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Research Article

Hamilton Paths and Cycles in Varietal Hypercube Networks with Mixed Faults

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This paper considers the varietal hypercube network VQ_n with mixed faults and shows that VQ_n contains a fault-free Hamilton cycle provided faults do not exceed $n-2$ for $n \geqslant 2$ and contains a fault-free Hamilton path between any pair of vertices provided faults do not exceed $n-3$ for $n \geqslant 3$. The proof is based on an inductive construction.

1. Introduction

As a topology of interconnection networks, the hypercube Q_n is the most simple and popular since it has many nice properties. The varietal hypercube VQ_n is a variant of Q_n and proposed by Cheng and Chuang [1] in 1994 and has many properties similar or superior to Q_n. For example, they have the same numbers of vertices and edges and the same connectivity and restricted connectivity (see Wang and Xu [2]), while all the diameter and the average distances, fault-diameter, and wide-diameter of VQ_n are smaller than those of the hypercube Q_n (see Cheng and Chuang [1], Jiang et al. [3]). Recently, Xiao et al. [4] have shown that VQ_n is vertex-transitive.

Embedding paths and cycles in various well-known networks, such as the hypercube and some well-known variations of the hypercube, have been extensively investigated in the literature (see, e.g., Tsai [5] for the hypercubes, Fu [6] for the folded hypercubes, Huang et al. [7] and Yang et al. [8] for the crossed cubes, Yang et al. [9] for the twisted cubes, Hsieh and Chang [10] for the Möbius cubes, Li et al. [11] for the star graphs and Xu and Ma [12] for a survey on this topic). Recently, Cao et al. [13] have shown that every edge of VQ_n is contained in cycles of every length from 4 to 2^n except 5, and every pair of vertices with distance d is connected by paths of every length from d to 2^n-1 except 2 and 4 if $d=1$, from which VQ_n contains a Hamilton cycle for $n \geqslant 2$ and a Hamilton path between any pair of vertices for $n \geqslant 3$. Huang and Xu [14] have improved this result by considering edge-faults and showing that VQ_n contains a fault-free Hamilton cycle provided faulty edges do not exceed $n-2$ for $n \geqslant 3$ and a fault-free Hamilton path between any pair of vertices provided faulty edges do not exceed $n-3$ for $n \geqslant 3$. In this paper, we will further improve these results by considering mixed faults of vertices and edges and proving that VQ_n contains a fault-free Hamilton cycle provided the number of mixed faults does not exceed $n-2$ for $n \geqslant 2$ and contains a fault-free Hamilton path between any pair of vertices provided the number of mixed faults does not exceed $n-3$ for $n \geqslant 3$.

The proofs of these results are in Section 3. The definition and some basic structural properties of VQ_n are given in Section 2.

2. Definitions and Structural Properties

We follow [15] for graph-theoretical terminology and notation not defined here. A graph $G=(V,E)$ always means a simple and connected graph, where $V=V(G)$ is the vertex-set and $E=E(G)$ is the edge-set of G. For $xy \in E(G)$, we call x (resp., y) a neighbor of y (resp., x).

Let G_k be a labeled graph with vertex set $V_k = \{x_k \cdots x_2 x_1 : x_i \in \{0,1\}, 1 \leqslant i \leqslant k\}$. For $j \geqslant 1$, let

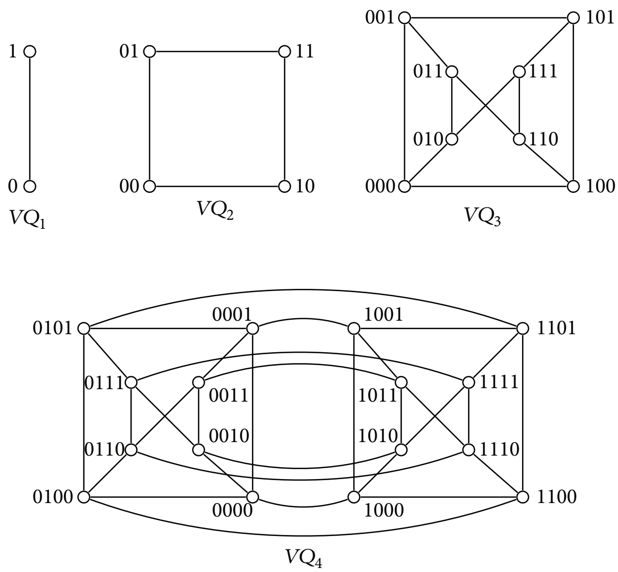


Figure 1: The varietal hypercubes VQ_1, VQ_2, VQ_3, and VQ_4.

