

Hamilton paths and cycles in fault-tolerant varietal hypercubes

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Abstract: The varietal hypercube VQ_n , a variant of the hypercube Q_n , was studied. It was proved that VQ_n contains a fault-free Hamilton cycle provided faulty edges do not exceed $n-2$, and that for two distinct vertices, x and y , there is a fault-free xy -Hamilton path in VQ_n provided faulty edges do not exceed $n-3$ for $n \geq 3$. The proof is based on an inductive construction.

Key words: graphs; Hamilton path; Hamilton cycle; varietal hypercube; fault-tolerant networks

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容错变形超立方体的圈和路

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摘要: 考虑包含故障边的 $n(n \geq 3)$ 维变形超立方体 VQ_n , 证明了: 如果故障边数不超过 $n-2$, 那么 VQ_n 包含非故障边的 Hamilton 圈; 如果故障边数不超过 $n-3$, 那么对任何两个不同顶点 x 和 y , VQ_n 包含非故障边的 xy -Hamilton 路. 该证明方法采用归纳法.

关键词: 图论; Hamilton 圈; Hamilton 路; 变形超立方体; 容错网络

0 Introduction

As a topology of interconnection networks, the hypercube Q_n is the simplest and most popular since it has many superior properties. The varietal hypercube, VQ_n , which is a variant of Q_n and was proposed in Ref. [1], has many properties similar

or superior to Q_n . For example, the connectivity and restricted connectivity of VQ_n and Q_n are the same^[2], while, all the diameter and the average distance, fault-diameter and wide-diameter of VQ_n are smaller than that of the hypercube^[1,3]. Recently, Ref. [4] has shown that VQ_n is vertex-transitive.

Embedding paths and cycles in various well-

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known networks, such as Q_n , have been extensively investigated in Ref. [5]. Recently, Cao et al. [6] have shown that every edge of VQ_n is contained in cycles of every length from 4 to 2^n except 5, and every pair of vertices with distance d is connected by paths of every length from d to $2^n - 1$ except 2 and 4 if $d = 1$. In this paper, we consider fault-tolerant varietal hypercubes and show that VQ_n contains a fault-free Hamilton cycle provided faulty edges do not exceed $n - 2$ for $n \geq 3$ and for two distinct vertices, x and y , there is a fault-free xy -Hamilton path in VQ_n provided faulty edges do not exceed $n - 3$ for $n \geq 3$.

The proofs of these results are in Section 2. The definition and some basic properties of VQ_n are given in Section 1.

1 Definitions and lemmas

We follow Ref. [7] for graph-theoretical terminology and notation not defined here. A graph $G = (V, E)$ always means a simple and connected graph, where $V = V(G)$ is the vertex-set and $E = E(G)$ is the edge-set of G . For $xy \in E(G)$, we call x (resp. y) a neighbor of y (resp. x).

The n -dimensional varietal hypercube VQ_n is the labeled graph defined recursively as follows. VQ_1 is the complete graph of two vertices labeled 0 and 1, respectively. Assume that VQ_{n-1} has been constructed. Let VQ_{n-1}^0 (resp. VQ_{n-1}^1) be a labeled graph obtained from VQ_{n-1} by inserting a zero (resp. 1) in front of each vertex-labeling in VQ_{n-1} . For $n > 1$, VQ_n is obtained by joining vertices in VQ_{n-1}^0 and VQ_{n-1}^1 , according to the rule: a vertex $x = 0x_{n-1}x_{n-2}x_{n-3} \cdots x_2x_1$ in VQ_{n-1}^0 and a vertex $y = 1y_{n-1}y_{n-2}y_{n-3} \cdots y_2y_1$ in VQ_{n-1}^1 are adjacent in VQ_n if and only if

① $x_{n-1}x_{n-2}x_{n-3} \cdots x_2x_1 = y_{n-1}y_{n-2}y_{n-3} \cdots y_2y_1$ if $n \neq 3k$, or

② $x_{n-3} \cdots x_2x_1 = y_{n-3} \cdots y_2y_1$ and $(x_{n-1}x_{n-2}, y_{n-1}y_{n-2}) \in I$ if $n = 3k$, where $I = \{(00, 00), (01, 01), (10, 11), (11, 10)\}$.

Fig. 1 shows the examples of varietal hypercubes VQ_n for $n = 1, 2, 3$ and 4.

