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Optimal *t*-pebbling on paths and cycles

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Abstract: A pebbling move removes two pebbles from a vertex and places one pebble on one of its neighbours. For $t\geqslant 1$, the optimal t-pebbling number of a graph G, $f'_t(G)$, is the minimum number of pebbles necessary so that from some initial distribution of them it is possible to move t pebbles to any target vertex by a sequence of pebbling moves. $f'(G)=f'_1(G)$ be the optimal pebbling number of G. Here the optimal t-pebbling numbers of the path P_n and the cycle C_5 were given, respectively. In the final section, it was obtained that $f'_{9t}(P_2\times P_3)=20t$, $f'_{9t+1}(P_2\times P_3)=20t+3$, and $20t+2r+1\leqslant f'_{9t+r}(P_2\times P_3)\leqslant 20t+2r+2$, for $2\leqslant r\leqslant 8$, the last equality holds for $r=5,6,7,8$.

Key words: optimal t-pebbling number; path; cycle; Cartesian product

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路与圈的优化 *t*-pebbling 数

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摘要：图上的一个 pebbling 移动，是从图的一个顶点同时移除 2 个 pebbles，并且在其某个邻点上放置 1 个 pebble. 图的优化 t-pebbling 数，记为 $f'_t(G)$，是指图 G 中所需要的 pebbled 的最小数目，使得存在该 $f'_t(G)$ 个 pebbles 在图上的一种分布，可以在经过一系列 pebbling 移动后，t 个 pebbles 可以移动到任意一个给定的目标顶点上. $f'(G)=f'_1(G)$ 称为图 G 的优化 pebbling 数. 这里给出了路 P_n 和圈 C_5 的优化 t-pebbling 数，证明了 $f'_{9t}(P_2\times P_3)=20t$; $f'_{9t+1}(P_2\times P_3)=20t+3$；当 $2\leqslant r\leqslant 8$ 时，$20t+2r+1\leqslant f'_{9t+r}(P_2\times P_3)\leqslant 20t+2r+2$，其中，当 $5\leqslant r\leqslant 8$ 时，最后一个不等式取到等号.

关键词：优化 t-pebbling 数；路；圈；笛卡尔乘积

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0 Introduction

Throughout this paper, $G=(V,E)$ denotes a simple, connected graph with n vertices. Let $P_n=u_1u_2\cdots u_n$ and $C_n=u_1u_2\cdots u_nu_1$ be the path and the cycle with n vertices, respectively. A function $D: V\rightarrow N\cup\{0\}$ is called a distribution on the vertices of G. Let $D(v)$ be the number of pebbles on the vertex $v\in V(G)$ ($D_i=D(u_i)$), $|D|$ be the total number of pebbles on $V(G)$ under D. A pebbling move consists of the removal of two pebbles from a vertex and placing one pebble at an adjacent vertex. The optimal t-pebbling number of G, denoted by $f'_t(G)$, is the least p such that, for some distribution of p pebbles on the vertices of G, a pebble can be moved to any vertex by a sequence of pebbling moves. Moreover, $f'(G)=f'_1(G)$ is called the optimal pebbling number of G.

Let $W(D,v)$ be the maximal number of pebbles on v by some (possibly empty) pebbling moves on G from the original distribution D. Then we call v t-reachable under D for some positive integer t if $W(D,v)\geqslant t$. A distribution D is called t-fold solvable if every vertex is t-reachable under D.

The known results about the optimal t-pebbling number of paths, cycles and the product of paths are given as follows.

Theorem 0.1[1,3] $f'(P_n)=f'(C_n)=\lceil 2n/3\rceil$.

Theorem 0.2[1] $f'_2(P_n)=n+1$.

Theorem 0.3[4] $f'_{3t+r}(P_2)=4t+2r$ if $r<2$, $f'_{3t+2}(P_2)=4t+3$.

The Cartesian product $G\times H$ is defined to be the graph with vertex set $V(G\times H)$ and edge set: the union of $\{((a,v),(b,v))\mid(a,b)\in E(G), v\in E(H)\}$ and $\{((u,x),(u,y))\mid u\in V(G),(x,y)\in E(H)\}$.