$\alpha_j = y_j \cdots y_1$, where $y_i \in \{0, 1\}$ for each $i = 1, \ldots, j$. Use $G_k^{\alpha_j}$ to denote a labeled graph obtained from G_k by inserting the string α_j in front of each vertex-labeling in G_k. Clearly, $G_k^{\alpha_j} \cong G_k$.

Definition 1. The n-dimensional varietal hypercube VQ_n is the labeled graph defined recursively as follows. VQ_1 is the complete graph of two vertices labeled with 0 and 1, respectively. Assume that VQ_{n-1} has been constructed. For $n > 1$, $VQ_n = VQ_{n-1}^0 \odot VQ_{n-1}^1$ is obtained from VQ_{n-1}^0 and VQ_{n-1}^1 by joining vertices between them, according to the rule: a vertex $x = 0x_{n-1}x_{n-2}x_{n-3} \cdots x_2x_1$ in VQ_{n-1}^0 and a vertex $y = 1y_{n-1}y_{n-2}y_{n-3} \cdots y_2y_1$ in VQ_{n-1}^1 are adjacent in VQ_n if and only if

(1) $x_{n-1}x_{n-2}x_{n-3} \cdots x_2x_1 = y_{n-1}y_{n-2}y_{n-3} \cdots y_2y_1$ if $n \neq 3k$, or

(2) $x_{n-3} \cdots x_2x_1 = y_{n-3} \cdots y_2y_1$ and $(x_{n-1}x_{n-2}, y_{n-1}y_{n-2}) \in I$ if $n = 3k$, where $I = \{(00, 00), (01, 01), (10, 11), (11, 10)\}$.

Figure 1 shows the examples of varietal hypercubes VQ_n for $n = 1, 2, 3$, and 4, respectively.

For convenience, we write $VQ_n = L \odot R$, where $L = VQ_{n-1}^0$ and $R = VQ_{n-1}^1$. Clearly, the set M of edges between L and R is a perfect matching of size 2^{n-1} in VQ_n. Use x_Lx_R to denote an edge in M joining $x_L \in L$ and $x_R \in R$. By the recursive definition of VQ_n, $VQ_{n-1}^0 = VQ_{n-2}^{00} \odot VQ_{n-2}^{01}$ and $VQ_{n-1}^1 = VQ_{n-2}^{10} \odot VQ_{n-2}^{11}$. Thus, VQ_n is of the recursive structure shown as in Figure 2.

Use U and W to denote two subgraphs of VQ_n induced by $V(VQ_{n-2}^{00}) \cup V(VQ_{n-2}^{10})$ and $V(VQ_{n-2}^{01}) \cup V(VQ_{n-2}^{11})$, respectively. It should be noted that U and W are not always isomorphic to VQ_{n-1}, although L and R are isomorphic to VQ_{n-1}.

Definition 2. The graph $G_n = G_{n-1}^0 \oplus_M G_{n-1}^1$ is the labeled graph defined recursively as follows. G_1 is the complete graph of two vertices labeled with 0 and 1, respectively. $G_2 = G_1^0 \oplus G_1^1$ is obtained from G_1^0 and G_1^1 plus two edges joining 00 and 10, 01, and 11. For $n \geqslant 3$, $G_n = G_{n-1}^0 \oplus_M G_{n-1}^1$ is obtained from G_{n-1}^0 and G_{n-1}^1 by adding a perfect matching M between G_{n-1}^0 and G_{n-1}^1, according to the following rule: M consists of two perfect matchings M_1 and M_2, where M_1 is a perfect matching between G_{n-2}^{00} and G_{n-2}^{10} and M_2 is a perfect matching between G_{n-2}^{01} and G_{n-2}^{11}.