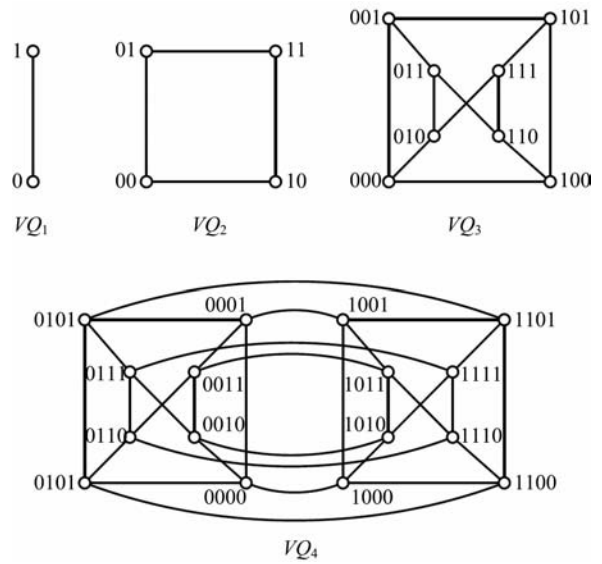


Fig. 1 The varietal hypercubes VQ_1 , VQ_2 , VQ_3 and VQ_4

An edge xy in VQ_n , where $x = x_nx_{n-1} \cdots x_2x_1$ and $y = y_ny_{n-1} \cdots y_2y_1$, is called the i -transversal edge if $x_n \cdots x_{i-1} = y_n \cdots y_{i-1}$ and $x_i \neq y_i$. For convenience, we express VQ_n as $VQ_n^0 \odot VQ_n^1$, where $VQ_n^0 \cong VQ_n^1 \cong VQ_{n-1}$. Then edges between VQ_n^0 and VQ_n^1 are n -transversal edges. The edges of Type ② are referred to as crossing edges when

$$(x_{n-1}x_{n-2}, y_{n-1}y_{n-2}) \in \{(10, 11), (11, 10)\}.$$

All the other edges are referred to as normal edges.

Let $VQ_n = L \odot R$, where $L = VQ_{n-1}^0$ and $R = VQ_{n-1}^1$, and denote by x_Lx_R the n -transversal edge joining $x_L \in L$ and $x_R \in R$. The recursive structure of VQ_n gives the following simple properties.

Lemma 1.1 Let $VQ_n = L \odot R$ with $n \geq 1$. Then VQ_n contains no triangles and every vertex $x_L \in L$ has exactly one neighbor x_R in R . Moreover, $x_Ly_L \in E(L)$ if and only if $x_Ry_R \in E(R)$ for $n \neq 3k$, where x_R and y_R are the neighbors of x_L and y_L in R .

Lemma 1.2 Let $VQ_n = L \odot R$ and xy be an n -transversal edge in VQ_n with $x \in L$ and $y \in R$. For $n \geq 3$, let $x = 0ab\beta$, where $\beta = x_{n-3} \cdots x_1$. Then $y = 1a'b'\beta$, where $ab = a'b'$ if xy is a normal edge, and $(ab, a'b') = (1b, 1\bar{b})$ if xy is a crossing edge, where $\bar{b} = \{0, 1\} \setminus b$.

Let VQ_{n-2}^{ab} be a labeled graph obtained from VQ_{n-2} by inserting ab in front of each vertex-

labeling in VQ_{n-2} , where $a, b \in \{0, 1\}$. Clearly, $VQ_{n-2}^{ab} \cong VQ_{n-2}$. By definition, $VQ_{n-1}^0 = VQ_{n-2}^{00} \odot VQ_{n-2}^{01}$ and $VQ_{n-1}^1 = VQ_{n-2}^{10} \odot VQ_{n-2}^{11}$. By Lemma 1.2, it is easy to see that when VQ_n is expressed as $(VQ_{n-2}^{00} \odot VQ_{n-2}^{01}) \odot (VQ_{n-2}^{10} \odot VQ_{n-2}^{11})$, VQ_n is of the recursive structure shown as Fig. 2.

