

Edge fault-tolerance of super restricted edge-connected Cartesian product graphs

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Abstract: A subset F of edges in a connected graph G is a restricted edge-cut if $G - F$ is disconnected and every component has at least two vertices. A graph G is super restricted edge-connected (super- λ' for short) if every minimum restricted edge-cut of G isolates at least one edge. The edge fault-tolerance $\rho'(G)$ of a super- λ' graph G is the maximum integer m for which $G - F$ is still super- λ' for any subset $F \subseteq E(G)$ with $|F| \leq m$. It was shown that

$$\min\{k_1 + k_2 - 1, v_1 k_2 - 2k_1 - 2k_2 + 1, v_2 k_1 - 2k_1 - 2k_2 + 1\} \leq \rho'(G_1 \times G_2) \leq k_1 + k_2 - 1,$$

where G_i is a k_i -regular k_i -edge-connected graph of order v_i with $k_i \geq 4$ for each $i \in \{1, 2\}$ and $G_1 \times G_2$ is the Cartesian product graph of G_1 and G_2 . And some sufficient conditions such that $\rho'(G_1 \times G_2) = k_1 + k_2 - 1$ were presented.

Key words: graphs; connectivity; fault tolerance; super restricted edge-connected; Cartesian product; regular graphs; networks

CLC number: O157.5; TP302.8 **Document code:** A doi:10.3969/j.issn.0253-2778.2014.12.001

AMS Subject Classification (2000): 05C40; 68M15; 68R10

Citation: Hong Zhenmu, Xu Junming. Edge fault-tolerance of super restricted edge-connected Cartesian product graphs[J]. Journal of University of Science and Technology of China, 2014, 44(12):967-974.

超限制边连通笛卡尔乘积图的边容错性

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摘要: 如果 $G - F$ 不连通且每个连通分支至少含有两个顶点, 则连通图 G 的边子集 F 称为限制边割. 如果图 G 的每个最小限制边割都孤立 G 中的一条边, 则称 G 是超限制边连通的 (简称超 λ'). 对于满足 $|F| \leq m$ 的任意子集 $F \subseteq E(G)$, 超 λ' 图 G 的边容错性 $\rho'(G)$ 是使得 $G - F$ 仍是超 λ' 的最大整数 m . 这里给出了

$$\min\{k_1 + k_2 - 1, v_1 k_2 - 2k_1 - 2k_2 + 1, v_2 k_1 - 2k_1 - 2k_2 + 1\} \leq \rho'(G_1 \times G_2) \leq k_1 + k_2 - 1,$$

其中, 对每个 $i \in \{1, 2\}$, G_i 是阶为 v_i 的 k_i 正则 k_i 边连通图且 $k_i \geq 4$, $G_1 \times G_2$ 是 G_1 和 G_2 的笛卡尔乘积. 并给

Received: 2013-12-12; **Revised:** 2014-03-19

Foundation item: Supported by NSFC (61272008).

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出了使得 $\rho'(G_1 \times G_2) = k_1 + k_2 - 1$ 的一些充分条件.

关键词:图;连通度;容错性;超限制边连通;笛卡尔乘积;正则图;网络

0 Introduction

We follow Ref. [1] for graph-theoretical terminology and notation not defined here. Let $G=(V, E)$ be a simple connected graph, where $V=V(G)$ is the vertex-set of G and $E=E(G)$ is the edge-set of G . It is well known that when the underlying topology of an interconnection network is modeled by a connected graph G , the edge-connectivity $\lambda(G)$ of G is an important measurement for reliability and fault tolerance of the network. In general, the larger $\lambda(G)$ is, the more reliable the network is. Because the connectivity has some shortcomings, Fàbrega and Fiol^[2-3] generalized the concept of the edge-connectivity to the h -extra edge-connectivity for a graph.

Definition 0.1 Let $h \geq 0$ be an integer. A subset $F \subseteq E(G)$ is an h -extra edge-cut if $G-F$ is disconnected and every component of $G-F$ has more than h vertices. The h -extra edge-connectivity of G , denoted by $\lambda^{(h)}(G)$, is defined as the minimum cardinality of an h -extra edge-cut of G .

Clearly, $\lambda^{(0)}(G) = \lambda(G)$ and $\lambda^{(1)}(G) = \lambda'(G)$ for any graph G , the latter is called the restricted edge-connectivity proposed by Esfahanian and Hakimi^[4], who proved that for a connected graph G of order $n \geq 4$, $\lambda'(G)$ exists if and only if G is not the star $K_{1,n-1}$. A graph G is said to be $\lambda^{(h)}$ -connected if $\lambda^{(h)}(G)$ exists, and to be not $\lambda^{(h)}$ -connected otherwise. For a $\lambda^{(h)}$ -connected graph G , an h -extra edge-cut F is called a $\lambda^{(h)}$ -cut if $|F| = \lambda^{(h)}(G)$.