Clearly, by Definition 1, in VQ_i, the set M of edges between VQ_{i-1}^0 and VQ_{i-1}^1 is a perfect matching between them satisfying the rule in Definition 2. Thus, VQ_n is a special example of G_n. We state this fact as a simple observation.

Observation 1. For each $i = 2, \ldots, n$, $VQ_i \cong VQ_{i-1}^0 \oplus_M VQ_{i-1}^1$ for the perfect matching M defined by the rule in Definition 1. Moreover, $G_3 \cong Q_3$ or VQ_3, where Q_3 is a 3-dimensional cube.

3. Main Results

Let G be a graph, and let x and y be two distinct vertices in G. A subgraph P of G is called an *xy-path*, if its vertex-set can be expressed as a sequence of adjacent vertices, written as $P = (x_0, x_1, x_2, \ldots, x_m)$, in which $x = x_0$, $y = x_m$, and all the vertices $x_0, x_1, x_2, \ldots, x_m$ are different from each other. For a path $P = (x_0, \ldots, x_i, x_{i+1}, \ldots, x_m)$, we can write $P = P(x_0, x_i) + x_ix_{i+1} + P(x_{i+1}, x_m)$, and the notation $P - x_ix_{i+1}$ denotes the subgraph obtained from P by deleting the edge x_ix_{i+1}. If P is an xy-path and $xy \in E(G)$, then $P + xy$ is called a cycle in G. A cycle is called a Hamilton cycle if it contains all vertices in G. An xy-path P is called an xy-Hamilton path if it contains all vertices in G. A graph G is Hamiltonian if it contains a Hamilton cycle and is called Hamilton-connected if it contains an xy-Hamilton path for any two vertices x and y in G. Clearly, if G has at least three vertices and is Hamilton-connected, then it certainly is Hamiltonian; moreover, every edge is contained in a Hamilton cycle.

Lemma 3 (Cao et al. [13]). *VQ_n is Hamilton-connected for $n \geqslant 3$, and so every edge of VQ_n is contained in a Hamilton cycle for $n \geqslant 2$.*

Let F be a subset of $V(G) \cup E(G)$. A subgraph H of G is called fault-free if H contains no elements in F. A graph G is called t-edge-fault-tolerant Hamiltonian (resp., t-edge-fault-free Hamilton-connected) if $G - F$ contains a Hamilton cycle (resp., is Hamilton-connected) for any $F \subset E(G)$ with $|F| \leqslant t$. G is called t-fault-tolerant Hamiltonian (resp., t-fault-free Hamilton-connected) if $G - F$ contains a Hamilton cycle (resp., is Hamilton-connected) for any $F \subset E(G) \cup V(G)$ with $|F| \leqslant t$.

Lemma 4 (Huang and Xu [14]). *VQ_n is $(n-2)$-edge-fault-tolerant Hamiltonian for $n \geqslant 2$ and $(n-3)$-edge-fault-tolerant Hamilton-connected for $n \geqslant 3$.*

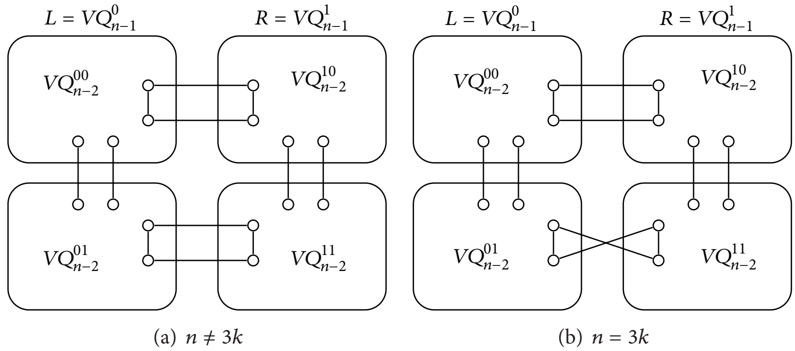


FIGURE 2: The recursive structure of VQ_n.