Theorem 0.4[1,4] $f'(P_m\times P_2)=f'(C_m\times P_2)=m$ for $m\geqslant 2$, except that $f'(P_2\times P_2)=3$ and $f'(P_5\times P_2)=6$. If $t=6q+r$,

$$f'_t(K_2\times K_3)=\begin{cases}12q & \text{if } r=0,\\ 12q+2r+1 & \text{otherwise.}\end{cases}$$

In this paper, we will give the optimal t-pebbling numbers of the path P_n and the cycle C_5. First, we give some lemmas.

Lemma 0.5[4] $f'_{s+t}(G)\leqslant f'_s(G)+f'_t(G)$.

For a given distribution D on $V(G)$, assume $d(v)=2$ and $D(v)\geqslant 3$. A smoothing move from v changes D by removing two pebbles from v, and add one pebble on each neighbour of v.

Lemma 0.6[1] Let D be a distribution on a graph G with distinct vertices u and v, where $d(v)=2$, $D(v)\geqslant 3$, and u is t-reachable under D, then u is t-reachable under the distribution D' obtained by making a smoothing move from v.

For more background and related topics of this article, we refer to [1-8].

1 The optimal t-pebbling number of path

First, we give an upper bound of $f'_t(P_n)$.

Theorem 1.1 $f'_{3t}(P_n)\leqslant t(n+2)$, $f'_{3t+1}(P_n)\leqslant t(n+2)+\lfloor n/2\rfloor+1$, $f'_{3t+2}(P_n)\leqslant t(n+2)+(n+1)$ for $t\geqslant 1$.

Proof From Lemma 0.5, we have that $f'_{3t}(P_n)\leqslant tf'_3(P_n)$, $f'_{3t+1}(P_n)\leqslant(t-1)f'_3(P_n)+f'_4(P_n)$ and $f'_{3t+2}(P_n)\leqslant tf'_3(P_n)+f'_2(P_n)$, for $t\geqslant 1$.

Clearly, it is sufficient to show that $f'_3(P_n)\leqslant n+2$, $f'_4(P_n)\leqslant n+\lfloor n/2\rfloor+3$.

Let $D_1=D_n=2$, and $D_i=1$ for $2\leqslant i\leqslant n-1$. Then it is a 3-fold solvable distribution on P_n with $|D|=n+2$. Thus $f'_3(P_n)\leqslant n+2$.

For $t=4$, we use induction on n to prove that there exists a 4-fold solvable distribution D on P_n with $D_n\geqslant 2$ and $|D|=n+\lfloor n/2\rfloor+3$.

If $n=1$, $D_1=4$; $n=2$, $D_1=D_2=3$. So it holds for $n=1,2$.

Assume that it holds for $n-2$, let D' be a 4-fold solvable distribution on P_{n-2}, so that $D'_{n-2}\geqslant 2$ and $|D'|=n-2+\lfloor(n-2)/2\rfloor+3$.

Let D be a distribution on P_n such that $D_i=D'_i$ for $i<n-2$, $D_{n-2}=D'_{n-2}-2$, $D_{n-1}=3$, $D_n=2$. It is clear that D is a 4-fold solvable distribution on P_n, $D_n\geqslant 2$, and

$|D|=|D'|+3=n-2+\lfloor(n-2)/2\rfloor+3+3=$

$$n+\lfloor n/2\rfloor+3,$$

this completes the proof. □

Lemma 0.6 shows that a smoothing move on any vertex v with degree 2 keeps a t-reachable vertex $u\neq v$ still be t-reachable, but it does not hold for v itself. The following lemma shows that v may still be t'-reachable for some $t'\leqslant t$.

Lemma 1.2 Let D be a distribution on P_3 with $W(D,u_i)\geqslant 3t+r$ for $1\leqslant i\leqslant 3$, $D_2\geqslant t+r+2-\lfloor r/2\rfloor$. If we make a smoothing move on u_2, then u_2 is at least $(3t+r)$-reachable.

Proof We only prove the cases $r=0,1$. The case $r=2$ can be proved similarly.

