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Cycles and paths embedded in varietal hypercubes

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Abstract: The varietal hypercube VQ_n is a variant of the hypercube Q_n and has better properties than Q_n with the same number of edges and vertices. It was proved that every edge of VQ_n is contained in cycles of every length from 4 to 2^n except 5, and that every pair of vertices with distance d is connected by paths of every length from d to 2^n-1 except 2 and 4 if $d=1$.

Key words: graphs; cycle; path; varietal hypercube; pancyclicity; panconnectivity

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变形超立方体的圈和路嵌入

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摘要:作为超立方体网络 Q_n 的变形,n 维变形超立方体 VQ_n 具有许多优于超立方体所具有的性质.这里证明了对任何整数 $\ell\in[4,2^n]$,VQ_n 中每条边被包含在长度为 ℓ 的圈中除非 $\ell=5$;对任何顶点对(x,y)和整数 $\ell\in[d,2^n-1]$,其中,d 为这两点之间的距离,VQ_n 中存在长度为 ℓ 的 xy 路除非当 $d=1$ 时 $\ell=2,4$.

关键词:图论;圈;路;变形超立方体;泛圈性;泛连通性

0 Introduction

The hypercube network Q_n has proved to be one of the most popular interconnection networks since it has a simple structure and has many nice properties. As a variant of Q_n, the varietal hypercube VQ_n, proposed in Ref. [1], has many properties similar or superior to Q_n. For example, the connectivity and restricted connectivity of VQ_n and Q_n are the same[2], while all the diameter and the average distance, fault-diameter and wide-diameter of VQ_n are smaller than those of the hypercube[1,3].

Several topological structures of multicomputer systems are commonly used in various applications such as image processing and scientific computing. Among them, the most common structures are paths and cycles.

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Embedding these structures in various well-known networks, such as Q_n, has been extensively investigated in the literatures (see, for example, Ref. [4]). However, no study has yet been conducted on embedding them in VQ_n. In this paper, we show that VQ_n should be capable of embedding these structures. The main results are stated as follows:

Every edge of VQ_n is contained in cycles of every length from 4 to 2^n except 5, and every pair of vertices with distance d is connected by paths of every length from d to 2^n-1 except 2 and 4 if $d=1$.

Some definitions and basic properties of VQ_n are given in Section 1. The proofs of the results are given in Section 2.

1 Definitions and lemmas

We follow Ref. [5] for graph-theoretical terminology and notation not defined here. A graph $G=(V, E)$ always means a simple and connected graph, where $V=V(G)$ is the vertex-set and $E=E(G)$ is the edge-set of G. For $uv\in E(G)$, we call u (resp. v) is a neighbor of v (resp. u). A uv-path is a sequence of adjacent vertices, written as $(v_0, v_1, v_2, \cdots, v_m)$, in which $u=v_0$, $v=v_m$ and all the vertices $v_0, v_1, v_2, \cdots, v_m$ are different from each other, u and v are called the end-vertices of P. If $u=v$, then a uv-path P is called a cycle. The length of a path P, denoted by $\varepsilon(P)$, is the number of edges in P. The length of the shortest uv-path in G is called the distance between u and v in G, denoted by $d_G(u, v)$. For a path

$$P=(v_0, v_1, \cdots, v_i, v_{i+1}, \cdots, v_m),$$

we can write

$$P=P(v_0, v_i)+v_iv_{i+1}+P(v_{i+1}, v_m),$$

and the notation $P-v_iv_{i+1}$ denotes the subgraph obtained from P by deleting the edge v_iv_{i+1}.

The n-dimensional varietal hypercube VQ_n is the labeled graph defined recursively as follows. VQ_1 is the complete graph of two vertices labeled 0 and 1, respectively. Assume that VQ_{n-1} has been constructed. Let VQ_{n-1}^0 (resp. VQ_{n-1}^1) be a labeled graph obtained from VQ_{n-1} by inserting a zero (resp. 1) in front of each vertex-labeling in VQ_{n-1}. For $n>1$, VQ_n is obtained by joining vertices in VQ_{n-1}^0 and VQ_{n-1}^1, according to the following rule: a vertex