Let x and y be two distinct vertices in a graph G . An xy -path is a sequence of adjacent vertices, written as $(x_0, x_1, x_2, \dots, x_m)$, in which $x = x_0$, $y = x_m$ and all the vertices $x_0, x_1, x_2, \dots, x_m$ are different from each other. For a path

$$P = (x_0, \dots, x_i, x_{i+1}, \dots, x_m),$$

we can write $P = P(x_0, x_i) + x_i x_{i+1} + P(x_{i+1}, x_m)$, and the notation $P - x_i x_{i+1}$ denotes the subgraph obtained from P by deleting the edge $x_i x_{i+1}$. An xy -path P is called a cycle if $x = y$; a cycle is called a Hamilton cycle if it contains all vertices in G . An xy -path P is called an xy -Hamilton path if it contains all vertices in G . A graph G is Hamiltonian if it contains a Hamilton cycle, and is called Hamilton-connected if it contains an xy -Hamilton path for any two vertices x and y in G . Clearly, if G is Hamilton-connected, then it certainly is Hamiltonian.

Lemma 1.3^[6] For $n \geq 2$, every edge of VQ_n is contained in a Hamilton cycle. For $n \geq 3$, VQ_n is Hamilton-connected.

Faults of some processors and/or communication lines in a large-scale system are inevitable. However, the presence of faults gives rise to a large number of problems that have to be

considered for some applications. Ref. [2] showed that VQ_n is n -connected. This fact implies that for any set of faults $F \subseteq E(VQ_n)$ with $|F| < n$, the remainder network $VQ_n - F$ is still connected. However, one does not know whether $VQ_n - F$ still remains Hamilton-connected or not.

Let $F \subseteq E(G)$ be a set of edge-faults of G . A subgraph H of G is called fault-free if H contains no edges in F , and G is called t -edge-fault-tolerant Hamiltonian (resp. t -edge-fault-free Hamiltonian) if $G - F$ contains a Hamilton cycle (resp. is Hamilton-connected) for any $F \subseteq E(G)$ with $|F| \leq t$.

The n -dimensional crossed cube CQ_n is such a graph: its vertex-set is the same as VQ_n , two vertices $x = x_n \dots x_2 x_1$ and $y = y_n \dots y_2 y_1$ are linked by an edge if and only if there exists some j ($1 \leq j \leq n$) such that ① $x_n \dots x_{j+1} = y_n \dots y_{j+1}$, ② $x_j \neq y_j$, ③ $x_{j-1} = y_{j-1}$ if j is even, and ④ $(x_{2i} x_{2i-1}, y_{2i} y_{2i-1}) \in I$ for each $i = 1, 2, \dots, \lceil \frac{1}{2}j \rceil - 1$, where

$$I = \{(00, 00), (01, 01), (10, 11), (11, 10)\}.$$

By definition, $VQ_n \cong CQ_n$ for each $n = 1, 2, 3$. The following results on CQ_n are used in the proofs of our main results for $n = 3$.

Lemma 1.4^[8-9] CQ_n is $(n - 2)$ -edge-fault-tolerant Hamiltonian for $n \geq 3$.

Lemma 1.5^[10] If each vertex is incident to at least two fault-free edges, then CQ_n is $(2n - 5)$ -edge-fault-tolerant Hamiltonian.

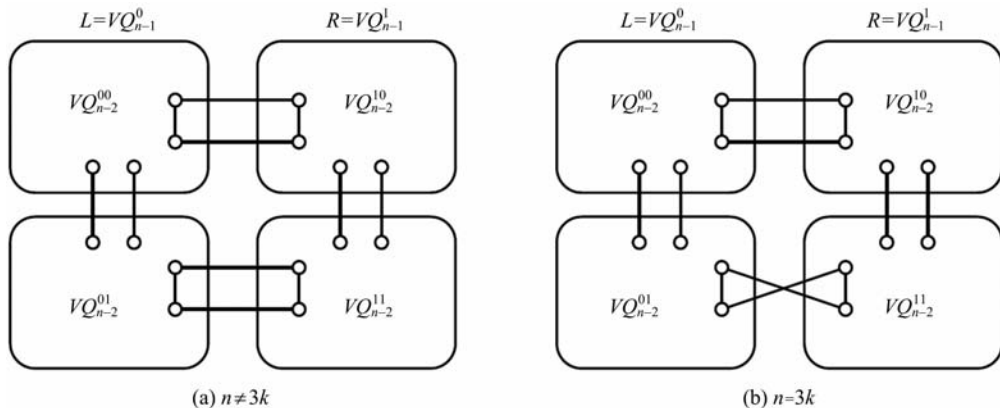


Fig. 2 The recursive structure of VQ_n

2 Main results

Theorem 2.1 VQ_n is $(n-3)$ -edge-faulty-tolerant Hamilton-connected for $n \geq 3$.

Proof We proceed by induction on $n \geq 3$.