In this paper, we will generalize this result by proving that VQ_n is $(n-2)$-fault-tolerant Hamiltonian for $n \geqslant 2$ and $(n-3)$-fault-tolerant Hamilton-connected for $n \geqslant 3$.

To prove our main results, we first prove the following result on the graph G_n.

Theorem 5. *For $n \geqslant 3$, $G_n = G^0_{n-1} \oplus_M G^1_{n-1}$ is $(n-3)$-fault-tolerant Hamilton-connected for any perfect matching M between G^0_{n-1} and G^1_{n-1} defined by the rule in Definition 2.*

Proof. We proceed by induction on $n \geqslant 3$.

Since $G_3 \cong Q_3$ or VQ_3, which is vertex-transitive, it is easy to check the conclusion is true for $n = 3$. Suppose now that $n \geqslant 4$ and the result holds for any integer less than n. Let $F \subset E(G_n) \cup V(G_n)$ with $|F| \leqslant n-3$, and let x and y be two distinct vertices in $G_n - F$. We need to prove that $G_n - F$ contains an xy-Hamilton path. Without loss of generality, we can assume $F \subset V(G_n)$. Let $G_n = L \oplus_M R$, where

$$L = G^{00}_{n-2} \oplus_{M_1} G^{01}_{n-2}, \qquad R = G^{10}_{n-2} \oplus_{M_2} G^{11}_{n-2}, \tag{1}$$

and let

$$F_L = F \cap L, \qquad F_R = F \cap R. \tag{2}$$

By symmetry of structure of G_n, we may assume $|F_L| \geqslant |F_R|$.

Case 1 ($|F_L| \leqslant n-4$). In this case, by the hypothesis, we have $|F_R| \leqslant |F_L| \leqslant n-4$.

Subcase 1.1 ($x, y \in L$ or $x, y \in R$). Without loss of generality, assume $x, y \in R$.

Since $R = G_{n-1}$ and $|F_R| \leqslant n-4 = (n-1)-3$, by the induction hypothesis $R - F_R$ contains an xy-Hamilton path, say P_R. Since $|V(P_R)| = 2^{n-1} - |F_R| \geqslant 2^{n-1} - (n-4) > 2(n-3) \geqslant 2|F|$, there is an edge $u_R v_R$ in P_R such that the neighbors u_L and v_L of u_R and v_R in L are not in F. Since $L = G_{n-1}$ and $|F_L| \leqslant n-4 = (n-1)-3$, by the induction hypothesis $L - F_L$ contains a $u_L v_L$-Hamilton path, say P_L. Thus, $P_R - u_R v_R + u_R u_L + v_R v_L + P_L$ is an xy-Hamilton path in $G_n - F$ (see Figure 3(a)).

Subcase 1.2 ($x \in L$ and $y \in R$). Since $|M| = 2^{n-1}$ and $2^{n-1} - 2 > 2(n-3) \geqslant 2|F|$, there is an edge $u_L u_R \in M$ such that u_L and u_R are not in $F \cup \{x, y\}$. By the induction hypothesis, let P_L be an xu_L-Hamilton path in $L - F_L$, and let P_R be a yu_R-Hamilton path in $R - F_R$. Then $P_L + u_L u_R + P_R$ is an xy-Hamilton path in $G_n - F$ (see Figure 3(b)).

Case 2 ($|F_L| = n-3$). In this case, $|F_R| = 0$.

