1}$ for short. We consider three cases depending on $|J_0| = 0, |J_0| = 1$ or $|J_0| \geqslant 2$.

**Case 1.** $|J_0| = 0$,

In this case, $J_X \cap J_Y = \emptyset$. By Lemma 3.3 (c), $|J_X| \geqslant 2$ or $|J_Y| \geqslant 2$ since $n \geqslant 4$. Clearly, $J_X \neq \emptyset$ and $J_Y \neq \emptyset$. Without loss of generality, assume $|J_X| \geqslant 2, \{i_1, i_2\} \subseteq J_X$ and $i_3 \in J_Y$. By Lemma 2.4, there are $\frac{(n-2)!}{(n-k)!}$ independent swap-edges between $S^{i_1}_{n-1,k-1}$ (resp. $S^{i_2}_{n-1,k-1}$) and $S^{i_3}_{n-1,k-1}$, each edge of which has at least one end-vertex in $S$. Since $J_X \cap J_Y = \emptyset$ and $S_{i_1} \cap S_{i_2} = \emptyset$, we have that

$$|S| \geqslant 2\ \frac{(n-2)!}{(n-k)!}. \tag{3.6}$$

Noting that, for $k=3$,

$$2\ \frac{(n-2)!}{(n-k)!} \geqslant 2(n-2) = n+(n-3)-1 \geqslant n+h(k-2)-1,$$

and, for $k \geqslant 4$,

$$2\ \frac{(n-2)!}{(n-k)!} \geqslant 2(n-2)(n-3) \geqslant n+(n-3)(n-3)-1 \geqslant n+h(k-2)-1,$$

we have that

$$2\,\frac{(n-2)!}{(n-k)!} \geqslant n+h(k-2)-1 \quad \text{for } k \geqslant 3. \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$\kappa_s^{(h)}(S_{n,k}) = |S| \geqslant 2\,\frac{(n-2)!}{(n-k)!} \geqslant n+h(k-2)-1.$$

Note that the proof of case $|J_X| \geqslant 2$ does not use the minimality of $X$. So, we can give the proof of case $|J_Y| \geqslant 2$ using the same way as the case of $|J_X| \geqslant 2$.

**Case 2.** $|J_0| = 1$,

Without loss of generality, assume $J_0 = \{1\}$. By Lemma 3.3 (a), $S_1$ is an $(h-1)$-cut of $S^1_{n-1,k-1}$. Let $S' = S \setminus S_1$.

If $|S'| \geqslant n-2$ then, by (3.5),

$$\begin{aligned} \kappa_s^{(h)}(S_{n,k}) &= |S| = |S_1| + |S'| \geqslant \kappa_s^{(h-1)}(S_{n-1,k-1}) + (n-2) \\ &\geqslant (n+(h-1)(k-3)-2) + (n-2) \\ &\geqslant (n+(h-1)(k-3)-2) + (h+k-2) \\ &= n+h(k-2)-1. \end{aligned}$$

Now assume $|S'| \leqslant n-3$. We claim $|J_X| = 1$. Suppose to the contrary $|J_X| \geqslant 2$.

If $|J_Y| = 1$, then $|J_X| = n$ by Lemma 3.3 (c). If there exists some $i \in J_X \setminus J_0$ such that $|S_i| = 0$, then $|X| > |X_i| = |V(S^i_{n-1,k-1})| > |Y_1| = |Y|$, which contradicts to the minimality of $X$. If $|S_i| \geqslant 1$ for each $i \in J_X \setminus J_0$, then $|S'| \geqslant n-1$, a contradiction.