By Lemma 1.3, the conclusion is true for $n=3$. Suppose now that $n \geq 4$ and the result holds for any integer less than n . Let $F \subset E(VQ_n)$ with $|F| \leq n-3$, x and y be two distinct vertices in VQ_n . We need to prove that $VQ_n - F$ contains an xy -Hamilton path. By Lemma 1.3, we can assume $|F| \geq 1$. Let $VQ_n = L \odot R$, and let

$$L = VQ_{n-2}^{00} \odot VQ_{n-2}^{01}, R = VQ_{n-2}^{10} \odot VQ_{n-2}^{11},$$

$$F_L = F \cap E(L), F_R = F \cap E(R),$$

$$F_n = F \setminus (F_L \cup F_R).$$

By symmetry of structure of VQ_n , we can assume $|F_L| \geq |F_R|$.

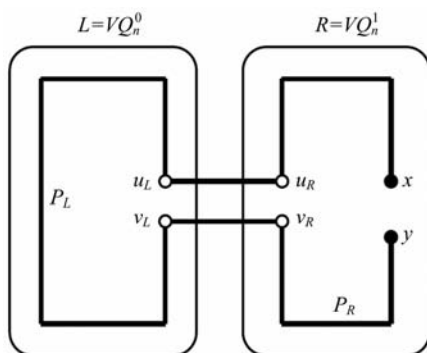
Case 1 $|F_L| \leq n-4$.

By the assumption, $|F_R| \leq n-4$ and $n \geq 5$.

Subcase 1.1 $x, y \in L$ or $x, y \in R$. Without loss of generality, assume $x, y \in R$.

Since $|F_R| \leq n-4 = (n-1)-3$, by the induction hypothesis $R - F_R$ contains an xy -Hamilton path, say P_R . Since $\varepsilon(P_R) = 2^{n-1}-1 > 2(n-3) \geq 2|F|$, there is an edge $u_R v_R$ in P_R such that the edges $u_R u_L$ and $v_R v_L$ are not in F , where u_L and v_L are neighbors of u_R and v_R in L . Since $|F_L| \leq n-4 = (n-1)-3$, by the induction hypothesis $L - F_L$ contains a $u_L v_L$ -Hamilton path, say P_L . Thus, $P_R - u_R v_R + u_R u_L + v_R v_L + P_L$ is an xy -Hamilton path in $VQ_n - F$ (see Fig. 3(a)).

Subcase 1.2 $x \in L$ and $y \in R$.



(a) $x, y \in R$

Since there are 2^{n-1} edges between L and R and $2^{n-1}-2 > n-3 \geq |F|$, there is an edge $u_L u_R \notin F_n$ such that $u_L \neq q_x$ and $u_R \neq q_y$. By the induction hypothesis, let P_L be an $x u_L$ -Hamilton path in $L - F_L$, and P_R be a $y u_R$ -Hamilton path in $R - F_R$. Then $P_L + u_L u_R + P_R$ is an xy -Hamilton path in $VQ_n - F$ (see Fig. 3a(b)).

Case 2 $|F_L| = n-3$.

In this case, $|F_R| = |F_n| = 0$. Let

$$F_{00} = F_L \cap E(VQ_{n-2}^{00}), F_{01} = F_L \cap E(VQ_{n-2}^{01}).$$

Without loss of generality, we can assume $F_{00} \neq \emptyset$.

Subcase 2.1 $x, y \in L$.

Arbitrarily take $e = u_L v_L \in F_L$. Since $|F_L - e| = n-4 = (n-1)-3$, by the induction hypothesis $L - (F_L - e)$ contains an xy -Hamilton path, say P_L . Without loss of generality, assume $e \in E(P_L)$. Let u_R and v_R be neighbors of u_L and v_L in R , respectively. By Lemma 1.3, R contains a $u_R v_R$ -Hamilton path, say P_R . Then $P_L - u_L v_L + u_L u_R + v_L v_R + P_R$ is an xy -Hamilton path in $VQ_n - F$.

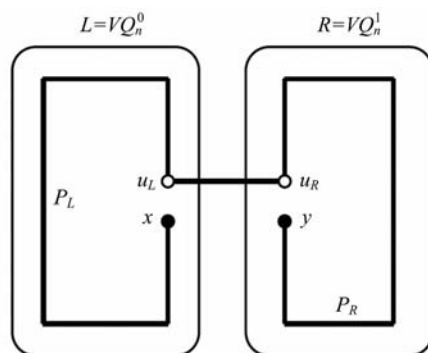
Subcase 2.2 $x \in L$ and $y \in R$.