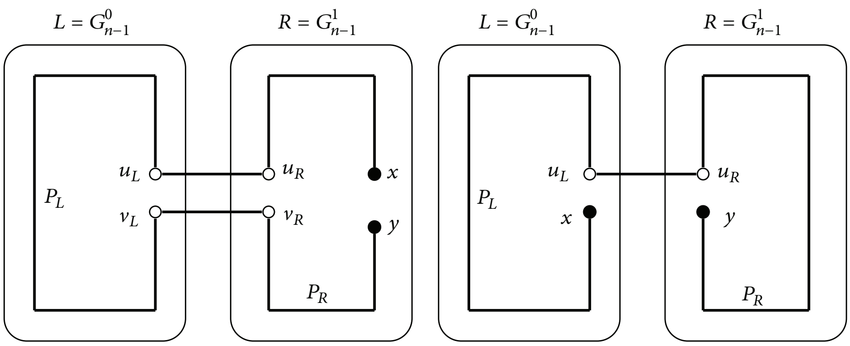
Subcase 2.1 ($x, y \in L$). Arbitrarily take a vertex $u \in F_L$. Since $|F_L - u| = n-4 = (n-1)-3$, by the induction hypothesis $L - (F_L - u)$ contains an xy-Hamilton path, say P_L. Without loss of generality, assume $u \in V(P_L)$. Let u_L and v_L be two neighbors of u in P_L, and let $u_L u_R, v_L v_R \in M$. By the induction hypothesis, R contains a $u_R v_R$-Hamilton path, say P_R. Then $P_L - u + u_L u_R + v_L v_R + P_R$ is an xy-Hamilton path in $G_n - F$.

Subcase 2.2 ($x \in L$ and $y \in R$). If $n = 4$, then $L \cong R \cong Q_3$ or VQ_3. Since $|F_L| = 1$ and L is vertex-transitive, we can assume $F_L = \{u\} = \{000\}$ unless $x = 000$. It is easy to check that $L - u$ contains a Hamilton cycle, say C_L. Choose a neighbor u_L of x in C_L such that its neighbor u_R in R is not y. By the induction basis, R contains a yu_R-Hamilton path, say P_R. Then, $C_L - xu_L + u_L u_R + P_R$ is an xy-Hamilton path in $G_4 - F$.

Assume now $n \geqslant 5$; that is, $n-2 \geqslant 3$. Let $F_{00} = F_L \cap V(G^{00}_{n-2})$, $F_{01} = F_L \cap V(G^{01}_{n-2})$. Without loss of generality, we can assume $F_{00} \neq \emptyset$.

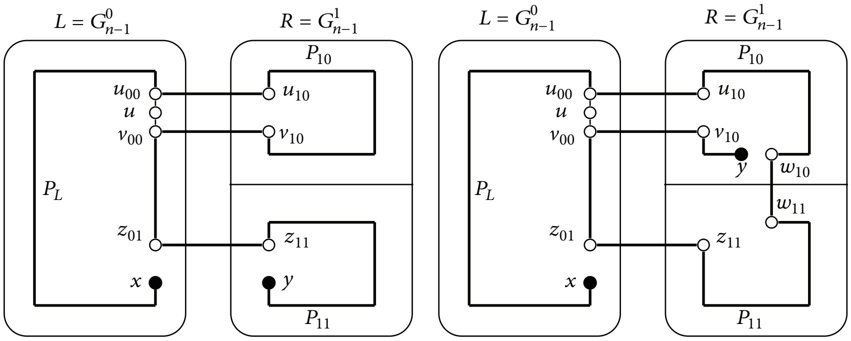
(a) $y \in G^{11}_{n-2}$ (See Figure 4(a)). Arbitrarily take $z_{11} \in G^{11}_{n-2}$ with $z_{11} \neq y$, and let $z_{01} z_{11} \in M$. Since $n-2 \geqslant 3$, by the induction hypothesis G^{11}_{n-2} contains a $z_{11} y$-Hamilton path, say P_{11}. Arbitrarily take a vertex $u \in F_{00}$. Since $n \geqslant 5$, by the induction hypothesis $L - (F_L - u)$ contains an xz_{01}-Hamilton path, say P_L. If u is in P_L, then let u_{00} and w_{00} be two neighbors of u in P_L; if u is not in P_L, then let $u_{00} v_{00}$ be an edge in P_L. Let $u_{00} u_{10}, v_{00} v_{10} \in M$. By the induction hypothesis, G^{10}_{n-2} contains a $u_{10} v_{10}$-Hamilton path, say P_{10}. Let $P'_L = P_L - u$ if u is in P_L and $P'_L = P_L - u_{00} v_{00}$ if u is not in P_L. Then $P_{10} + u_{00} u_{10} + v_{00} v_{10} + P'_L + z_{01} z_{11} + P_{11}$ is an xy-Hamilton path in $G_n - F$ (see Figure 4(a)).

