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Embedded connectivity of recursive networks ☆

Xiang-Jun Li [a], Qi-Qi Dong [a], Zheng Yan [a], Jun-Ming Xu [b,*]

[a] School of Information and Mathematics, Yangtze University, Jingzhou, Hubei, 434023, China
[b] School of Mathematical Sciences, University of Science and Technology of China, Hefei, 230026, China

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ABSTRACT

Let G_n be an n-dimensional recursive network. The h-embedded connectivity $\zeta_h(G_n)$ (resp. edge-connectivity $\eta_h(G_n)$) of G_n is the minimum number of vertices (resp. edges) whose removal results in disconnected and each vertex is contained in an h-dimensional subnetwork G_h. This paper determines ζ_h and η_h for the hypercube Q_n and the star graph S_n, and η_3 for the bubble-sort network B_n.

1. Introduction

It is well known that interconnection networks play an important role in parallel computing/communication systems. An interconnection network can be modeled by a graph $G = (V, E)$, where V is the set of processors and E is the set of communication links in the network.

The *connectivity* $\kappa(G)$ (resp. *edge-connectivity* $\lambda(G)$) of G is defined as the minimum number of vertices (resp. edges) whose removal from G results in a disconnected graph. The connectivity $\kappa(G)$ and edge-connectivity $\lambda(G)$ of a graph G are two important measurements for fault tolerance of the network since the larger $\kappa(G)$ or $\lambda(G)$ is, the more reliable the network is.

However, the definitions of $\kappa(G)$ and $\lambda(G)$ are implicitly assumed that any subset of system components is equally likely to be faulty simultaneously, which may not be true in real applications, thus they underestimate the reliability of the network. To overcome such a shortcoming, Harary [2] introduced the concept of conditional connectivity by appending some requirements on connected components, Latifi et al. [3] specified requirements and proposed the concept of the restricted h-connectivity. These parameters can measure fault tolerance of an interconnection network more accurately than the classical connectivity. The concepts stated here are slightly different from theirs.

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* Corresponding author.
E-mail address: xujm@ustc.edu.cn (J.-M. Xu).

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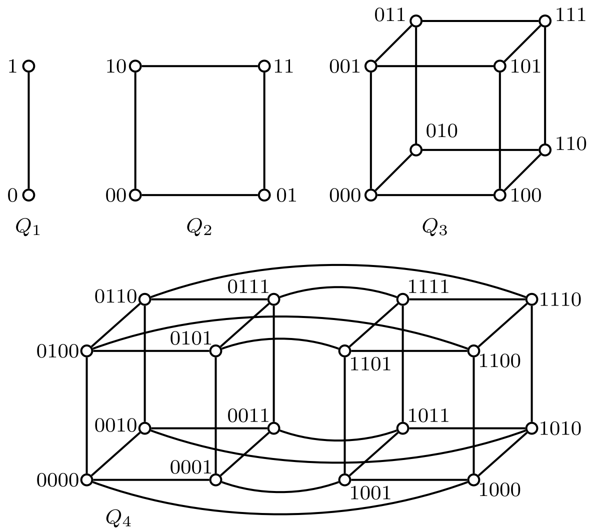


Fig. 1. The n-cubes Q_1, Q_2, Q_3 and Q_4.

For a graph G, $\delta(G)$ denotes its minimum degree. A subset $S \subset V(G)$ (resp. $F \subset E(G)$) is called an *h-vertex-cut* (resp. *h-edge-cut*), if $G - S$ (resp. $G - F$) is disconnected and $\delta(G - S) \geq h$. The *h-super connectivity* $\kappa^h(G)$ (resp. *h-super edge-connectivity* $\lambda^h(G)$) of G is defined as the cardinality of a minimum h-vertex-cut (resp. h-edge-cut) of G.