For two disjoint subsets X and Y in $V(G)$, use $[X, Y]$ to denote the set of edges between X and Y in G . In particular, $E_G(X) = [X, \bar{X}]$ and let $d_G(X) = |E_G(X)|$, where $\bar{X} = V(G) \setminus X$. For a subset $X \subset V(G)$, use $G[X]$ to denote the subgraph of G induced by X . For a $\lambda^{(h)}$ -connected

graph G , there is certainly a subset $X \subset V(G)$ with $|X| \geq h+1$ such that $E_G(X)$ is a $\lambda^{(h)}$ -cut and, both $G[X]$ and $G[\bar{X}]$ are connected. Such an X is called a $\lambda^{(h)}$ -fragment of G . Let

$$\xi_h(G) = \min\{d_G(X) : X \subset V(G), |X| = h+1 \text{ and } G[X] \text{ is connected}\}.$$

Clearly, $\xi_0(G) = \delta(G)$, the minimum vertex-degree of G , and $\xi_1(G) = \xi(G)$, the minimum edge-degree of G defined as $\min\{d_G(x) + d_G(y) - 2 : xy \in E(G)\}$. For a $\lambda^{(h)}$ -connected graph G , Whitney's inequality shows $\lambda^{(0)}(G) \leq \xi_0(G)$; Esfahanian and Hakimi^[4] showed $\lambda^{(1)}(G) \leq \xi_1(G)$; Bonsma et al.^[5] showed $\lambda^{(2)}(G) \leq \xi_2(G)$. For $h \geq 3$, Bonsma et al.^[5] found that the inequality $\lambda^{(h)}(G) \leq \xi_h(G)$ is no longer true in general. Zhang and Yuan^[6] proved that $\lambda^{(h)}(G)$ exists and $\lambda^{(h)}(G) \leq \xi_h(G)$ for any graph G with $h \leq \delta(G)$ except for a class of special graphs.

Definition 0.2 A $\lambda^{(h)}$ -connected graph G is super h -extra edge-connected (super- $\lambda^{(h)}$ for short), if every $\lambda^{(h)}$ -cut of G isolates at least one connected subgraph of order $h+1$.

Definition 0.3 The edge fault-tolerance of a super- $\lambda^{(h)}$ graph G on super- $\lambda^{(h)}$ property, denoted by $\rho^{(h)}(G)$, is the maximum integer m for which $G-F$ is still super- $\lambda^{(h)}$ for any subset $F \subseteq E(G)$ with $|F| \leq m$.

It is clear that the edge fault-tolerance $\rho^{(h)}(G)$ is a measurement for vulnerability of a super- $\lambda^{(h)}$ graph G when its edge failures appear. For convenience, we write $\lambda, \lambda', \lambda'', \rho$ and ρ' for $\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}, \rho^{(0)}$ and $\rho^{(1)}$, respectively. In this paper, we only focus on $\rho'(G)$ for a super- λ' Cartesian product graph G .

Very recently, Hong et al.^[7] have shown

$$\min\{\delta_1 + \delta_2 - 1, u_1 \delta_2 - \delta_1 - \delta_2 - 1, u_2 \delta_1 - \delta_1 - \delta_2 - 1\} \leq \rho(G_1 \times G_2) \leq \delta_1 + \delta_2 - 1,$$

where G_i is a δ_i -edge-connected graph of order u_i with $\delta(G_i) = \delta_i \geq 2$ for each $i \in \{1, 2\}$ and $G_1 \times G_2$ is

the Cartesian product graph of G_1 and G_2 . The edge fault-tolerance ρ of super edge-connectivity of three families of networks has been discussed by Wang and Lu^[8].

In this paper, we consider ρ' for Cartesian product of regular graphs and obtain

$$\begin{aligned} &\min\{k_1 + k_2 - 1, v_1 k_2 - 2k_1 - 2k_2 + 1, \\ &\quad v_2 k_1 - 2k_1 - 2k_2 + 1\} \leq \\ &\rho'(G_1 \times G_2) \leq k_1 + k_2 - 1, \end{aligned}$$

where G_i is a k_i -regular k_i -edge-connected graph of order v_i with $k_i \geq 4$ for each $i \in \{1, 2\}$. We also show that the bounds on $\rho'(G_1 \times G_2)$ are best possible and give some sufficient conditions such that $\rho'(G_1 \times G_2) = k_1 + k_2 - 1$.

1 Preliminaries

The Cartesian product is a very effective method for constructing a larger graph from several specified graphs, and plays an important role in designing and analysing large-scale interconnection networks^[9, Section 2.2].

The Cartesian product of graphs G_1 and G_2 is the graph $G_1 \times G_2$ with vertex-set $V(G_1) \times V(G_2)$, two vertices $x_1 x_2$ and $y_1 y_2$, where $x_1, y_1 \in V(G_1)$ and $x_2, y_2 \in V(G_2)$, being adjacent in $G_1 \times G_2$ if and only if either $x_1 = y_1$ and $x_2 y_2 \in E(G_2)$, or $x_2 = y_2$ and $x_1 y_1 \in E(G_1)$. For any $y \in V(G_2)$ (resp. $x \in V(G_1)$), we use G_1^y (resp. G_2^x) to denote the subgraph of $G_1 \times G_2$ induced by $V(G_1) \times \{y\}$ (resp. $\{x\} \times V(G_2)$). Clearly, G_1^y (resp. G_2^x) is isomorphic to G_1 (resp. G_2).