$$x=0\,x_{n-1}\,x_{n-2}\,x_{n-3}\cdots x_2\,x_1$$

in VQ_{n-1}^0 and a vertex

$$y=1\,y_{n-1}\,y_{n-2}\,y_{n-3}\cdots y_2\,y_1$$

in VQ_{n-1}^1 are adjacent in VQ_n if and only if ①

$$x_{n-1}\,x_{n-2}\,x_{n-3}\cdots x_2\,x_1=y_{n-1}\,y_{n-2}\,y_{n-3}\cdots y_2\,y_1$$

if $n\neq 3k$, or ②

$$x_{n-3}\cdots x_2\,x_1=y_{n-3}\cdots y_2\,y_1$$

and $(x_{n-1}x_{n-2}, y_{n-1}y_{n-2})\in I$ if $n=3k$, where

$$I=\{(00,00),(01,01),(10,11),(11,10)\}.$$

Fig. 1 shows the examples of varietal hypercubes VQ_n for $n=1$, 2, 3 and 4, respectively.

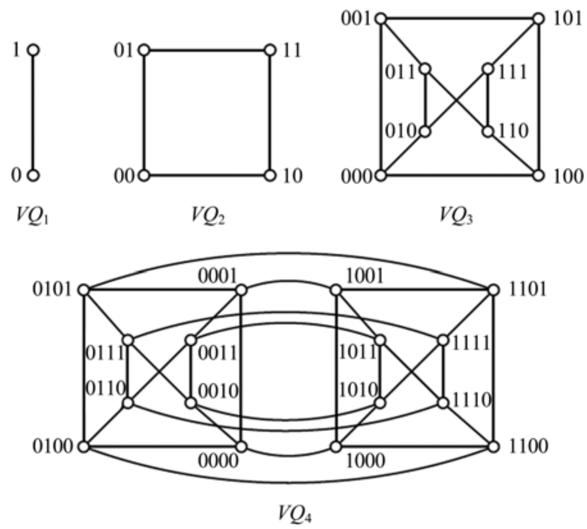

Fig. 1 The varietal hypercubes VQ_1, VQ_2, VQ_3 **and** VQ_4

The edges of Type ② are referred to as crossing edges when

$$(x_{n-1}x_{n-2}, y_{n-1}y_{n-2})\in\{(10,11),(11,10)\}.$$

All the other edges are referred to as normal edges.

The varietal hypercube VQ_n is proposed in Ref. [1] as an attractive alternative to the n-dimensional hypercube Q_n when they are used to model the topological structure of a large-scale parallel processing system. Like Q_n, VQ_n is an n-regular graph with 2^n vertices and $2^{n-1}n$ edges.

For convenience, we express VQ_n as $VQ_n = L\odot R$, where $L = VQ^0_{n-1}$ and $R = VQ^1_{n-1}$, and denote by $x_L x_R$ the n-transversal edge joining $x_L \in L$ and $x_R \in R$. The recursive structure of VQ_n gives the following simple properties.

Lemma 1.1 Let $VQ_n = L\odot R$ with $n\geqslant 1$. Then VQ_n contains no triangles and every vertex $x_L \in L$ has exactly one neighbor x_R in R joined by the n-transversal edge $x_L x_R$.

Lemma 1.2 Let $VQ_n = L\odot R$ and xy be an n-transversal edge in VQ_n with $x\in L$ and $y\in R$. For $n\geqslant 3$, let $x=0abx_{n-3}\cdots x_1$ and $\beta = x_{n-3}\cdots x_1$. Then $y=1a'b'\beta$, where $ab=a'b'$ if xy is a normal edge, and $(ab, a'b')=(1b, 1\bar{b})$ if xy is a crossing edge, where $\bar{b}=\{0,1\}\setminus b$.

Lemma 1.3 Any edge in VQ_n ($n\geqslant 2$) is contained in a cycle of length 4.

Proof Clearly, the conclusion is true for $n=2$. Assume $n\geqslant 3$ and let xy be any edge in VQ_n. Then by definition of VQ_n there is some m with $2\leqslant m\leqslant n$ such that xy is an m-transversal edge. Let $VQ_m = L\odot R$, $x\in L$ and $y\in R$.

If xy is a normal edge, let u_L be a neighbor of x in L and u_R be the neighbor of u_L in R, then y and u_R are adjacent and so (x, u_L, u_R, y) is a cycle of length 4.