Let $a=D_1$, $b=D_2$, $c=D_3$. The distribution, after a smoothing move on u_2, is denoted by D'. Without loss of generality, we assume that $a\geqslant c$.

If $a>c+2$, then we remove two pebbles from u_1 and add them onto u_3 to get D^*. Clearly, we have that $W(D^*,u_i)\geqslant 3t+r$, and $W(D^{*\prime},u_2)=W(D',u_2)$ ($D^{*\prime}$ is the distribution after a smoothing move on u_2 from D^*).

So we only need to deal with $c\leqslant a\leqslant c+2$.

Case 1 $a=c$.

If a is odd, then

$$W(D',u_2)\geqslant D_2-2+2\lfloor (a+1)/2\rfloor = D_2+2\lfloor a/2\rfloor = W(D,u_2)\geqslant 3t+r.$$

If a is even, then $W(D,u_3)=\lfloor (a/2+b)/2\rfloor + a\geqslant 3t+r$. Note that $\min\{a+b\}$ can be achieved while b is at its minimum $\min b=t+r+2$, then $a\geqslant 2t$. Thus

$$W(D',u_2)=\lfloor a/2\rfloor+\lfloor a/2\rfloor+b-2 = a+b-2\geqslant 3t+r.$$

Case 2 $a=c+1$. $W(D,u_3)=\lfloor (a/2+b)/2\rfloor + a-1\geqslant 3t+r$. Then $\min\{a+b\}$ can be achieved while $b=\min b=t+r+2$, then $a\geqslant 2t$.

$$W(D',u_2)=\lfloor (a+1)/2\rfloor+\lfloor a/2\rfloor+b-2 = a+b-2\geqslant 3t+r.$$

Case 3 $a=c+2$. If a is odd, then we are done. If a is even,

$$W(D,u_3)=\lfloor (a/2+b)/2\rfloor+a-2\geqslant 3t+r.$$

Then $\min\{a+b\}$ can be achieved if $b=\min b=t+r+2$, then $a\geqslant 2t+1$. Moreover,

$$W(D',u_2)=\lfloor (a-2)/2\rfloor+\lfloor a/2\rfloor+b-2 = a+b-3\geqslant 3t+r.$$ □

Corollary 1.3 There exists a $(3t+r)$-fold solvable distribution with $f'_{3t+r}(P_n)$ pebbles on P_n so that $D_1\geqslant 2t+\lfloor r/2\rfloor$, $D_n\geqslant 2t+\lfloor r/2\rfloor$, $D_i\leqslant t+r+1-\lfloor r/2\rfloor$ for $1<i<n$.

Proof We only prove the cases $r=0,1$. The case $r=2$ can be similarly proved.

By Lemma 1.2, we can make a smoothing move on vertex u_i if $D_i\geqslant t+r+2$ for $1<i<n$, and the smoothing moves must be finished. Hence we can make sure that $D_i\leqslant t+r+1$. Then if we can move at least $t+r+1$ pebbles from $P_n\setminus u_n$ to u_n, then there are at least $t+r+1$ pebbles that can be moved from $P_n\setminus\{u_{n-1},u_n\}$ to u_{n-1}, and so on. Then the number of pebbles on P_n is at least

$$2t+2r+2+(n-2)(t+r+1)+2t-1 = nt+nr+n+2t-1,$$

which is incompatible with Theorem 1.1. Thus there are at most $t+r$ pebbles that can be moved from $P_n\setminus u_n$ to u_n, but $W(D,u_n)\geqslant 3t+r$, so $D_n\geqslant 2t$, and similarly $D_1\geqslant 2t$. □

Lemma 1.4 $f'_{3t+r}(P_n)\leqslant f'_{3t+r}(P_{n-2})+2t+r$ for $t\geqslant 1$.

Proof Let D' be a $(3t+r)$-fold solvable distribution with $f'_{3t+r}(P_{n-2})$ pebbles on P_{n-2} so that $D'_{n-2}\geqslant 2t+\lfloor r/2\rfloor$.