To investigate the edge fault-tolerance ρ' for Cartesian product of regular graphs, the following lemmas are needed.

Lemma 1.1^[10, Corollary 4] Let G_i be a k_i -regular k_i -edge-connected graph for each $i=1, 2$. Then

$$\lambda(G_1 \times G_2) = k_1 + k_2.$$

The following result gives a necessary and sufficient condition for a graph to be super- λ' .

Lemma 1.2^[11, Theorem 2.3] Let G be a λ' -connected graph and $\lambda''(G)$ exists. Then G is super- λ' if and only if $\lambda''(G) > \xi(G)$.

In Ref. [12], Ou obtained a sufficient and

necessary condition for the Cartesian product of regular graphs to be super- λ' .

Lemma 1.3^[12, Theorem 2.2, 1] Let G_i be a k_i -regular k_i -edge-connected graph with $k_i \geq 2$ for each $i=1, 2$. Then $G_1 \times G_2$ is super- λ' if and only if it is not isomorphic to the Cartesian product of a complete graph and a cycle.

Lemma 1.4 Let G_i be a k_i -regular k_i -edge-connected graph with $k_i \geq 4$ for each $i=1, 2$ and $G = G_1 \times G_2$. Then $d_{G-F}(X) > \xi(G-F)$ holds for any $F \subseteq E(G)$ and any $X \subseteq V(G)$ such that $|F| \leq k_1 + k_2 - 1$, $|X| \geq 3$, $G[X] - F$ is connected, and $X \subseteq V(G_1^{y_0})$ with $y_0 \in V(G_2)$ or $X \subseteq V(G_2^{x_0})$ with $x_0 \in V(G_1)$.

Proof Suppose, to the contrary, that there exist a subset F of $E(G)$ and a subset X of $V(G)$ satisfying the specified conditions, but

$$d_{G-F}(X) \leq \xi(G-F) \tag{1}$$

By Lemma 1.1, $\lambda(G) = k_1 + k_2$. Since

$$|F| \leq k_1 + k_2 - 1 = \lambda(G) - 1,$$

we have $G-F$ is connected and

$$\xi(G-F) \leq \xi(G) \tag{2}$$

Since G is $(k_1 + k_2)$ -regular, $\xi(G) = 2k_1 + 2k_2 - 2$.

By (1) and (2),

$$\begin{aligned} 2k_1 + 2k_2 - 2 &\geq d_{G-F}(X) \geq \\ d_G(X) - |F| &\geq d_G(X) - (k_1 + k_2 - 1). \end{aligned}$$

Thus

$$d_G(X) \leq 3k_1 + 3k_2 - 3 \tag{3}$$

Let $|X| = a$. Without loss of generality, assume that $X \subseteq V(G_1^{y_0})$ with $y_0 \in V(G_2)$. Then

$$\begin{aligned} d_G(X) &= d_{G_1^{y_0}}(X) + |[X, V(G) \setminus V(G_1^{y_0})]| = \\ d_{G_1^{y_0}}(X) &+ \sum_{z=(x, y_0) \in X} d_{G_2^x}(z) = d_{G_1^{y_0}}(X) + ak_2. \end{aligned}$$

Combining this with (3) and noting that $a \geq 3$ and $k_2 \geq 4$, we have

$$d_{G_1^{y_0}}(X) \leq 3k_1 - (a-3)k_2 - 3 \leq 3k_1 - 4a + 9 \tag{4}$$

Since

$$\begin{aligned} d_{G_1^{y_0}}(X) &= \sum_{z \in X} (d_{G_1^{y_0}}(z) - d_{G_1^{y_0}[X]}(z)) \geq \\ &a(k_1 - a + 1), \end{aligned}$$

by (4), we have

$$(a-3)k_1 \leq a^2 - 5a + 9 \tag{5}$$

Since G_1^b is isomorphic to G_1 , $|X| \geq 3$ and

$$X \subset V(G_1^b), d_{G_1^b}(X) \geq \lambda(G_1) = k_1.$$

By (4), $k_1 \leq 3k_1 - 4a + 9$ and so $2k_1 \geq 4a - 9$.

Combining this with (5) and $a \geq 3$, we have $(a-1)(2a-9) \leq 0$ and so $3 \leq a \leq 4$. Consider the following two cases.

Case 1 $a=4$.

Note that $k_1 + k_2 \geq 8$. Since G is $(k_1 + k_2)$ -regular,

$$d_G(X) = \sum_{z \in X} (d_G(z) - d_{G[X]}(z)) = 4(k_1 + k_2) - 2 |E(G[X])|.$$

Therefore, $d_G(X)$ is even and $d_G(X) \geq 4k_1 + 4k_2 - 12$. If $d_G(X) > 4k_1 + 4k_2 - 11$, then, by $k_1 + k_2 \geq 8$, $d_G(X) > 3k_1 + 3k_2 - 3$, a contradiction to (3).