If xy is a crossing edge, let $x=01b\beta$, then $y=11\bar{b}\beta$. Choose $u_L=01\bar{b}\beta$. Then $u_R=11b\beta$ by Lemma 1.2, and so (x, u_L, u_R, y) is a cycle of length 4. □

Lemma 1.4 Any n-transversal edge must be contained in some cycle of length 5 unless $n\neq 3k$ for $k\geqslant 1$.

Proof Let $VQ_n = L\odot R$ and xy be an n-transversal edge in VQ_n, where $x\in L$ and $y\in R$. We first prove that xy is not contained in any cycle of length 5 if $n\neq 3k$ for $k\geqslant 1$. The conclusion is true for $n=1$ or 2 clearly. Assume $n\geqslant 3$ below.

Suppose that there is a cycle $C=(x, u, z, v, y)$ of length 5 containing the edge xy. Then C contains two n-transversal edges. Since $n\neq 3k$, xy is a normal edge. Let $x=0ab\beta$, where $\beta=x_{n-3}\cdots x_1$. Then $y=1ab\beta$. Since every vertex in L has exactly one neighbor in R by Lemma 1.1, $u\in L$ and $v\in R$. Without loss of generality, assume $z\in L$. Then x and z differ in exactly two positions. Without loss of generality, let $z=0\bar{a}\,\bar{b}\beta$. Since zv is an n-transversal edge and $n\neq 3k$, $v=1\bar{a}\,\bar{b}\beta$. Thus, y and v differ in exactly two positions, which implies that y and v are not adjacent, a contradiction.

We now show that the n-transversal edge xy must be contained in some cycle of length 5 if $n=3k$ for $k\geqslant 1$ by constructing such a cycle. Let $x=0ab\beta\in L$ and $y=1a'b'\beta\in R$, where $(ab, a'b')\in I$. A required cycle $C=(x, u, z, v, y)$ can be constructed as follows.

If xy is a normal edge, then $ab=a'b'=0b$. Let $u=00\bar{b}\beta$, $z=01\bar{b}\beta$ and $v=11b\beta$ (where zv is a crossing edge).

If xy is a crossing edge, then

$$(ab, a'b') = (1b, 1\bar{b}).$$

Let $u=01\bar{b}\beta$, $z=00\bar{b}\beta$ and $v=10\bar{b}\beta$ (where zv is a normal edge).

The lemma follows. □

Lemma 1.5 Any n-transversal edge in VQ_n is contained in cycles of lengths 6 and 7 for $n\geqslant 3$.

Proof Let $VQ_n = L\odot R$ and xy be an n-transversal edge in VQ_n, where $x\in L$ and $y\in R$.

We first show that xy is contained in a cycle of length 6. By Lemma 1.3, there is a cycle C of length 4. Let $C=(x, u, v, y)$, where $u\in L$ and $v\in R$. Also by Lemma 1.3, there is a cycle C' of length 4 containing the xu in L. Clearly, $C\cap C'=\{xu\}$. Thus, $C\cup C' - xu$ is a cycle of length 6 containing the edge xy.

We now show that xy is contained in a cycle of length 7. If $n=3k$ for $k\geqslant 1$ then, by Lemma 1.4, there is a cycle C of length 5 containing the edge xy. Let $C=(x, u, z, v, y)$, where $x, u, z\in L$ and $v\in R$, without loss of generality. By Lemma 1.3, there is a cycle C' of length 4 containing the edge yv in R. Clearly, $C\cap C'=\{yv\}$. Thus $C\cup C' - yv$ is a cycle of length 7 containing the edge xy.

Assume $n\neq 3k$ for $k\geqslant 1$ below. In this case, all n-transversal edges are normal edges. We can choose a cycle $C=(x, u, v, y)$ such that the edge

xu lies on some subgraph H that is isomorphic to VQ_3. By Lemma 1.4, there is a cycle C' of length 5 containing the edge xu in $H\subseteq L$. Then $C\cup C'-xu$ is a cycle of length 7 containing the edge xy.

The lemma follows. □

The n-dimensional crossed cube CQ_n is such a graph, whose vertex-set is the same as VQ_n, with its two vertices $x=x_n\cdots x_2x_1$ and $y=y_n\cdots y_2y_1$ are linked by an edge if and only if there exists some j ($1\leqslant j\leqslant n$) such that

(a) $x_n\cdots x_{j+1}=y_n\cdots y_{j+1}$,

(b) $x_j\neq y_j$,

(c) $x_{j-1}=y_{j-1}$ if j is even,

(d) $(x_{2i}x_{2i-1},\ y_{2i}y_{2i-1})\in I$ for each $i=1,2,\cdots,\left\lceil \frac{1}{2}j\right\rceil-1$.