If $n=4$, then $L \cong R \cong VQ_3 \cong CQ_3$. Since $|F_L|=1$, by Lemma 1.5 $L - F_L$ contains a Hamilton cycle, say C_L . Choose a neighbor u_L of x in C_L such that its neighbor u_R in R is not y . By Lemma 1.3, R contains a $y u_R$ -Hamilton path, say P_R . Then, $C_L - x u_L + u_L u_R + P_R$ is an xy -Hamilton path in $VQ_4 - F$.

Assume now $n \geq 5$, that is, $n-2 \geq 3$.

(a) $y \in VQ_{n-2}^{11}$ (see Fig. 4(a)).

Arbitrarily take $z_{11} \in VQ_{n-2}^{11}$ with $z_{11} \neq q_y$, and let z_{01} be the neighbor of z_{11} in L . By Lemma 1.3,



(b) $x \in L$ and $y \in R$

Fig. 3 Illustrations of Case 1 in the proof of Theorem 2.1

VQ_{n-2}^{11} contains a $z_{11}y$ -Hamilton path, say P_{11} . Arbitrarily take $e \in F_{00}$. Since $n \geq 5$, by the induction hypothesis $L - (F_L - e)$ contains an xz_{01} -Hamilton path, say P_L . Without loss of generality, assume $e = u_{00}v_{00}$ in P_L . Let u_{10} and v_{10} be neighbors of u_{00} and v_{00} in VQ_{n-2}^{10} , respectively. By Lemma 1.3, VQ_{n-2}^{10} contains a $u_{10}v_{10}$ -Hamilton path, say P_{10} . Then $P_{10} + u_{00}u_{10} + v_{00}v_{10} + P_L - u_{00}v_{00} + z_{01}z_{11} + P_{11}$ is an xy -Hamilton path in $VQ_n - F$ (see Fig. 4(a)).

(b) $y \in VQ_{n-2}^{10}$ (see Fig. 4(b)).

Arbitrarily take a vertex z_{01} in VQ_{n-2}^{01} with $z_{01} \neq x$. Let z_{11} be the neighbor of z_{01} in VQ_{n-2}^{11} , $e = u_{00}v_{00} \in F_{00}$ such that their neighbors u_{10} and v_{10} in VQ_{n-2}^{10} do not contain y . Since $n \geq 5$, by the induction hypothesis $L - (F_L - e)$ contains an xz_{01} -Hamilton path, say P_L . Without loss of generality, assume that e is in P_L . By Lemma 1.3, VQ_{n-2}^{10} contains a $u_{10}v_{10}$ -Hamilton path, say P_{10} . Since $y \in P_{10}$, we can write $P_{10} = P_{10}(v_{10}, y) + yw_{10} + P_{10}(w_{10}, u_{10})$. Let w_{11} be the neighbor of w_{10} in VQ_{n-2}^{11} . By Lemma 1.3, VQ_{n-2}^{11} contains a $z_{11}w_{11}$ -Hamilton path, say P_{11} . Then $P_L - u_{00}v_{00} + u_{00}u_{10} + v_{00}v_{10} + P_{10} - yw_{10} + w_{10}w_{11} + P_{11} + z_{01}z_{11}$ is an xy -Hamilton path in $VQ_n - F$ (see Fig. 4(b)).

Subcase 2.3 $x, y \in R$.

If $n=4$, then $L \cong R \cong VQ_3 \cong CQ_3$. By Lemma 1.3, R contains an xy -Hamilton path, say P_R . Since $L \cong CQ_3$ and $|F_L|=1$, by Lemma 1.5 $L - F_L$ contains a Hamilton cycle, say C_L . Since L and R are 3-regular and isomorphic, there is an edge u_Rv_R in P_R which is not incident with x and y such that

the corresponding edge e_L in L is contained in C_L . Since $n=4$, by Lemma 1.1 $e = u_Lv_L$, where u_L and v_L be neighbors of u_R and v_R in L , respectively. Thus, $P_R - u_Rv_R + u_Lu_R + v_Lv_R + C_L$ is an xy -Hamilton path in $VQ_4 - F$ (as a reference, see Fig. 3(a)).

Assume $n \geq 5$ below, that is, $n-2 \geq 3$.

(a) $x, y \in VQ_{n-1}^{11}$ (see Fig. 5(a)).