Thus

$$4k_1 + 4k_2 - 12 \leq d_G(X) \leq 4k_1 + 4k_2 - 11.$$

Since $d_G(X)$ is even, we have

$$d_G(X) = 4k_1 + 4k_2 - 12.$$

Thus $G[X] \cong K_4$.

If

$$|F \cap [X, \bar{X}]| < k_1 + k_2 - 2,$$

then, by $k_1 + k_2 \geq 8$,

$$\begin{aligned} d_{G-F}(X) &\geq d_G(X) - |F \cap [X, \bar{X}]| > \\ &3k_1 + 3k_2 - 10 \geq 2k_1 + 2k_2 - 2 = \\ &\xi(G) \geq \xi(G-F), \end{aligned}$$

a contradiction to (1).

$$\text{If } k_1 + k_2 - 2 \leq |F \cap [X, \bar{X}]| \leq k_1 + k_2 - 1,$$

then

$$d_{G-F}(X) \geq 3k_1 + 3k_2 - 11.$$

Since

$|F \cap [X, \bar{X}]| \geq k_1 + k_2 - 2 \geq 6$, $|F \cap E(G[X])| \leq 1$ and $G[X] \cong K_4$, there exists an edge in $G[X] - F$, say uv , such that

$$|F \cap [\{u, v\}, \bar{X}]| \geq 3.$$

Therefore,

$$\begin{aligned} \xi(G-F) &\leq \\ d_G(u) + d_G(v) - 2 - |F \cap [\{u, v\}, \bar{X}]| &\leq \\ 2(k_1 + k_2) - 2 - 3 &< 3k_1 + 3k_2 - 11 \leq \\ d_{G-F}(X), & \end{aligned}$$

a contradiction to (1).

Case 2 $a=3$.

Since G is $(k_1 + k_2)$ -regular,

$$d_G(X) = 3(k_1 + k_2) - 2 |E(G[X])|.$$

Since $G[X]$ is connected, $2 \leq |E(G[X])| \leq 3$ and so $d_G(X)$ is equal to $3k_1 + 3k_2 - 6$ or $3k_1 + 3k_2 - 4$.

Subcase 2.1 $d_G(X) = 3k_1 + 3k_2 - 6$.

In this case, $G[X] \cong K_3$. If

$$|F \cap [X, \bar{X}]| < k_1 + k_2 - 4,$$

then

$$\begin{aligned} d_{G-F}(X) &\geq d_G(X) - |F \cap [X, \bar{X}]| > \\ 2k_1 + 2k_2 - 2 &= \xi(G) \geq \xi(G-F), \end{aligned}$$

a contradiction to (1). Next suppose that

$$k_1 + k_2 - 4 \leq |F \cap [X, \bar{X}]| \leq k_1 + k_2 - 1.$$

If

$$k_1 + k_2 - 4 \leq |F \cap [X, \bar{X}]| \leq k_1 + k_2 - 2,$$

then

$$\begin{aligned} d_{G-F}(X) &\geq d_G(X) - |F \cap [X, \bar{X}]| \geq \\ 2k_1 + 2k_2 - 4. & \end{aligned}$$

Since $G[X] - F$ is connected,

$$0 \leq |E(G[X]) \cap F| \leq 1.$$

Note that

$$|F \cap [X, \bar{X}]| \geq k_1 + k_2 - 4 \geq 4$$

and $G[X] \cong K_3$. If

$$|E(G[X]) \cap F| = 0,$$

then there exists an edge in $G[X]$, say uv , such that

$$|F \cap [\{u, v\}, \bar{X}]| \geq 3.$$

Thus,

$$\begin{aligned} \xi(G-F) &\leq \\ d_G(u) + d_G(v) - 2 - |F \cap [\{u, v\}, \bar{X}]| &\leq \\ 2k_1 + 2k_2 - 2 - 3 &< 2k_1 + 2k_2 - 4 \leq \\ d_{G-F}(X), & \end{aligned}$$

a contradiction to (1). If

$$|E(G[X]) \cap F| = 1,$$

say

$$E(G[X]) \cap F = \{uv\},$$

then there exists an edge in $G[X] - uv$, say uw , such that

$$|F \cap [\{u, w\}, \bar{X}]| \geq 2.$$

Thus,

$$\begin{aligned} \xi(G-F) &\leq \\ d_G(u) + d_G(w) - 2 - |F \cap [\{u, w\}, \bar{X}]| - |\{uw\}| &\leq \\ 2k_1 + 2k_2 - 2 - 2 - 1 &< \\ 2k_1 + 2k_2 - 4 &\leq d_{G-F}(X), \end{aligned}$$

a contradiction to (1).