(b) $y \in G^{10}_{n-2}$ (See Figure 4(b)). Arbitrarily take a vertex z_{01} in $G^{01}_{n-2} - F_L$ with $z_{01} \neq x$. Let z_{11} be the neighbor of z_{01} in G^{11}_{n-2}. Arbitrarily take a vertex $u \in F_{00}$. Since $n \geqslant 5$, by the induction hypothesis $L - (F_L - u)$ contains an xz_{01}-Hamilton



(a) $x, y \in R$ (b) $x \in L$ and $y \in R$

FIGURE 3: Illustrations of Case 1 in the proof of Theorem 5.



(a) $x \in L$ and $y \in G^{11}_{n-2}$ (b) $x \in L$ and $y \in G^{10}_{n-2}$

FIGURE 4: Illustrations of Subcase 2.2 in the proof of Theorem 5.

path, say P_L. If u is in P_L, then let u_{00} and w_{00} be two neighbors of u in P_L; if u is not in P_L, then let $u_{00}v_{00}$ be an edge in P_L. Let $u_{00}u_{10}, v_{00}v_{10} \in M$. By the induction hypothesis, G^{10}_{n-2} contains a $u_{10}v_{10}$-Hamilton path, say P_{10}. Since $y \in P_{10}$, we can write $P_{10} = P_{10}(v_{10}, y) + yw_{10} + P_{10}(w_{10}, u_{10})$. Let w_{11} be the neighbor of w_{10} in G^{11}_{n-2}. By the induction hypothesis, G^{11}_{n-2} contains a $z_{11}w_{11}$-Hamilton path, say P_{11}. Let $P'_L = P_L - u$ if u is in P_L and $P'_L = P_L - u_{00}v_{00}$ if u is not in P_L. Then $P'_L + u_{00}u_{10} + v_{00}v_{10} + P_{10} - yw_{10} + w_{10}w_{11} + P_{11} + z_{01}z_{11}$ is an xy-Hamilton path in $G_n - F$ (see Figure 4(b)).

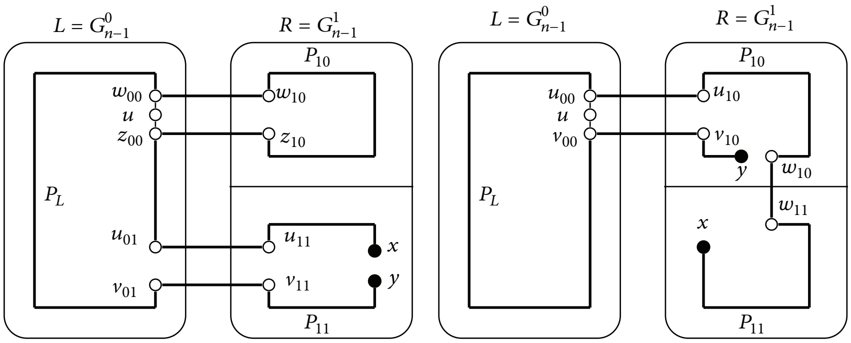
Subcase 2.3 ($x, y \in R$). If $n = 4$, then $L \cong R \cong G_3$. By the induction basis, R contains an xy-Hamilton path, say P_R. Since G_3 is vertex-transitive and $|F_L| = 1$, it is easy to check that $L - F_L$ contains a Hamilton cycle, say C_L. Since L and R are 3-regular and isomorphic, there is an edge u_Rv_R in P_R which is not incident with x and y such that the corresponding edge e_L in L is contained in C_L. By Definition 2 $e_L = u_Lv_L$, where u_L and v_L are neighbors of u_R and v_R in L, respectively. Thus, $P_R - u_Rv_R + u_Lu_R + v_Lv_R + C_L - e_L$ is an xy-Hamilton path in $G_4 - F$ (as a reference, see Figure 3(a)).