If $2t+\lfloor r/2\rfloor\geqslant t+r$, then let $D_{n-2}=D'_{n-2}-t-r$, $D_{n-1}=t+2r$, $D_n=2t$, and $D_i=D'_i$ for $i<n-2$.

If $2t+\lfloor r/2\rfloor<t+r$, we must have that $t=1$, $r=2$. Then let $D_i=D'_i$ for $i<n-2$,

$$D_{n-2}=D'_{n-2}-2,\ D_{n-1}=3,\ D_n=3.$$

It is easy to see that the new distribution D is $(3t+r)$-fold solvable on P_n with $|D|\leqslant|D'|+2t+r$. Therefore this lemma holds. □

Lemma 1.5 $f'_{3t+r}(P_n)\geqslant f'_{3t+r}(P_{n-2})+2t+r$ for $t\geqslant 1$ and $r=0,1$.

Proof Assume that D is a $(3t+r)$-fold solvable distribution with $f'_{3t+r}(P_n)$ pebbles on P_n which was provided by Corollary 1.3. Then we let D' be a new distribution such that $D'_i=D_i$ for $i<n-2$ and $D'_{n-2}=D_{n-2}+D_{n-1}+D_n-2t-r$.

First we note that

$$D_{n-1}+D_n\geqslant 2t+r \tag{1}$$

Second we show that

$$W(D', u_{n-2}) \geqslant W(D, u_{n-2}) \quad (2)$$

It is sufficient to show that

$$D_{n-1} + D_n - 2t - r \geqslant \lfloor (D_{n-1} + \lfloor D_n/2 \rfloor)/2 \rfloor \quad (3)$$

Let $a = D_n$, $b = D_{n-1}$. From Corollary 1.3 and its proof, it follows that $a \geqslant 2t$ and there are at most $t+r$ pebbles that can be moved from $P_n \backslash \{u_{n-1}, u_n\}$ to u_{n-1}, and $W(D, u_{n-1}) \geqslant 3t+r$, $W(D, u_n) \geqslant 3t+r$, we have

$$\begin{cases} b + \lfloor a/2 \rfloor + t + r \geqslant 3t + r \\ \quad \Rightarrow b + a/2 \geqslant 2t, \\ \lfloor (b + t + r)/2 \rfloor + a \geqslant 3t + r \\ \quad \Rightarrow b/2 + a \geqslant 5t/2 + r/2. \end{cases}$$

Let $g = a + b - 2t - r - \lfloor (b + \lfloor a/2 \rfloor)/2 \rfloor$ (similarly let $g_i = a_i + b_i - 2t - r - \lfloor (b_i + \lfloor a_i/2 \rfloor)/2 \rfloor$). Then $g = 3a/4 + b/2 - 2t - r + \delta$ for some $\delta \in \{0, 1/4, 1/2, 3/4\}$ and min g can be reached in a small neighbourhood of

$$\{(a,b) \mid b + a/2 = 2t,\ b/2 + a = 5t/2 + r/2\} = (2t + 2r/3, t - r/3).$$

If $a = 2t$, $b = t + r$, then $g \geqslant 0$.

If $a > 2t$, then min g can be reached along the line $b + a/2 = 2t$. Note that if $a_1 = a_2 + 2$, $b_1 = b_2 - 1$, then $g_1 > g_2$. So we only need to consider $a = 2t+1$, $b = t$ and $a = 2t+2$, $b = t-1$. In both cases, $g \geqslant 0$.

Therefore, D' is $(3t+r)$-fold solvable on P_{n-2}. □

Similarly, we have the following lemma.

Lemma 1.6 $f'_{3t+2}(P_n) \geqslant f'_{3t+2}(P_{n-2}) + 2t + 2$ for $t \geqslant 1$.

Proof Assume that D is a $(3t+2)$-fold solvable distribution with $f'_{3t+2}(P_n)$ pebbles on P_n.

Let $D'_i = D_i$ for $i < n-2$ and $D'_{n-2} = D_{n-2} + D_{n-1} + D_n - 2t - 2$.