By definition, $VQ_n\cong CQ_n$ for each $n=1,2,3$. The following results on CQ_n are used in the proofs of our main results for $n=3$.

Lemma 1.6[6-8] For any two vertices x and y with distance d in CQ_n with $n\geqslant 2$, CQ_n contains xy-paths of every length from d to 2^n-1 except 2 when $d=1$.

Lemma 1.7 For $n\geqslant 3$ and any integer ℓ with $2^n-2\leqslant\ell\leqslant 2^n-1$, there exists an xy-path of length ℓ between any pair of vertices x and y in VQ_n.

Proof We proceed by induction on $n\geqslant 3$. By Lemma 1.6, the conclusion is true for $n=3$ since $VQ_3\cong CQ_3$. Assume the induction hypothesis for $n-1$ with $n\geqslant 4$. Let $VQ_n=L\odot R$, x and y be two distinct vertices in VQ_n.

If $x,\ y\in L$ (or R), then, by the induction hypothesis, there exists an xy-path P_L of length ℓ_0 in L, where $\ell_0\in\{2^{n-1}-2,2^{n-1}-1\}$. Let u be the neighbor of y in P_L, u_R and y_R be the neighbors of u and y in R, respectively. By the induction hypothesis, there exists a u_Ry_R-path P_R of length $2^{n-1}-1$ in R. Then $P_L-uy+uu_R+P_R+y_Ry$ is an xy-path of length ℓ_0+2^{n-1} in VQ_n.

If $x\in L$ and $y\in R$, let u be a vertex in L rather than x such that its neighbor u_R in R is different from y, then, by the induction hypothesis, there exist an xu-path P_L of length ℓ_0' in L and a u_Ry-path P_R of length $2^{n-1}-1$ in R, where $\ell_0'\in\{2^{n-1}-2,2^{n-1}-1\}$. Then $P_L+uu_R+P_R$ is an xy-path of length $\ell_0'+2^{n-1}$ in VQ_n.

The lemma follows. □

Lemma 1.8 Let $VQ_n=L\odot R$, x_L and y_L be two vertices in L. Then

$$d_L(x_L,y_L)=d_R(x_R,y_R)$$

if $n\neq 3k$ and

$$|d_L(x_L,y_L)-d_R(x_R,y_R)|\leqslant 2$$

if $n=3k$ for $k\geqslant 1$.

Proof Without loss of generality, assume $d_L(x_L,y_L)\leqslant d_R(x_R,y_R)$. Let P_L be the shortest x_Ly_L-path in L and P_R a path in R obtained from P_L by replacing the first position 0 by 1 in every vertices. Clearly, $\varepsilon(P_R)=\varepsilon(P_L)$.

Note that for an edge u_Lv_L in P_L, if u_Lv_R is a crossing edge, then v_Lu_R is also a crossing edge. For convenience, we call the edge u_Lv_L an induced crossing edge, u_L and v_L induced crossing vertices.

If both x_L and y_L are not induced crossing vertices, then P_R is an x_Ry_R-path in R, and so

$$d_R(x_R,y_R)\leqslant\varepsilon(P_R)=d_L(x_L,y_L),$$

and so

$$d_R(x_R,y_R)=d_L(x_L,y_L).$$

Assume below that $\{x_L,\ y_L\}$ contains induced crossing vertices. Then $n=3k$.

Let x_L be an induced crossing vertex, x_Lu_L an induced crossing edge. Then, x_R is not an end-vertex of P_R, while u_R is an end-vertex of P_R. Similarly, if y_L is an induced crossing vertex, y_Lv_L an induced crossing edge, then y_R is not an end-vertex of P_R, while v_R is an end-vertex of P_R. Thus, an x_Ry_R-path $P_R'\subseteq P_R$ has length

$$\varepsilon(P_R')=\varepsilon(P_L)-\begin{cases}0 & \text{if neither } x_L \text{ and } y_L \text{ are induced crossing vertices;}\\ 1 & \text{if either } x_L \text{ or } y_L \text{ is an induced crossing vertex;}\\ 2 & \text{if both } x_L \text{ and } y_L \text{ are induced crossing vertices,}\end{cases}$$