Assume $n \geqslant 5$ below; that is, $n - 2 \geqslant 3$.

(a) $x, y \in G^{11}_{n-1}$ (See Figure 5(a)). By the induction hypothesis, G^{11}_{n-2} contains an xy-Hamilton path, say P_{11}. Take $u_{11}v_{11} \in E(P_{11})$, and let u_{01} and v_{01} be neighbors of u_{11} and v_{11} in G^{01}_{n-2}, respectively. Take a vertex u in F_{00}. By the induction hypothesis, $L - (F_L - u)$ contains a $u_{01}v_{01}$-Hamilton path, say P_L. If u is in P_L, then let w_{00} and z_{00} be two neighbors of u in P_L; if u is not in P_L, then let $w_{00}z_{00}$ be an edge in P_L. Let w_{10} and z_{10} be neighbors of w_{00} and z_{00} in G^{10}_{n-2}, respectively. By the induction hypothesis, G^{10}_{n-2} contains a $w_{10}z_{10}$-Hamilton path, say P_{10}. Let $P'_L = P_L - u$ if u is in P_L and $P'_L = P_L - w_{00}z_{00}$ if u is not in P_L. Thus, $P_{10} + w_{00}w_{10} + z_{00}z_{10} + P'_L + P_{11} - u_{11}v_{11} + u_{01}u_{11} + v_{01}v_{11}$ is an xy-Hamilton path in $G_n - F$ (see Figure 5(a)).

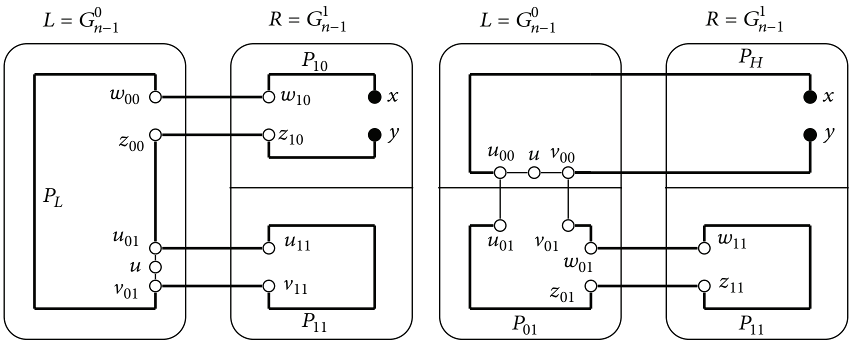
(b) $x \in G^{11}_{n-1}$ and $y \in G^{10}_{n-2}$ (See Figure 5(b)). Arbitrarily take a vertex u in F_{00} and an edge $u_{00}v_{00}$ in G^{00}_{n-2}. By the induction hypothesis, $L - (F_L - u)$ contains a $u_{00}v_{00}$-Hamilton path, say P_L. If u is in P_L, then let $P' = P_L - u + u_{00}v_{00}$; if u is not in P_L, then let $P' = P_L$. Without loss of generality, assume that u is in P_L and let u_{00} and v_{00} be two neighbors of u in P_L.

Let u_{10} and v_{10} be neighbors of u_{00} and v_{00} in G^{10}_{n-2}, respectively. By the induction hypothesis, G^{10}_{n-2} contains a $u_{10}v_{10}$-Hamilton path, say P_{10}. Since y is in P_{10}, we can write $P_{10} = P_{10}(v_{10}, y) + yw_{10} + P_{10}(w_{10}, u_{10})$ (see Figure 5(b)). Let w_{11} be the neighbor of w_{10} in G^{11}_{n-2}. By the induction hypothesis, G^{11}_{n-2} contains an xw_{11}-Hamilton path, say P_{11}.