First we show that $D_{n-1} + D_n \geqslant 2t + 2$, we know that $D_n \geqslant 2t+1$, if $D_n \geqslant 2t+2$, then we are done; if $D_n = 2t+1$, then, similar to the proof in Corollary 1.3, at most $t+1$ pebbles that can be moved from $P_n \backslash \{u_{n-1}, u_n\}$ to u_{n-1}. But $W(D, u_{n-1}) \geqslant 3t+2$, so $D_{n-1} \geqslant t+1$.

Second we show that $W(D', u_{n-2}) \geqslant W(D, u_{n-2})$, we only need to show

$$D_{n-1} + D_n - 2t - 2 \geqslant \lfloor (D_{n-1} + \lfloor D_n/2 \rfloor)/2 \rfloor.$$

Let $a = D_n$, $b = D_{n-1}$, from Corollary 1.3, assume that $a \geqslant 2t+1$. For at most $t+1$ pebbles that can be moved from $P_n \backslash \{u_{n-1}, u_n\}$ to u_{n-1}, and $W(D, u_{n-1}) \geqslant 3t+2$, $W(D, u_n) \geqslant 3t+2$, they imply that

$$\begin{cases} b + \lfloor a/2 \rfloor + t + 1 \geqslant 3t + 2 \\ \quad \Rightarrow b + a/2 \geqslant 2t + 1, \\ \lfloor (b + t + 1)/2 \rfloor + a \geqslant 3t + 2 \\ \quad \Rightarrow b/2 + a \geqslant 5t/2 + 3/2. \end{cases}$$

Let $g = a + b - 2t - 2 - \lfloor (b + \lfloor a/2 \rfloor)/2 \rfloor$ (similarly let $g_i = a_i + b_i - 2t - 2 - \lfloor (b_i + \lfloor a_i/2 \rfloor)/2 \rfloor$). Then we should prove that $g \geqslant 0$.

Let $g = 3a/4 + b/2 - 2t - 2 + \delta$ for some $\delta \in \{0, 1/4, 1/2, 3/4\}$. It is not hard to see that min g can be reached in a small neighbourhood of

$$\{(a,b) \mid b + a/2 = 2t+1,\ b/2 + a = 5t/2 + 3/2\} = (2t + 4/3, t + 1/3).$$

If $a = 2t+1$, $b = t+1$, then $g \geqslant 0$.

If $a > 2t+1$, then min g can be reached along the line $b + a/2 = 2t+1$. Note that if $a_1 = a_2 + 2$, $b_1 = b_2 - 1$, then $g_1 > g_2$. So we only need to consider $a = 2t+2$, $b = t$ and $a = 2t+3$, $b = t$. In both cases, $g \geqslant 0$.

So D' is $(3t+2)$-fold solvable on P_{n-2}, and we are done. □

By Lemmas 1.4 ~ 1.6, we can get the following theorem immediately.

Theorem 1.7 $f'_{3t+r}(P_n) = f'_{3t+r}(P_{n-2}) + 2t + r$, for $t \geqslant 1$.

From Theorems 1.7 and 0.3, also note that $f'_{3t+r}(P_1) = 3t + r$, we can get the following theorem.

Theorem 1.8 $f'_{3t}(P_n) = t(n+2)$,

$$f'_{3t+1}(P_n) = t(n+2) + \lfloor n/2 \rfloor + 1,$$

$$f'_{3t+2}(P_n) = t(n+2) + (n+1),$$

for $t \geqslant 1$.

2 The optimal t-pebbling number of cycle

The optimal t-pebbling numbers of C_3 and C_4 were obtained in Ref. [4]. In this section, we will give the optimal t-pebbling number of C_5.

A distribution D is smooth if it has at most two pebbles on every vertex of degree 2.

Theorem 2.1 $f_2'(C_n)=n$ for $n>3$, $f_3'(C_n)=n+2$.

Proof If $f_2'(C_n)\leqslant n-1$, then it is not hard to see that $f'(C_n\times K_2)\leqslant n-1$, a contradiction to Theorem 6.6 in Ref.[1], so $f_2'(C_n)\geqslant n$.

Let D be a distribution on C_n such that $D_i=2$ if i is odd and $D_i=0$ if i is even, except for $D_n=1$ if n is odd. Then D is 2-fold solvable and $|D|=n$, so $f_2'(C_n)\leqslant n$.