If

$$|F \cap [X, \bar{X}]| = k_1 + k_2 - 1,$$

then $F \subseteq [X, \bar{X}]$ and

$$d_{G-F}(X) = d_G(X) - |F \cap [X, \bar{X}]| = 2k_1 + 2k_2 - 5.$$

Since $|F \cap [X, \bar{X}]| \geq 7$ and $G[X] \cong K_3$, there exists an edge in $G[X]$, say uv , such that

$$|F \cap [\{u, v\}, \bar{X}]| \geq 5.$$

Thus,

$$\begin{aligned} \xi(G-F) &\leq d_G(u) + d_G(v) - 2 - |F \cap [\{u, v\}, \bar{X}]| \leq \\ &2k_1 + 2k_2 - 2 - 5 < \\ &2k_1 + 2k_2 - 5 = d_{G-F}(X), \end{aligned}$$

a contradiction to (1).

Subcase 2.2 $d_G(X) = 3k_1 + 3k_2 - 4$.

In this case, $G[X] \cong P_3$ and $F \cap E(G[X]) = \emptyset$. If

$$|F \cap [X, \bar{X}]| < k_1 + k_2 - 2,$$

then

$$d_{G-F}(X) = d_G(X) - |F \cap [X, \bar{X}]| > 2k_1 + 2k_2 - 2 = \xi(G) \geq \xi(G-F),$$

a contradiction to (1). Next suppose that

$$k_1 + k_2 - 2 \leq |F \cap [X, \bar{X}]| \leq k_1 + k_2 - 1.$$

Then

$$d_{G-F}(X) = d_G(X) - |F \cap [X, \bar{X}]| \geq 2k_1 + 2k_2 - 3.$$

Since

$$|F \cap [X, \bar{X}]| \geq k_1 + k_2 - 2 \geq 6$$

and $G[X] \cong P_3$, there exists an edge in $G[X]$, say uv , such that $|F \cap [\{u, v\}, \bar{X}]| \geq 3$. Thus,

$$\begin{aligned} \xi(G-F) &\leq d_G(u) + d_G(v) - 2 - |F \cap [\{u, v\}, \bar{X}]| \leq \\ &2k_1 + 2k_2 - 2 - 3 < \\ &2k_1 + 2k_2 - 3 \leq d_{G-F}(X), \end{aligned}$$

a contradiction to (1). □

Lemma 1.5 Let G_i be a k_i -regular k_i -edge-connected graph with $k_i \geq 4$ for $i = 1, 2$ and $G = G_1 \times G_2$. Then

$$d_{G-F}(X_1 \cup X_2) > \xi(G-F)$$

holds for any $F \subseteq E(G)$ and any $X_1, X_2 \subseteq V(G)$ such that $|F| \leq k_1 + k_2 - 1, X_1 \cap X_2 = \emptyset, |X_1| \geq$

$1, |X_2| \geq 1, |X_1 \cup X_2| \geq 3, G[X_1 \cup X_2] - F$ is connected, $X_1 \subseteq V(G_1^{y_1})$ (resp. $X_1 \subseteq V(G_2^{y_1})$) and $X_2 \subseteq V(G_1^{y_2})$ (resp. $X_2 \subseteq V(G_2^{y_2})$) with $\{y_1, y_2\} \subseteq V(G_2)$ (resp. $\{x_1, x_2\} \subseteq V(G_1)$).

Proof Suppose, to the contrary, that there exist a subset F of $E(G)$ and two subsets X_1 and X_2 of $V(G)$ satisfying the specified conditions, but

$$d_{G-F}(X_1 \cup X_2) \leq \xi(G-F) \quad (6)$$

By Lemma 1.1, $\lambda(G) = k_1 + k_2$. Since

$$|F| \leq k_1 + k_2 - 1 = \lambda(G) - 1,$$

we have $G-F$ is connected and

$$\xi(G-F) \leq \xi(G) \quad (7)$$

Since G is $(k_1 + k_2)$ -regular, $\xi(G) = 2k_1 + 2k_2 - 2$.

By (6) and (7),

$$\begin{aligned} 2k_1 + 2k_2 - 2 &\geq d_{G-F}(X_1 \cup X_2) \geq \\ &d_G(X_1 \cup X_2) - |F| \geq \\ &d_G(X_1 \cup X_2) - (k_1 + k_2 - 1). \end{aligned}$$

Thus

$$d_G(X_1 \cup X_2) \leq 3k_1 + 3k_2 - 3 \quad (8)$$

Let $|X_1| = a_1$ and $|X_2| = a_2$. Then $a_1, a_2 \geq 1$ and $a_1 + a_2 \geq 3$. Without loss of generality, assume that $X_1 \subseteq V(G_1^{y_1})$ and $X_2 \subseteq V(G_1^{y_2})$ with $\{y_1, y_2\} \subseteq V(G_2)$. Then

$$\begin{aligned} d_G(X_1 \cup X_2) &\geq d_{G_1^{y_1}}(X_1) + d_{G_1^{y_2}}(X_2) + \\ &|[X_1 \cup X_2, V(G) \setminus (V(G_1^{y_1}) \cup V(G_1^{y_2}))]| \geq \\ &d_{G_1^{y_1}}(X_1) + d_{G_1^{y_2}}(X_2) + |X_1 \cup X_2| (k_2 - 1) = \\ &d_{G_1^{y_1}}(X_1) + d_{G_1^{y_2}}(X_2) + (a_1 + a_2)(k_2 - 1). \end{aligned}$$