and $d_R(x_R,y_R)\leqslant\varepsilon(P_R')$. If

$$d_R(x_R,y_R)\leqslant d_L(x_L,y_L)-3$$

then, using the above method, we can prove that there is an $x_L y_L$-path P'_L with length

$$\varepsilon(P'_L) \leqslant d_R(x_R, y_R) + 2,$$

from which we have

$$d_L(x_L, y_L) \leqslant \varepsilon(P'_L) \leqslant d_R(x_R, y_R) + 2 \leqslant d_L(x_L, y_L) - 1,$$

a contradiction. Thus,

$$d_R(x_R, y_R) \geqslant d_L(x_L, y_L) - 2.$$

And so

$$| d_L(x_L, y_L) - d_R(x_R, y_R) | \leqslant 2.$$

The lemma follows. □

Corollary 1.9 Let $VQ_n = L \odot R$, x and y be two vertices in H, where $H \in \{L, R\}$. Then

$$d_H(x, y) = d_{VQ_n}(x, y).$$

Proof Let x and y be in L and P the shortest xy-path in VQ_n. If $P \cap R \neq \emptyset$, then $P \cap L$ consists of several sections of P. Without loss of generality, assume that $P \cap L$ consists of two sections, P_{xu_L} and $P_{v_L y}$. Then $u_R v_R$-section $P_{u_R v_R}$ of P from u_R to v_R is in R. By Lemma 1.8,

$$d_L(u_L, v_L) \leqslant d_R(u_R, v_R) + 2 = \varepsilon(P_{u_R v_R}) + 2.$$

Since P is the shortest xy-path in VQ_n, we have that

$$d_{VQ_n}(x, y) \leqslant d_L(x, y) = \varepsilon(P_{xu_L}) + d_L(u_L, v_L) + \varepsilon(P_{v_L y}) \leqslant \varepsilon(P_{xu_L}) + \varepsilon(P_{u_R v_R}) + 2 + \varepsilon(P_{v_L y}) = \varepsilon(P) = d_{VQ_n}(x, y),$$

which implies $d_L(x, y) = d_{VQ_n}(x, y)$. The corollary follows. □

Corollary 1.10 Let $VQ_n = L \odot R$, $x \in L$ and $y \in R$. Then there is an n-transversal edge $u_L u_R$ such that

$$d_{VQ_n}(x, y) = d_L(x, u_L) + 1 + d_R(u_R, y).$$

2 Main results

A graph G of order n is said to be ℓ-pancyclic (resp. ℓ-vertex-pancyclic, ℓ-edge-pancyclic) if it contains (resp. each of its vertices, edges is contained in) cycles of every length from ℓ to n. Clearly, an ℓ-edge-pancyclic graph must be ℓ-vertex-pancyclic and ℓ-pancyclic.

We consider edge-pancyclity of VQ_n. Since VQ_n contains no triangles, no edge is contained in a cycle of length 3. Lemma 1.3 shows that any edge in $VQ_n (n \geqslant 2)$ is contained in a cycle of length 4. Lemma 1.4 shows that no n-transversal edge is contained in a cycle of length 5 if $n \neq 3k$ for $k \geqslant 1$. In general, we have the following result.

Theorem 2.1 For $n \geqslant 2$, every edge of VQ_n is contained in cycles of every length from 4 to 2^n except 5 and, hence, VQ_n is 6-edge-pancyclic for $n \geqslant 3$.

Proof By Lemma 1.3, we only need to show that every edge of VQ_n is contained in cycles of every length from 6 to 2^n for $n \geqslant 3$. Let ℓ be an integer with $6 \leqslant \ell \leqslant 2^n$ and xy be an edge in VQ_n. In order to prove the theorem, we only need to show that xy lies on a cycle of length ℓ. We proceed by induction on $n \geqslant 3$.

Since $VQ_3 \cong CQ_3$, by Lemma 1.6, the conclusion is true for $n=3$. Assume the induction hypothesis for $n-1$ with $n \geqslant 4$. Let $VQ_n = L \odot R$. There are two cases.

Case 1 x, $y \in L$ or $x, y \in R$. Without loss of generality, let $x, y \in L$.

By the induction hypothesis, we only need to consider ℓ with $2^{n-1}+1 \leqslant \ell \leqslant 2^n$.