By Lemma 1.3, VQ_{n-2}^{11} contains an xy -Hamilton path, say P_{11} . Take $u_{11}v_{11} \in E(P_{11})$, and let u_{01} and v_{01} be neighbors of u_{11} and v_{11} in VQ_{n-2}^{01} , respectively. Take $e = w_{00}z_{00} \in F_{00}$. By the induction hypothesis, $L - (F_L - e)$ contains a $u_{01}v_{01}$ -Hamilton path, say P_L . Without loss of generality, assume that e is in P_L , and let w_{10} and z_{10} be neighbors of w_{00} and z_{00} in VQ_{n-2}^{10} , respectively. By Lemma 1.3, VQ_{n-2}^{10} contains a $w_{10}z_{10}$ -Hamilton path, say P_{10} . Thus, $P_{10} + w_{00}w_{10} + z_{00}z_{10} + P_L - w_{00}z_{00} + P_{11} - u_{11}v_{11} + u_{01}u_{11} + v_{01}v_{11}$ is an xy -Hamilton path in $VQ_n - F$ (see Fig. 5(a)).

(b) $x \in VQ_{n-1}^{11}$ and $y \in VQ_{n-2}^{10}$ (see Fig. 5(b)).

Choose $e = u_{00}v_{00} \in F_{00}$ such that their neighbors u_{10} and v_{10} in VQ_{n-2}^{10} do not contain y . By the induction hypothesis, $L - (F_L - e)$ contains a $u_{00}v_{00}$ -Hamilton path, say P_L . Without loss of generality, assume e in P_L . By Lemma 1.3, VQ_{n-2}^{10} contains a $u_{10}v_{10}$ -Hamilton path, say P_{10} . Since $y \in P_{10}$, we can write

$$P_{10} = P_{10}(v_{10}, y) + yw_{10} + P_{10}(w_{10}, u_{10}).$$

Let w_{11} be the neighbor of w_{10} in VQ_{n-2}^{11} . By Lemma 1.3, VQ_{n-2}^{11} contains an xw_{11} -Hamilton

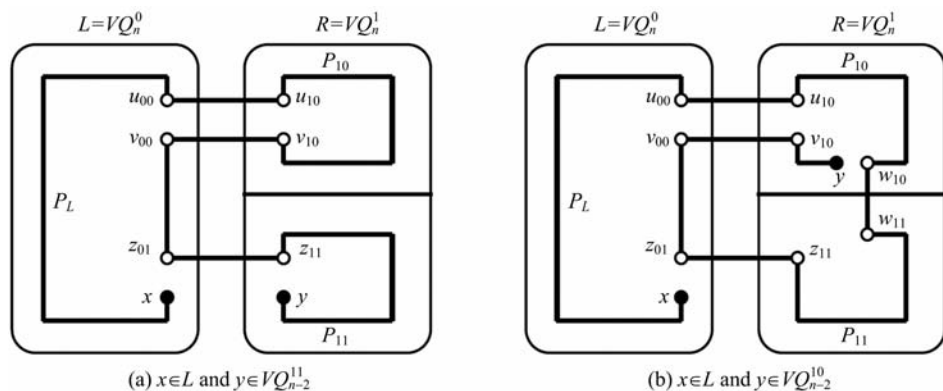


Fig. 4 Illustrations of Subcase 2.2 in the proof of Theorem 2.1

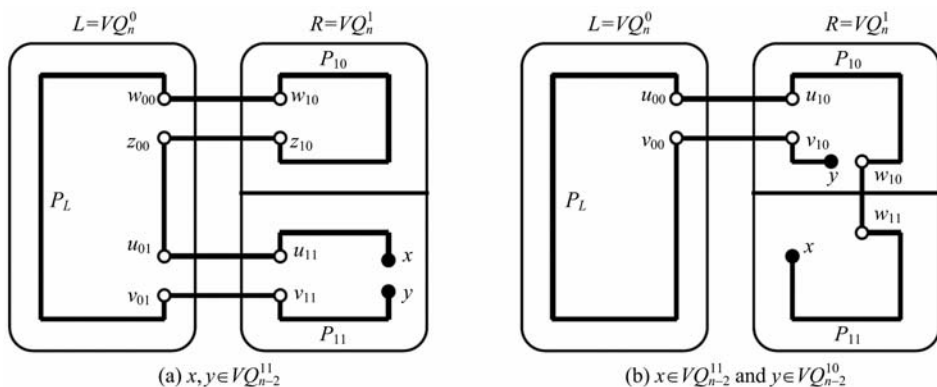


Fig. 5 Illustrations of Subcase 2.3 in the proof of Theorem 2.1

path, say P_{11} . Then $P_L - u_{00}v_{00} + u_{00}u_{10} + v_{00}v_{10} + P_{10} - yw_{10} + w_{10}w_{11} + P_{11}$ is an xy -Hamilton path in $VQ_n - F$ (see Fig. 5(b)).