For any graph G and any integer h, determining $\kappa^h(G)$ and $\lambda^h(G)$ is quite difficult, no polynomial algorithm to compute them has been yet known so far. In fact, the existence of $\kappa^h(G)$ and $\lambda^h(G)$ is an open problem for $h \geq 1$. Only a little knowledge of results has been known on κ^h and λ^h for some special classes of graphs for any h, such as the hypercube Q_n and the star graph S_n.

In order to facilitate the expansion of the network, and to use the same routing algorithm or maintenance strategy as used in the original one, large-scale parallel computing systems always take some networks of recursive structures as underlying topologies, such as the hypercube Q_n, the star graph S_n, the bubble-sort graph B_n and so on. Since the presence of vertex and/or edge failures maybe disconnects the entire network, one hopes that every remaining component has undamaged subnetworks (i.e., smaller networks with same topological properties as the original one). Under this consideration, Yang et al. [12] proposed the concept of embedded connectivity.

Let G_n be an n-dimensional recursive network. For a positive integer h with $h \leq n-1$, there is a sub-network $G_h \subset G_n$. Let $\delta_h = \delta(G_h)$.

A subset $F \subset V(G_n)$ (resp. $F \subset E(G_n)$) is an *h-embedded vertex-cut* (resp. *h-embedded edge-cut*) if $G_n - F$ is disconnected and each vertex is contained in an h-dimensional subnetwork G_h. The *h-embedded connectivity* $\zeta_h(G_n)$ (resp. *edge-connectivity* $\eta_h(G_n)$) of G_n is defined as the cardinality of a minimum h-embedded vertex-cut (resp. h-embedded edge-cut) of G_n.

By definition, if S is an h-embedded vertex-cut of G_n with $|S| = \zeta_h(G_n)$, then $G_n - S$ contains a sub-network G_h, and so $\delta(G_n - S) \geq \delta_h$, which implies that S is a δ_h-vertex-cut of G_n. Thus, $\kappa^{\delta_h}(G_n) \leq |S| = \zeta_h(G_n)$. Similarly, $\lambda^{\delta_h}(G_n) \leq \eta_h(G_n)$. These facts are useful and we write them as the following lemma.

Lemma 1.1. *For $h \leq n-1$, $\zeta_h(G_n) \geq \kappa^{\delta_h}(G_n)$ if $\zeta_h(G_n)$ exists, and $\eta_h(G_n) \geq \lambda^{\delta_h}(G_n)$ if $\eta_h(G_n)$ exists.*

Using Lemma 1.1, for a star graph S_n and a bubble-sort graph B_n, Yang et al. [12,13] determined $\zeta_2(S_n) = 2n-4$ for $n \geq 3$, $\eta_2(S_n) = 2n-4$ for $n \geq 3$ and $\eta_3(S_n) = 6(n-3)$ for $n \geq 4$; and $\zeta_2(B_n) = 2n-4$ for $n \geq 3$. In this paper, we will determine ζ_h and η_h for Q_n and S_n for any $h \leq n-1$ and determine $\eta_3(B_n)$.

The rest of the paper is organized as follows. In Section 2, we determine $\zeta_h(Q_n) = 2^h(n-h)$ for $h \leq n-2$ and $\eta_h(Q_n) = 2^h(n-h)$ for $h \leq n-1$. In Section 3, we determine $\zeta_h(S_n) = \eta_h(S_n) = h!(n-h)$ for $1 \leq h \leq n-1$. In Section 4, we determine $\eta_3(B_n) = 6(n-3)$ for $n \geq 4$ and point out a flaw in the proof of this conclusion in [13]. A conclusion is in Section 5.

For graph terminology and notation not defined here we follow Xu [10]. For a subset X of vertices in G, we do not distinguish X and the induced subgraph $G[X]$.

2. Hypercubes

The hypercube Q_n has the vertex-set consisting of 2^n binary strings of length n, two vertices being linked by an edge if and only if they differ in exactly one position. Hypercubes Q_1, Q_2, Q_3, Q_4 are shown in Fig. 1.