Let D be a distribution on C_n so that $D_1=D_2=2$ and $D_i=1$ otherwise. Then D is 3-fold solvable and $|D|=n+2$, so $f_3'(C_n)\leqslant n+2$.

Let D' be a distribution on C_n with $f_3'(C_n)$ pebbles such that it has at most two pebbles on every vertex. If all the vertices of C_n are occupied and there exists one vertex u_i with $D_i\geqslant 3$, then from the upper bound $n+2$ it follows that $D_j=1$ for every $j\neq i$, and $D_i=3$. It is easy to see that it is not a 3-fold solvable distribution. If $D_i=0$ for some i, then from Lemma 4.4 in Ref. [1] it follows that at most two pebbles can be moved to u_i, a contradiction. Thus $D_i\geqslant 1$ for $1\leqslant i\leqslant n$, but no distribution with $n+1$ pebbles can be 3-fold solvable and hence $f_3'(C_n)\geqslant n+2$. □

Definition 2.2 Assume u is the target vertex, a pebbling move from v to w is greedy if $d(w,u)<d(v,u)$.

Theorem 2.3 $f'_t(C_{2n})\geqslant \dfrac{2^{n+1}nt}{3\cdot 2^n-3}$, the equality holds if and only if $(3\cdot 2^n-3)\mid t$; $f_t'(C_{2n+1})\geqslant\dfrac{2^{n-1}(2n+1)t}{3\cdot 2^{n-1}-1}$, the equality holds if and only if $(3\cdot 2^n-2)\mid t$.

Proof We only prove that for even cycle, the case for the odd cycle can be similarly proved.

Let D be a t-fold solvable distribution with $f_t'(C_{2n})$ pebbles on $C_{2n}=u_1u_2\cdots u_{2n}u_1$. For simplicity, let $a_i=D_i$ for $1\leqslant i\leqslant 2n$ ($a_{2n+i}=a_i$), $\tilde{a}_i$ be the number of pebbles on u_i after some pebbling moves.

Then we have

$$t\leqslant W(D,u_i)\leqslant a_i+(a_{i+1}/2+a_{i-1}/2)+(a_{i+2}/4+a_{i-2}/4)+\cdots+(a_{i+n-1}/2^{n-1}+a_{i-n+1}/2^{n-1})+a_{i+n}/2^n.$$

Adding these $2n$ inequalities, we can get the inequality.

If $t=(3\cdot 2^n-3)m$ for some integer m, then we put 2^nm pebbles on each vertex, which is a t-fold solvable distribution, so the equality holds.

Conversely, if the equality holds, then for $1\leqslant i\leqslant 2n$,

$$W(D,u_i)=a_i+(a_{i+1}/2+a_{i-1}/2)+(a_{i+2}/4+a_{i-2}/4)+\cdots+(a_{i+n-1}/2^{n-1}+a_{i-n+1}/2^{n-1})+a_{i+n}/2^n=t.$$

This means:

① The pebbling moves must be greedy.

② In the sequence of pebbling moves, we can not lose any one pebble. In other words, if t pebbles have been moved to u_i, then the number of pebbles left on any other vertex must be 0.

First we prove that a_i is a constant for all $1\leqslant i\leqslant 2n$.

Let $d=\min\{a_j\mid 1\leqslant j\leqslant 2n\}$. Without loss of generality, we assume that $a_1=d$. Let $\alpha=a_2+a_3/2+\cdots+a_n/2^{n-2}$ and $\beta=a_{2n}+a_{2n-1}/2+\cdots+a_{n+2}/2^{n-2}$. Then from $W(D,u_{2n})=W(D,u_2)$, $W(D,u_{2n})=W(D,u_1)$, we can get