Combining this with (8), we obtain

$$d_{G_1^{y_1}}(X_1) + d_{G_1^{y_2}}(X_2) \leq 3k_1 - (a_1 + a_2 - 3)(k_2 - 1) \quad (9)$$

Since $a_1 + a_2 \geq 3$ and $k_2 \geq 4$, by (9) we have

$$d_{G_1^{y_1}}(X_1) + d_{G_1^{y_2}}(X_2) \leq 3k_1 - 3(a_1 + a_2) + 9 \quad (10)$$

Note that

$$d_{G_1^{y_1}}(X_1) = \sum_{z \in X_1} (d_{G_1^{y_1}}(z) - d_{G_1^{y_1}}[X_1](z)) \geq$$

$$a_1(k_1 - a_1 + 1)$$

and $d_{G_1^{y_2}}(X_2) \geq a_2(k_1 - a_2 + 1)$. Combining this with (10), we have

$$(a_1 + a_2 - 3)k_1 \leq a_1^2 + a_2^2 - 4(a_1 + a_2) + 9 \quad (11)$$

Since $d_{G_1^{y_1}}(X_1) \geq \lambda(G_1) = k_1$ and $d_{G_1^{y_2}}(X_2) \geq$

$\lambda(G_1) = k_1$, by (10), $2k_1 \leq 3k_1 - 3(a_1 + a_2) + 9$ and so $k_1 \geq 3(a_1 + a_2 - 3)$. Combining this with (11), we get

$$(a_1 + a_2)^2 - 7(a_1 + a_2) + 9 + a_1 a_2 \leq 0,$$

that is,

$$(a_1 + a_2 - 1)(a_1 + a_2 - 6) + a_1 a_2 + 3 \leq 0 \tag{12}$$

Since $a_1, a_2 \geq 1$ and $a_1 + a_2 \geq 3$, by (12), we have

$$(a_1 + a_2 - 1)(a_1 + a_2 - 6) < 0$$

and so $3 \leq a_1 + a_2 \leq 5$.

If $a_1 + a_2 = 5$, then, by (12), we have $a_1 a_2 \leq 1$, a contradiction to $a_1 + a_2 \geq 3$.

If $a_1 + a_2 = 4$, then $\{a_1, a_2\}$ equals $\{1, 3\}$ or $\{2, 2\}$. Since $G[X_1 \cup X_2]$ is a connected subgraph of $G = G_1 \times G_2$, $|E(G[X_1 \cup X_2])| \leq 4$. By

$$k_1 + k_2 \geq 8,$$

$$\begin{aligned} d_G(X_1 \cup X_2) &= \\ 4(k_1 + k_2) - 2 |E(G[X_1 \cup X_2])| &\geq \\ 4k_1 + 4k_2 - 8 &> 3k_1 + 3k_2 - 3, \end{aligned}$$

a contradiction to (8).

If $a_1 + a_2 = 3$, then $\{a_1, a_2\} = \{1, 2\}$. Since $G[X_1 \cup X_2]$ is connected, $G[X_1 \cup X_2]$ is isomorphic to P_3 . Let $X = X_1 \cup X_2$, then

$$d_G(X) = 3k_1 + 3k_2 - 4.$$

Since $G[X] - F$ is connected, $F \cap E(G[X]) = \emptyset$. If

$$|F \cap [X, \bar{X}]| < k_1 + k_2 - 2,$$

then

$$\begin{aligned} d_{G-F}(X) &= d_G(X) - |F \cap [X, \bar{X}]| > \\ 2k_1 + 2k_2 - 2 &= \xi(G) \geq \xi(G-F), \end{aligned}$$

a contradiction to (6). Next suppose that

$$k_1 + k_2 - 2 \leq |F \cap [X, \bar{X}]| \leq k_1 + k_2 - 1.$$

Then

$$d_{G-F}(X) \geq 2k_1 + 2k_2 - 3.$$

Since

$$|F \cap [X, \bar{X}]| \geq k_1 + k_2 - 2 \geq 6$$

and $G[X] \cong P_3$, there exists an edge in $G[X]$, say uv , such that $|F \cap [\{u, v\}, \bar{X}]| \geq 3$. Thus,

$$\begin{aligned} \xi(G-F) &\leq \\ d_G(u) + d_G(v) - 2 - |F \cap [\{u, v\}, \bar{X}]| &\leq \\ 2k_1 + 2k_2 - 2 - 3 &< \\ 2k_1 + 2k_2 - 3 &\leq d_{G-F}(X), \end{aligned}$$

a contradiction to (6). □

2 Main results

In this section, we give the lower and upper bounds on $\rho'(G)$ for Cartesian product of regular graphs and show that the bounds are best possible.