If $|J_Y| \geqslant 2$, say $i_1 \in J_X \setminus J_0$ and $i_2 \in J_Y \setminus J_0$, then $X_{i_1} \neq \emptyset, Y_{i_1} = \emptyset, X_{i_2} = \emptyset, Y_{i_2} \neq \emptyset$. By Lemma 2.4, there are $\frac{(n-2)!}{(n-k)!}$ independent swap-edges between $S^{i_1}_{n-1,k-1}$ and $S^{i_2}_{n-1,k-1}$, each edge of which must have one end-vertex in $S'$. Thus,

$$|S'| \geqslant \frac{(n-2)!}{(n-k)!} \geqslant n-2 \quad \text{for } k \geqslant 3,$$

a contradiction.

Thus, $|J_X| = 1$, that is, $J_X = \{1\}$ since $\{1\} = J_0 \subseteq J_X$. Thus, $X_1 = X$ and $|X_1| \geqslant h+1$. By the choice of $t$, the $i$th ($i \neq 1$) bits of all vertices in $X_1$ are same, and so $X_1$ is a complete graph. Thus, as computed in (3.1),

$$\kappa_s^{(h)}(S_{n,k}) = |S| = n + (|X_1|-1)(k-2) - 1 \geqslant n+h(k-2)-1.$$

**Case 3.** $|J_0| \geqslant 2$.

By Lemma 3.3 (b) and (3.5), we have that

$$\begin{aligned} \kappa_s^{(h)}(S_{n,k}) = |S| &\geqslant |J_0|\kappa_s^{(h-1)}(S_{n-1,k-1}) \\ &\geqslant 2(n+(h-1)(k-3)-2) \\ &\geqslant n+(h+k)+2(h-1)(k-3)-4 \\ &\geqslant n+h(k-2)-1. \end{aligned}$$

By the induction principle, the theorem follows. □

**Corollary 3.5.** (Yang et al. [26]) *If* $2 \leqslant k \leqslant n-2$ *then* $\kappa_s^{(1)}(S_{n,k}) = n+k-3$ *for* $n \geqslant 3$ *and* $\kappa_s^{(2)}(S_{n,k}) = n+2k-5$ *for* $n \geqslant 4$.

## 4. Conclusions and Remarks

In this paper, we consider a refined measure for the fault tolerance of a network, called the $k$-super connectivity $\kappa_s^{(h)}$. For the $(n,k)$-star graph $S_{n,k}$, which is an attractive alternative network to the hypercube, we prove that $\kappa_s^{(h)}(S_{n,k}) = n+h(k-2)-1$ for $2 \leqslant k \leqslant n-1$ and $0 \leqslant h \leqslant n-k$.

This result shows that at least $n+h(k-2)-1$ vertices of $S_{n,k}$ have to be removed to get a disconnected graph without vertices of degree less than $h$. When the $(n,k)$-star graph is used to model the topological structure of a large-scale parallel processing system, this result can provide a more accurate measure for the fault tolerance of the system.

We should notice that the condition $k \geqslant 2$ is necessary in our result since if $k=1$ then $S_{n,1} \cong K_n$, for which $\kappa_s^{(h)}(K_n)$ does not exist for any $h$ with $0 \leqslant h \leqslant n-1$.

We should also notice that when $k = n-1, S_{n,n-1}$ is isomorphic to the star graph $S_n$. In this case, the condition $0 \leqslant h \leqslant n-k$ implies that $0 \leqslant h \leqslant 1$. Akers and Krishnamurthy [1] determined $\kappa(S_n) = n-1$ for $n \geqslant 2$; Hu and Yang [13], Nie et al. [20] and Rouskov et al. [21], independently, determined $\kappa_s^{(1)}(S_n) = 2n-4$ for $n \geqslant 3$. All these results are special cases of our result by setting $k = n-1$ and $h = 0, 1$, respectively. However, Wan and Zhang [23] determined

$\kappa_s^{(2)}(S_n) = 6(n-3)$ for $n \geqslant 4$, which cannot be deduced from our result. In fact, very recently we have shown that $\kappa_s^{(h)}(S_n) = (h+1)!(n-h-1)$ for any $h$ with $0 \leqslant h \leqslant n-2$ (see Li and Xu [17]).

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