$$\alpha+\beta/4+a_{n+1}/2^{n-1}+d/2=\beta+\alpha/4+a_{n+1}/2^{n-1}+d/2,$$

$$d+\beta/2+\alpha/2+a_{n+1}/2^n=\beta+\alpha/4+a_{n+1}/2^{n-1}+d/2.$$

So $\alpha=\beta=2d-a_{n+1}/2^{n-2}\leqslant 2d-\frac{1}{2^{n-2}}d$. From

$$\alpha=a_2+a_3/2+\cdots+a_n/2^{n-2},$$

it follows that

$$\alpha\geqslant d+\left(1-\frac{1}{2^{n-2}}\right)d=2d-\frac{1}{2^{n-2}}d,$$

where equality holds if and only if $a_2=a_3=\cdots=a_n=d$. Again from

$$2d-\frac{1}{2^{n-2}}d\leqslant\alpha=2d-a_{n+1}/2^{n-2}\leqslant 2d-\frac{1}{2^{n-2}}d$$

it follows that $a_{n+1}=d$. In the same way, we know that $a_{n+2}=a_{n+3}=\cdots=a_{2n-1}=a_{2n}=d$.

Now we prove that $2^n | d$.

If $a_i = d$ is odd and u_{i+n+1} is the target vertex, there is at least one pebble on u_i which can not be moved, so $2 | n$.

If $a_i = d = 4k + 2$ and u_{i+n+1} is the target vertex, after all the pebbles are removed from u_i, one of $\tilde{a}_{i+1}$, $\tilde{a}_{i-1}$ must be odd. So $4 | d$.

If $2^j | d$ for some $j < n$, but then $2^{j+1} \nmid d$, namely, there is some integer k such that $a_i = 2^{j+1}k + 2^j$. Let the target vertex be u_{i+n+1}, then when we move all pebbles off u_s for $i-j < s < i+j$, then one of $\tilde{a}_{i+j}$, $\tilde{a}_{i-j}$ must be odd, a contradiction. So $2^{j+1} | d$.

From the above argument, it follows that $2^n | d$. Assume that $d = 2^n m$ for some integer m, then $t = (3 \cdot 2^n - 3)m$, and hence $(3 \cdot 2^n - 3) | t$. □

Now, we give the optimal t-pebbling number of C_5.

From Theorem 2.3, we have the following corollary.

Corollary 2.4 $f'_t(C_5) \geqslant 2t + 1$ if $10 \nmid t$, $f'_t(C_5) = 2t$ if $10 | t$.

Let $C_5 = u_1 u_2 u_3 u_4 u_5$, $a_i = D(u_i)$. First we give the optimal t-pebbling numbers of C_5 for $2 \leqslant t \leqslant 11$, which were obtained by the direct calculation.

Lemma 2.5 For $2 \leqslant t \leqslant 11$, the optimal t-pebbling number of C_5 is given in Tab. 1.

Theorem 2.6 The optimal t-pebbling number of C_5 is

$$f'_t(C_5) = \begin{cases} 4, & \text{if } t = 1; \\ 2t, & \text{if } 10 \mid t; \\ 2t+1, & \text{otherwise}. \end{cases}$$

Proof For $t \leqslant 10$, it follows from Tab. 1.

For $t > 10$, assume that $t = 10n + r$, where $0 \leqslant r \leqslant 9$.

If $r = 0$, it follows from Corollary 2.4.

If $r = 1$, then from Lemma 0.5 and Corollary 2.4, we have

$$\begin{aligned} 2t + 1 \leqslant\ & f'_t(C_5) \leqslant \\ & (n-1)f'_{10}(C_5) + f'_{11}(C_5) = \\ & 20(n-1) + 23 = 2t + 1. \end{aligned}$$

So $f'_t(C_5) = 2t + 1$ if $t = 10n + 1$, where $n \geqslant 1$.

If $r \neq 1$, then from Lemma 0.5 and Corollary 2.4,

$$\begin{aligned} 2t + 1 \leqslant\ & f'_t(C_5) \leqslant \\ & nf'_{10}(C_5) + f'_r(C_5) = \\ & 20n + 2r + 1 = 2t + 1. \end{aligned}$$

So $f'_t(C_5) = 2t + 1$ if $t = 10n + r$, where $r \neq 0, 1$. □

3 Optimal pebbling on product of paths

In this section, we give the optimal t-pebbling number of $P_2 \times P_3$. Let $D = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix}$ be a t-fold solvable distribution on

$$P_2 \times P_3 = \begin{pmatrix} u_1 & u_2 & u_3 \\ u_4 & u_5 & u_6 \end{pmatrix}.$$

First we give a lower bound of $f'_t(P_2 \times P_3)$.