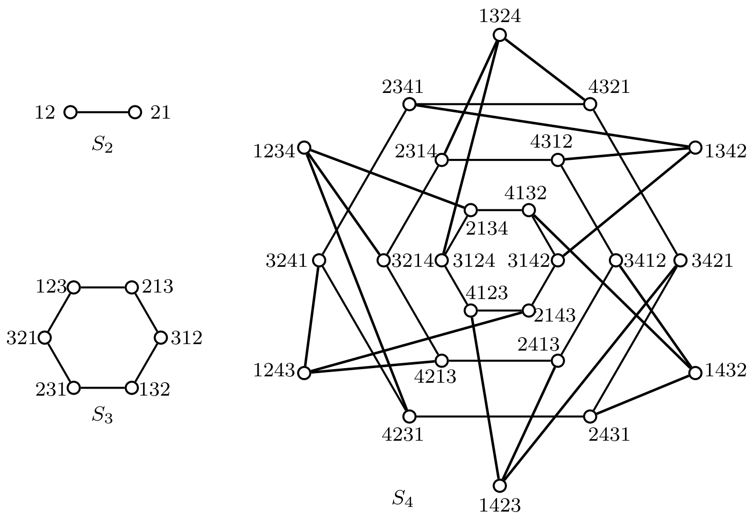


Fig. 2. The star graphs S_2, S_3 and S_4.

The hypercube Q_n is also defined as Cartesian product $K_2 \times K_2 \times \cdots \times K_2$ of n complete graph K_2's. Thus, $Q_n = Q_h \times Q_{n-h}$ for $1 \le h \le n-1$, and Q_n is a Cayley graph with degree n (see Xu [10]). Oh et al. [5] and Wu et al. [7] independently determined $\kappa^h(Q_n)$, and Xu [8] determined $\lambda^h(Q_n)$.

Lemma 2.1. (See [5,7,8].) *$\kappa^h(Q_n) = 2^h(n-h)$ for any h with $0 \le h \le n-2$, and $\lambda^h(Q_n) = 2^h(n-h)$ for any h with $0 \le h \le n-1$.*

Since $\delta_h = \delta(Q_h) = h$, by Lemma 1.1 and Lemma 2.1, the following corollary holds.

Corollary 2.2. *$\zeta_h(Q_n) \ge 2^h(n-h)$ for any h with $1 \le h \le n-2$, and $\eta_h(Q_n) \ge 2^h(n-h)$ for any h with $1 \le h \le n-1$.*

Lemma 2.3. *$\zeta_h(Q_n) \le 2^h(n-h)$ for any h with $1 \le h \le n-2$, and $\eta_h(Q_n) \le 2^h(n-h)$ for any h with $1 \le h \le n-1$.*

Proof. For $h \le n-2$, let $Q_n = Q_h \times Q_{n-h}$. Let x be a vertex in Q_{n-h}, $N(x)$ be the neighbor-set of x in Q_{n-h}, and $S = Q_h \times N(x)$. Then $Q_n - S$ is disconnected, and isomorphic to $Q_h \times (Q_{n-h} - N(x))$, so each vertex of $Q_n - S$ is in some Q_h. It follows that $\zeta_h(Q_n) \le |S| = 2^h(n-h)$.

For $h \le n-1$, $Q_n = Q_h \times Q_{n-h}$, let F be the set of edges between Q_h and $Q_n - Q_h$. It is easy to see that $Q_n - F$ is disconnected, and isomorphic to $Q_h \times (Q_{n-h} - E(y))$, where $E(y)$ is the set of edges in Q_{n-h} incident with a vertex y of Q_{n-h}. Thus each vertex of $Q_n - F$ is in some Q_h. It follows that $\eta_h(Q_n) \le |F| = 2^h(n-h)$. □

Combining Corollary 2.2 with Lemma 2.3, we obtain the following conclusion.