(c) $x, y \in VQ_{n-2}^{10}$ (see Fig. 6(a)).

By Lemma 1.3, VQ_{n-2}^{10} contains an xy -Hamilton path, say P_{10} .

(c1) $|F_{01}| \neq 0$.

Take $w_{10}z_{10} \in E(P_{10})$, and let w_{00} and z_{00} be neighbors of w_{10} and z_{10} in VQ_{n-2}^{00} , respectively. Take $e = u_{01}v_{01} \in F_{01}$. By the induction hypothesis, $L - (F_L - e)$ contains a $w_{00}z_{00}$ -Hamilton path, say P_L . Without loss of generality, assume that e is in P_L , and let u_{11} and v_{11} be neighbors of u_{01} and v_{01} in VQ_{n-2}^{11} , respectively. By Lemma 1.3, VQ_{n-2}^{11} contains a $u_{11}v_{11}$ -Hamilton path, say P_{11} . Thus, $P_{10} - w_{10}z_{10} + w_{00}w_{10} + z_{00}z_{10} + P_L - u_{01}v_{01} + u_{01}u_{11} + v_{01}v_{11} + P_{11}$ is an xy -Hamilton path in $VQ_n - F$ (see Fig. 6(a)).

(c2) $|F_{01}| = 0$. Then $|F_{00}| = n - 3 \geq 2$ since $n \geq 5$.

Let P'_{00} be an $x_{00}y_{00}$ -Hamilton path in VQ_{n-2}^{00}

that corresponds to P_{10} obtained from P_{10} by changing the left-most coordinate 1 of every vertex into 0, where x_{00} (resp. y_{00}) is a vertex corresponding to x (resp. y). Arbitrarily take an edge $u_{00}v_{00}$ in P'_{00} . Let u_{10} and v_{10} be neighbors of u_{00} and v_{00} in VQ_{n-2}^{10} , respectively. Then $u_{10}v_{10}$ is an edge in P_{10} , $u_{00}u_{10}$ and $v_{00}v_{10}$ are edges in VQ_n (see Fig. 2).

If P'_{00} contains at most one edge in F_{00} . Without loss of generality, take $e_1 = u_{00}v_{00} \in F_{00}$, and let $P_{00} = P'_{00}$.

If P'_{00} contains exactly two edges e_1 and e_2 in F_{00} and $n = 5$. Let $e_1 = u_{00}v_{00}$. By Lemma 1.3, there is a Hamilton cycle C_L in VQ_3^{00} containing the edge e_1 . Without loss of generality, assume that $e_2 = x_{00}y_{00}$ is in C_R , and let $P_{00} = C_L - e_2$.

If P'_{00} contains at least three edges in F_{00} . Then $n - 3 = |F| = |F_{00}| \geq 3$, that is, $n - 2 \geq 4$. Let $e_1 = u_{00}v_{00}$ and $e_2 = x_{00}y_{00}$ be two edges in F_{00} . Since $|F_{00} - e_1 - e_2| = (n - 2) - 3$, by the induction hypothesis $VQ_{n-2}^{00} - (F_{00} - e_1 - e_2)$ contains an

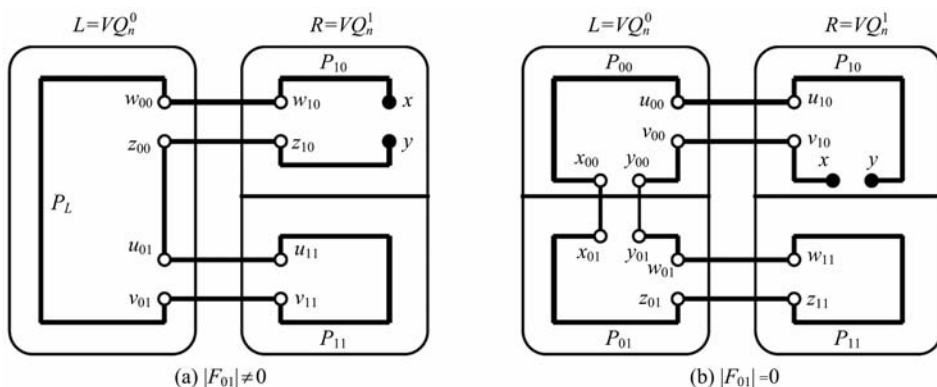


Fig. 6 Other illustrations of Subcase 2.3 in the proof of Theorem 2.1

$x_{00}y_{00}$ -Hamilton path, say P_{00} . Without loss of generality, assume that e_1 is in P_{00} .