If $\ell = 2^{n-1}+1$, then let x_R and y_R be the neighbors of x and y in R, respectively. By Lemma 1.7, there exists an $x_R y_R$-path $P_{x_R y_R}$ of length $2^{n-1}-2$ in R. Then $xx_R + P_{x_R y_R} + y_R y + xy$ is a cycle of length $2^{n-1}+1$.

If $2^{n-1}+2 \leqslant \ell \leqslant 2^n$, let $\ell_0 = \ell - 2^{n-1} - 1$, then $1 \leqslant \ell_0 \leqslant 2^{n-1}-1$. By Lemma 1.7, there exists a cycle C of length 2^{n-1} containing the edge xy in L. We choose an xz-path P_{xz} of length ℓ_0 in C that contains xy. Let x_R and z_R be the neighbors of x and z in R, respectively. By Lemma 1.7, there exists an $x_R z_R$-path $P_{x_R z_R}$ of length $2^{n-1}-1$ in R. Then $xx_R + P_{x_R z_R} + z_R z + P_{xz}$ is a cycle of length ℓ containing the edge xy in VQ_n.

Case 2 $x \in L$ and $y \in R$.

In this case, xy is an n-transversal edge. By Lemma 1.5, the conclusion is true for each $\ell = 6$, 7. Assume $\ell \geqslant 8$ below.

If $\ell \leqslant 2^{n-1}+2$, let $\ell_0 = \ell - 2$, then $6 \leqslant \ell_0 \leqslant 2^{n-1}$.

By Lemma 1.3, there exists a 4-cycle $C = (x, u_L, u_R, y)$. By the induction hypothesis, there exists a cycle C_L of length ℓ_0 that contains xu_L in L. Then, $C \cap C_L = \{xu_L\}$, and so $C \cap C_L - \{xu_L\}$ is a cycle of length ℓ containing xy.

If $2^{n-1}+3 \leqslant \ell \leqslant 2^n$, let $\ell_0 = \ell - 2^{n-1} - 1$, then $2 \leqslant \ell_0 \leqslant 2^{n-1}-1$. Choose a vertex u in L rather than x, By Lemma 1.7, there exists an xu-path P_{xu} of length $2^{n-1}-1$ in L, from which we can choose an xz-path P_{xz} of length ℓ_0. Let z_R be the neighbor of z in R. By Lemma 1.7, there exists a z_Ry-path P_{z_Ry} of length $2^{n-1}-1$. Thus, $P_{xz} + zz_R + P_{z_Ry} + xy$ is a cycle of length ℓ containing xy in VQ_n.

The theorem follows. □

A graph G of order n is said to be panconnected if for any two distinct vertices x and y with distance d in G there are xy-paths of every length from d to $n-1$.

We now consider panconnectivity of VQ_n. Since VQ_n contains no triangles, there exist no xy-paths of length 2 if x and y are adjacent. Lemma 1.4 shows that there exist no xy-paths of length 4 if xy is an n-transversal edge in VQ_n if $n \neq 3k$ for $k \geqslant 1$. In general, we have the following result.

Theorem 2.2 For $n \geqslant 3$, any two vertices x and y in VQ_n with distance d, there exist xy-paths of every length from d to 2^n-1 except 2, 4 if $d=1$.

Proof Let x and y be any two vertices in VQ_n with distance d. First, we note that if $d=1$ then the theorem is true by Theorem 2.1. In the following discussion, we assume $d \geqslant 2$. We only need to prove that there exist xy-paths of every length from $d+1$ to 2^n-1.

We proceed by induction on $n \geqslant 3$. Since $VQ_3 \cong CQ_3$, by Lemma1.6, the conclusion is true for $n=3$. Assume the induction hypothesis for $n-1$ with $n \geqslant 4$. Let $VQ_n = L \odot R$.

Case 1 $x, y \in L$ or $x, y \in R$. Without loss of generality, let $x, y \in L$.

By Corollary 1.9, $d_L(x, y) = d$. By the induction hypothesis, we only need to consider ℓ with $2^{n-1} \leqslant \ell \leqslant 2^n - 1$.

If $2^{n-1} \leqslant \ell \leqslant 2^{n-1}+1$, then

$$2^{n-1} - 2 \leqslant \ell - 2 \leqslant 2^{n-1} - 1.$$

Let x_R and y_R be the neighbors of x and y in R, respectively. By Lemma 1.7, there exists an x_Ry_R-path P_R of length $\ell-2$ in R. Then $x_Ry_R + P_R + yy_R$ is an xy-path of length ℓ in VQ_n.