Theorem 2.1 Let G_i be a k_i -regular k_i -edge-connected graph of order v_i with $k_i \geq 4$ for $i = 1, 2$. Then

$$\begin{aligned} \min\{k_1 + k_2 - 1, v_1 k_2 - 2k_1 - 2k_2 + 1, \\ v_2 k_1 - 2k_1 - 2k_2 + 1\} &\leq \\ \rho'(G_1 \times G_2) &\leq k_1 + k_2 - 1. \end{aligned}$$

Proof Let $G = G_1 \times G_2$. Then G is super- λ' by Lemma 1.3. Note that G is $(k_1 + k_2)$ -regular and $k_1 + k_2 \geq 8$. Let F be a set of edges incident with some vertex of degree $k_1 + k_2$. Since $G - F$ is disconnected, $G - F$ is not super- λ' . By definition of $\rho'(G)$, we have

$$\rho'(G) \leq k_1 + k_2 - 1.$$

Let

$$\begin{aligned} m &= \min\{k_1 + k_2 - 1, v_1 k_2 - 2k_1 - 2k_2 + 1, \\ v_2 k_1 - 2k_1 - 2k_2 + 1\}. \end{aligned}$$

To show that $\rho'(G) \geq m$, we only need to show that for any $F \subseteq E(G)$ with $|F| \leq m$, $G' = G - F$ is super- λ' . By Lemma 1.1, $\lambda(G) = k_1 + k_2$. Since

$$|F| \leq m \leq k_1 + k_2 - 1 = \lambda(G) - 1,$$

G' is connected. It is easy to find that G' is not a star, since G' is a spanning subgraph of $G_1 \times G_2$ with $v_i > 4$. Thus, G' is λ' -connected. If G' is not λ'' -connected, then every λ' -cut of G' isolates one edge and so G' is super- λ' . Next, suppose that G' is λ'' -connected. Then, by Lemma 1.2, we need to show $\lambda''(G') > \xi(G')$. Let X be any λ'' -fragment of G' such that $|\bar{X}| \geq |X| \geq 3$. By $d_{G'}(X) = \lambda''(G')$ and the arbitrary of X , it suffices to prove

$$d_{G'}(X) > \xi(G') \tag{13}$$

Since

$$\begin{aligned} d_{G'}(X) &\geq d_G(X) - |F| \geq d_G(X) - m \\ \text{and } \xi(G') &\leq \xi(G), \text{ if } d_G(X) > \xi(G) + m, \text{ then } d_{G'}(X) > \xi(G'). \end{aligned}$$

In the following, we assume that

$$d_G(X) \leq \xi(G) + m \tag{14}$$

and prove that (13) holds.

Let

$$I_1 = \{x : x \in V(G_1) \text{ and}$$

$G_2^x - [X, \bar{X}]$ is disconnected),

$$I_2 = \{y: y \in V(G_2) \text{ and}$$

$G_1^y - [X, \bar{X}]$ is disconnected}.

Claim 1 $|I_i| < v_i$ for $i=1,2$.

Suppose, without loss of generality, that $|I_1| = v_1$. Then $G_2^x - [X, \bar{X}]$ is disconnected for any $x \in V(G_1)$ and so $d_G(X) \geq v_1 \lambda(G_2) = v_1 k_2$. Combining this with (14), we have

$$\begin{aligned} v_1 k_2 &\leq \xi(G) + m \leq \\ &2k_1 + 2k_2 - 2 + v_1 k_2 - 2k_1 - 2k_2 + 1 = \\ &v_1 k_2 - 1, \end{aligned}$$

a contradiction.

Claim 2 $|I_i| \geq 1$ for $i=1,2$.

Suppose, without loss of generality, that $|I_1| = 0$. Then $G_2^x - [X, \bar{X}]$ is connected for any $x \in V(G_1)$. By Claim 1, there exists at least one vertex $y \in V(G_2)$ such that $G_1^y - [X, \bar{X}]$ is connected. Thus we have that $G - [X, \bar{X}]$ is connected, a contradiction.

Claim 3 $|I_1| \leq 2$ or $|I_2| \leq 2$.

Suppose that $|I_1| \geq 3$ and $|I_2| \geq 3$. Then

$$\begin{aligned} d_G(X) &\geq 3k_1 + 3k_2 = \\ &k_1 + k_2 - 1 + 2k_1 + 2k_2 - 2 + 3 \geq \\ &m + \xi(G) + 3, \end{aligned}$$

a contradiction to (14).

By Claim 2 and Claim 3, we assume, without loss of generality, that $1 \leq |I_2| \leq 2$. Since $G'[X]$ and $G'[\bar{X}]$ are connected, $G[X]$ and $G[\bar{X}]$ are also connected. We consider the following two cases.

Case 1 $|I_2| = 1$.