Lemma 3.1 $f'_t(P_2 \times P_3) \geqslant \lceil 20t/9 \rceil$, equality holds if $t = 0 \pmod 9$.

Proof Let $D = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix}$ be a distribution on $P_2 \times P_3$ with $f'_t(P_2 \times P_3)$ pebbles. Then we can get $W(1) \doteq a_1 + a_2/2 + a_3/4 + a_4/2 + a_5/4 + a_6/8 \geqslant t$. For the other 5 vertices, we can get similar inequalities according to pebbling moves.

Since $W(1) + W(3) + W(4) + W(6) \geqslant 4t$,

$$\frac{15}{8}(a_1 + a_3 + a_4 + a_6) + \frac{3}{2}(a_2 + a_5) \geqslant 4t \quad (4)$$

Since $W(2) + W(5) \geqslant 2t$,

$$\frac{3}{4}(a_1 + a_3 + a_4 + a_6) + \frac{3}{2}(a_2 + a_5) \geqslant 2t \quad (5)$$

$(4) \times 2 + (5)$, we can get

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 \geqslant 20t/9,$$

Tab. 1 $f'_t(C_5)$ **for** $2 \leqslant t \leqslant 11$

t	2	3	4	5	6	7	8	9	10	11
$f'_t(C_5)$	5	7	9	11	13	15	17	19	20	23
a_1, a_2, a_3	2,0,2	2,2,1	1,3,1	2,2,2	3,3,2	3,3,3	3,4,3	4,4,4	4,4,4	5,4,5
a_4, a_5	0,1	1,1	2,2	2,3	3,2	3,3	3,4	4,3	4,4	4,5

Tab. 2 $f'_t(P_2\times P_3)$ **for** $1\leqslant t\leqslant 9$

t	1	2	3	4	5	6	7	8	9
$\lceil 20t/9\rceil$	3	5	7	9	12	14	16	18	20
$f'_t(P_2\times P_3)$	3	6	8	10	12	14	16	18	20
a_1,a_2,a_3	0,2,0	1,2,1	1,2,1	2,2,2	2,2,2	2,3,2	3,2,3	3,2,3	4,2,4
a_4,a_5,a_6	0,1,0	0,2,0	1,2,1	1,2,1	2,2,2	2,3,2	3,2,3	4,2,4	4,2,4

so $f'_t(P_2\times P_3)\geqslant 20t/9$.

If $t\equiv 0 \pmod 9$, then assume $t=9m$ for some integer m. Let $a_1=a_3=a_4=a_6=4m$, $a_2=a_4=2m$, which is a t-fold solvable distribution with $20m$ pebbles, so the equality holds. □

From direct computation, we can get

Lemma 3.2 $f'_t(P_2\times P_3)=2t+2$ for $2\leqslant t\leqslant 9$.

Proof We use Tab. 2, where the last row is a t-fold solvable distribution D with $f'_t(P_2\times P_3)$ pebbles on $P_2\times P_3$. □

Theorem 3.3 $f'_{9t}(P_2\times P_3)=20t$,

$$f'_{9t+1}(P_2\times P_3)=20t+3,$$

and

$$20t+2r+1\leqslant f'_{9t+r}(P_2\times P_3)\leqslant 20t+2r+2,\ \text{for } 2\leqslant r\leqslant 8,$$

the last equality holds for $r=5,6,7,8$.

Proof From Lemma 3.1, we have

$$f'_{9t+r}(P_2\times P_3)\geqslant 20t+\lceil 20r/9\rceil.$$

By Lemma 0.5, we know that

$$f'_{9t+r}(P_2\times P_3)\leqslant tf'_9(P_2\times P_3)+f'_r(P_2\times P_3)=20t+f'_r(P_2\times P_3).$$

By Tab. 2, we are done. □

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