If $2^{n-1}+2 \leqslant \ell \leqslant 2^n-1$, let $\ell_0 = \ell - 2^{n-1} - 1$. then $1 \leqslant \ell_0 \leqslant 2^{n-1}-2$. By Lemma 1.7, there exists an xy-path P_{xy} of length $2^{n-1}-1$ in L. We choose an xz-path P_{xz} of length ℓ_0 in P_{xy}. Clearly, $z \notin \{x, y\}$. Let z_R and y_R be the neighbors of z and y in R, respectively. By Lemma 1.7, there exists a z_Ry_R-path P_R of length $2^{n-1}-1$ in R. Then $P_{xz} + zz_R + P_R + yy_R$ is an xy-path of length ℓ in VQ_n.

Case 2 $x \in L$ and $y \in R$.

By Corollary 1.10, there is the shortest xy-path P_{xy} in VQ_n such that

$$P_{xy} = P_{xu_L} + u_Lu_R + Pu_Ry,$$

where $u_L \in L$ and $u_R \in R$, $\varepsilon(P_{xu_L}) = d_L(x, u_L)$ and $\varepsilon(P_{u_Ry}) = d_R(u_R, y)$. Thus,

$$d = \varepsilon(P_{xu_L}) + 1 + \varepsilon(P_{u_Ry}) =$$

$$d_L(x, u_L) + 1 + d_R(u_R, y).$$

Since $d \geqslant 2$, without loss of generality, assume $d_L(x, u_L) \geqslant d_R(u_R, y)$.

If $d+1 \leqslant \ell \leqslant 2^{n-1}$, let $\ell_0 = \ell - d_R(u_R, y) - 1$, then $d_L(x, u_L) + 1 \leqslant \ell_0 \leqslant 2^{n-1}-1$. By the induction hypothesis, there exists an xu_L-path P' of length ℓ_0 in L. Then $P' + u_Lu_R + P_{u_Ry}$ is an xy-path of length ℓ in VQ_n.

If $2^{n-1}+1 \leqslant \ell \leqslant 2^n-1$, let $\ell_0 = \ell - 2^{n-1}$, then $1 \leqslant \ell_0 \leqslant 2^{n-1}-1$. Let y_L be the neighbor of y in L. Then $y_L \neq x$ since x and y are not adjacent. By Lemma 1.7, there exists an xy_L-path P_{xy_L} of length $2^{n-1}-1$ in L. We choose an xz-path P_{xz} of length ℓ_0 in P_{xy_L}. Let z_R be the neighbor of z in R. By Lemma 1.7, there exists a z_Ry-path P_{z_Ry} of length $2^{n-1}-1$ in R. Then $P_{xz} + zz_R + P_{z_Ry}$ is an xy-path of length ℓ in VQ_n.

The theorem follows. □

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$$K=-\frac{d\omega_i^j-\sum_k \omega_i^k \wedge \omega_k^j}{\varepsilon_i \omega^i \wedge \omega^j}=-1 \qquad (19)$$

Notice that

$$K=-1<0,$$

according to Ref. [5, Theorem 3.3], the normal bundle $\vartheta(M_1^n)$ is flat, i.e. $\Omega_\alpha^\beta=0$. Furthermore, similarly we could get

$$d\tilde{f}=df-de_n=\omega^n e_n-\sum_\alpha \omega_n^\alpha e_\alpha,$$

$$\langle d\tilde{f},d\tilde{f}\rangle=(\omega^n)^2+\sum_{\alpha=n+1}^{2n-1}(\omega_n^\alpha)^2$$

and $\widetilde{M}$ is a space-like submanifold in $\mathbb{R}_1^{2n-1}$ of the constant sectional curvature -1. □

4 Conclusion

In this paper, we have derived three kinds of Bianchi transformations in $\mathbb{R}_1^{2n-1}$. In comparison with Bäcklund transformation in $\mathbb{R}_1^{2n-1}$ in Ref. [5], we know that only the Bianchi transformation in Case (ii) could be regarded as a special Bäcklund transformation. Further more, we expect to generalize the Bianchi transformation to $\mathbb{R}_k^{2n-1}$ where $k>1$.

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