Let $I_2 = \{y_0\}$. Then $G_1^{y_0} - [X, \bar{X}]$ is disconnected and $G_1^y - [X, \bar{X}]$ is connected for each $y \in V(G_2) \setminus \{y_0\}$. By Claim 1, there exists at least one vertex $x \in V(G_1)$ such that $G_2^x - [X, \bar{X}]$ is connected. Thus $G_2^x \cup (\cup_{y \in V(G_2) \setminus \{y_0\}} G_1^y) - [X, \bar{X}]$ is connected and it is completely contained in $G[\bar{X}]$ since $|\bar{X}| \geq |X|$. It follows that $X \subseteq V(G_1^{y_0})$. Since $G_1^{y_0} - [X, \bar{X}]$ is disconnected, $X \subset V(G_1^{y_0})$. Note that $G'[X] = G[X] - F$ is connected. By Lemma 1.4, we have $d_{G-F}(X) > \xi(G-F)$. Hence, (13) is proved.

Case 2 $|I_2| = 2$.

Let $I_2 = \{y_1, y_2\}$. Then $G_1^{y_j} - [X, \bar{X}]$ is disconnected for each $j=1,2$ and $G_1^y - [X, \bar{X}]$ is connected for each $y \in V(G_2) \setminus \{y_1, y_2\}$. By Claim 1, there exists at least one vertex $x \in V(G_1)$ such that $G_2^x - [X, \bar{X}]$ is connected. Thus $G_2^x \cup (\cup_{y \in V(G_2) \setminus \{y_1, y_2\}} G_1^y) - [X, \bar{X}]$ is connected and it is completely contained in $G[\bar{X}]$ since $|\bar{X}| \geq |X|$. It follows that $X \subseteq V(G_1^{y_1} \cup G_1^{y_2})$. Let $X_1 = X \cap V(G_1^{y_1})$ and $X_2 = X \cap V(G_1^{y_2})$. Then $X = X_1 \cup X_2$ with $X_1 \cap X_2 = \emptyset$ and $|X_1 \cup X_2| \geq 3$. Since $G_1^{y_j} - [X, \bar{X}]$ is disconnected, $|X_j| \geq 1$ and $X_j \subset V(G_1^{y_j})$ for each $j=1,2$. Note that $G'[X_1 \cup X_2] = G[X_1 \cup X_2] - F$ is connected. By Lemma 1.5, we have

$$d_{G-F}(X_1 \cup X_2) > \xi(G-F).$$

Hence, (13) is proved. \square

The following result presents some sufficient conditions for ρ' of a Cartesian product graph to attain the maximum value. This result also shows that the lower and upper bounds on ρ' given in Theorem 2.1 are best possible.

Corollary 2.2 Let G_i be a k_i -regular k_i -edge-connected graph of order v_i with $k_i \geq 4$ for $i=1,2$. Then

$$\rho'(G_1 \times G_2) = k_1 + k_2 - 1,$$

if one of the following conditions holds:

- ① G_1 and G_2 are not complete graphs;
- ② $k_1 + k_2 \geq 10$;
- ③ $k_1 = 4, k_2 = 5$ and G_2 is not complete.

Proof By Theorem 2.1, it suffices to prove that

$$v_1 k_2 - 2k_1 - 2k_2 + 1 \geq k_1 + k_2 - 1,$$

$$v_2 k_1 - 2k_1 - 2k_2 + 1 \geq k_1 + k_2 - 1,$$

that is

$$v_1 k_2 \geq 3k_1 + 3k_2 - 2 \tag{15}$$

$$v_2 k_1 \geq 3k_1 + 3k_2 - 2 \tag{16}$$

If G_1 is not complete, then $v_1 \geq k_1 + 2$. Combining this with $k_1, k_2 \geq 4$, we have

$$v_1 k_2 \geq (k_1 + 2)k_2 =$$

$$3k_1 + 3k_2 - 2 + (k_1 - 1)(k_2 - 3) - 1 >$$

$$3k_1 + 3k_2 - 2.$$

Similarly, if G_2 is not complete, then (16) holds. Therefore, (15) and (16) hold under

Condition ①.

If $k_1 + k_2 \geq 10$, then $(k_1 - 2)(k_2 - 3) \geq 4$ and $(k_1 - 3)(k_2 - 2) \geq 4$ and so

$$\begin{aligned} v_1 k_2 &\geq (k_1 + 1)k_2 = \\ &3k_1 + 3k_2 - 2 + (k_1 - 2)(k_2 - 3) - 4 \geq \\ &3k_1 + 3k_2 - 2 \end{aligned} \quad (17)$$

By substituting v_2 for v_1 and exchanging k_1 with k_2 in (17), we can get (16). Therefore, (15) and (16) hold under Condition ②.

If $k_1 = 4$ and $k_2 = 5$, then

$$v_1 k_2 \geq (k_1 + 1)k_2 = 3k_1 + 3k_2 - 2.$$

If G_2 is not complete, then (16) holds. Thus, (15) and (16) hold under Condition ③. \square

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