Let x_{01} and y_{01} be neighbors of x_{00} and y_{00} in VQ_{n-2}^{01} , respectively. By Lemma 1.3, VQ_{n-2}^{01} contains an $x_{01}y_{01}$ -Hamilton path, say P_{01} . Arbitrarily take an edge $w_{01}z_{01}$ in P_{01} . Let w_{11} and z_{11} be neighbors of w_{01} and z_{01} in VQ_{n-2}^{11} , respectively. By Lemma 1.3, VQ_{n-2}^{11} contains a $w_{11}z_{11}$ -Hamilton path, say P_{11} . Then $P_{10} - u_{10}v_{10} + u_{00}u_{10} + v_{00}v_{10} + P_{00} - u_{00}v_{00} + x_{00}x_{01} + y_{00}y_{01} + P_{01} - w_{01}z_{01} + w_{01}w_{11} + z_{01}z_{11} + P_{11}$ is an xy -Hamilton path in $VQ_n - F$ (see Fig. 6(b)).

The theorem follows. \square

Theorem 2.2 VQ_n is $(n-2)$ -edge-fault-tolerant Hamiltonian for $n \geq 3$.

Proof We proceed by induction on $n \geq 3$.

Since $VQ_3 \cong CQ_3$, by Lemma 1.4, the conclusion is true for $n=3$. Assume the induction hypothesis for $n-1$ with $n \geq 4$. Let $F \subset E(VQ_n)$ with $|F| \geq 1$, $VQ_n = L \odot R$, and let

$$F_L = F \cap L, F_R = F \cap R, F_n = F \setminus (F_L \cup F_R).$$

Without loss of generality, assume $|F_L| \geq |F_R|$.

If $|F_L| = n-2$, then $|F_R| = |F_n| = 0$. For any $e \in F_L$, $|F_L - e| = n-3$. By the induction hypothesis, $L - (F_L - e)$ contains a Hamilton cycle, say C_L . Without loss of generality, assume that $e = u_Lv_L$ is in C_L . Let u_R and v_R be neighbors of u_L and v_L in R , respectively. Then $u_Rv_R \in E(L)$. By Theorem 2.1, R contains a u_Rv_R -Hamilton path, say P_R . Then, $C_L - u_Lv_L + u_Lu_R + v_Lv_R + P_R$ is a Hamilton cycle in $VQ_n - F$.

We now assume $|F_L| \leq n-3$. If $n=4$, then $|F_R| + |F_n| \leq 1$. Since $L \cong R \cong CQ_3$, by Lemma 1.4 both L and R contain fault-free Hamilton cycles, say C_L and C_R , respectively. Since CQ_3 is 3-regular, any two Hamilton cycles have at least one edge in common. Thus, assume $u_Lv_L \in E(C_L)$ and $u_Rv_R \in E(C_R)$. Then $C_L \cup C_R - u_Lv_L - u_Rv_R + u_Lu_R + v_Lv_R$ is a Hamilton cycle in $VQ_4 - F$.

We now assume $n \geq 5$. Since $|F_L| \leq n-3$, $|F_R| \leq n-4$, otherwise $n-2 \geq |F| \geq |F_L| + |F_R| \geq 2n-6$, which contradicts the hypothesis of $n \geq 5$. By the induction hypothesis, $L - F_L$ contains a

Hamilton cycle, say C_L . Choose $e = u_Lv_L \in C_L$ such that it is not incident with any edge in F_n if $F_n \neq \emptyset$. Let u_R and v_R be the neighbors of u_L and v_L in R . By Theorem 2.1, R contains a u_Rv_R -Hamilton path, say P_R . Then, $C_L - u_Lv_L + u_Lu_R + v_Lv_R + P_R$ is a Hamilton cycle in $VQ_n - F$.

The theorem follows. \square

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