(a) $x, y \in G_{n-2}^{11}$ (b) $x \in G_{n-2}^{11}$ and $y \in G_{n-2}^{10}$

Figure 5: Illustrations of Subcase 2.3 in the proof of Theorem 5.



(a) $|F_{01}| \neq 0$ (b) $|F_{01}| = 0$

Figure 6: Illustrations of Subcase 2.3(c) in the proof of Theorem 5.

Then $P_L' + P_{10} - yw_{10} + w_{10}w_{11} + P_{11}$ is an xy-Hamilton path in $G_n - F$ (see Figure 5(b)).

(c) $x, y \in G_{n-2}^{10}$ (See Figure 6)

(c1) $|F_{01}| \neq 0$. By the induction hypothesis, G_{n-2}^{10} contains an xy-Hamilton path, say P_{10}. Take $w_{10}z_{10} \in E(P_{10})$, and let w_{00} and z_{00} be neighbors of w_{10} and z_{10} in G_{n-2}^{00}, respectively. Take a vertex u in F_{01}. By the induction hypothesis, $L - (F_L - u)$ contains a $w_{00}z_{00}$-Hamilton path, say P_L. If u is in P_L, let u_{00} and v_{00} be two neighbors of u in P_L; if u is not in P_L, let $u_{00}v_{00}$ be an edge in P_L. Let $P_L' = P_L - u$ if u is in P_L and $P_L' = P_L - u_{00}v_{00}$ if u is not in P_L.

Let u_{11} and v_{11} be neighbors of u_{01} and v_{01} in G_{n-2}^{11}, respectively. By the induction hypothesis, G_{n-2}^{11} contains a $u_{11}v_{11}$-Hamilton path, say P_{11}. Thus, $P_{10} - w_{10}z_{10} + w_{00}w_{10} + z_{00}z_{10} + P_L' + u_{01}u_{11} + v_{01}v_{11} + P_{11}$ is an xy-Hamilton path in $G_n - F$ (see Figure 6(a)).

(c2) $|F_{01}| = 0$. In this case, $|F_{00}| = |F| = n - 3 \geqslant 2$ since $n \geqslant 5$. Consider the subgraph H of G_n induced by $V(G_{n-2}^{00}) \cup V(G_{n-2}^{10})$. By Definition 2, it is easy to check that $H = G_{n-2}^{00} \oplus_M G_{n-2}^{10}$. Let $u \in F$. By the induction hypothesis, $H - (F - u)$ contains an xy-Hamilton path, say P_H. Without loss of generality, assume that u is in P_H. Let u_{00} and v_{00} be two neighbors of u in P_H, and let u_{01} and v_{01} be two neighbors of u_{00} and v_{00} in G_{n-2}^{01}. Then there is a $u_{01}v_{01}$-Hamilton path in G_{n-2}^{01}, say P_{01}. Take an edge $w_{01}z_{01}$ in P_{01}, and let w_{11} and z_{11} be neighbors of w_{01} and z_{01} in G_{n-2}^{11}. Then there is a $w_{11}z_{11}$-Hamilton path in G_{n-2}^{11}, say P_{11}. Thus, $P_H - u + P_{01} - w_{01}z_{01} + P_{11}$ is an xy-Hamilton path in $G_n - F$ (see Figure 6(b)).

The theorem follows. □

By Observation 1 and Theorem 5, we have the following results immediately.

Corollary 6. *VQ_n is $(n-3)$-fault-tolerant Hamilton-connected for $n \geqslant 3$.*

Corollary 7. *Every fault-free edge of VQ_n is contained in a fault-free Hamilton cycle if the number of faults does not exceed $n - 2$ and $n \geqslant 2$.*

Proof. If $n = 2$, then the conclusion holds clearly. Assume now $n \geqslant 3$. Let xy be a fault-free edge in VQ_n. Let F be a set of faults in VQ_n with $|F| \leqslant n-2$ and containing the edge xy. By Corollary 6, there is an xy-Hamilton path P in $VQ_n - (F - xy)$. Then $P + xy$ is a required cycle. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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