Theorem 2.4. *$\zeta_h(Q_n) = 2^h(n-h)$ for any h with $1 \le h \le n-2$, and $\eta_h(Q_n) = 2^h(n-h)$ for any h with $1 \le h \le n-1$.*

3. Star graphs

For a given integer n with $n \ge 2$, let $I_n = \{1, 2, \ldots, n\}$, $I'_n = \{2, \ldots, n\}$ and $P(n) = \{p_1 p_2 \ldots p_n : \; p_i \in I_n, p_i \ne p_j, 1 \le i \ne j \le n\}$, the set of permutations on I_n. Clearly, $|P(n)| = n!$. For $p = p_1 \ldots p_j \ldots p_n \in P(n)$, the digit p_j is called the j-th digit of p.

The n-dimensional star graph S_n has vertex-set $P(n)$ and has an edge between any two vertices if and only if one can be obtained from the other by swapping the 1-th digit and the i-th digit for $i \in I'_n$, that is, two vertices $x = p_1 p_2 \ldots p_i \ldots p_n$ and y are adjacent if and only if $y = p_i p_2 \ldots p_{i-1} p_1 p_{i+1} \ldots p_n$ for some $i \in I'_n$. The star graphs S_2, S_3, S_4 are shown in Fig. 2. It is shown that the star graph S_n is a Cayley graph with degree $(n-1)$ (see Akers and Krishnamurthy [1]).

For a fixed $i \in I_n$, let $S_n^{j:i}$ denote a subgraph of S_n induced by all vertices whose the j-th digit is i for each $j \in I_n$. By definition, it is easy to see that $S_n^{j:i}$ is isomorphic to S_{n-1} for each $j \in I'_n$ and $S_n^{1:i}$ is an edgeless graph with $(n-1)!$ vertices. As shown in Fig. 1, $S_4^{1:1}$ is an edgeless graph with 6 vertices, $S_4^{j:1}$ is isomorphic to S_3 for each j with $2 \le j \le 4$.

There are two different hierarchical structures of S_n depending on different partition methods. The first one is partitioned along a fixed dimension, which is well-known and used frequently. The second one is partitioned along a fixed digit in I_n, which is a new structure proposed by Shi et al. [6].

Lemma 3.1. (Shi et al. [6], 2012.) *For a fixed $i \in I_n$, S_n can be partitioned into n subgraphs $S_n^{j:i}$, which is isomorphic to S_{n-1} for each $j \in I'_n$ and $S_n^{1:i}$ is an edgeless graph with $(n-1)\,!$ vertices. Moreover, there are $(n-2)!$ independent edges between $S_n^{1:i}$ and $S_n^{j:i}$ for any $j \in I'_n$, and there is no edge between $S_n^{j_1:i}$ and $S_n^{j_2:i}$ for any $j_1, j_2 \in I'_n$ with $j_1 \neq j_2$.*

Clearly, S_1, S_2, S_3 are isomorphic to K_1, K_2, C_6, respectively. As shown in Fig. 1, S_4 is partitioned along digit 1.

Lemma 3.2. (Li and Xu [4], 2014.) $\kappa^h(S_n) = \lambda^h(S_n) = (h+1)\,!\,(n-h-1)$ *for any h with* $0 \le h \le n-2$.

Since $\delta_h = \delta(S_h) = h-1$, by Lemma 1.1 and Lemma 3.2, the following corollary holds.

Corollary 3.3. $\zeta_h(S_n) \ge h\,!\,(n-h)$ *and* $\eta_h(S_n) \ge h\,!\,(n-h)$ *for any h with* $1 \le h \le n-1$.

To determine $\zeta_h(S_n)$ and $\eta_h(S_n)$, we investigate their upper bounds.

Lemma 3.4. (Yang et al. [12].) $\eta_h(S_n) \le h\,!\,(n-h)$ *for any h with* $1 \le h \le n-1$.

Now we establish the upper bound on $\zeta_h(S_n)$ by Lemma 3.1.

Lemma 3.5. $\zeta_h(S_n) \le h\,!\,(n-h)$ *for any h with* $1 \le h \le n-1$.

Proof. Let

$$X = \{\, p_1 p_2 \cdots p_h 12 \cdots (n-h) \in V(S_n) : \ p_i \in I_n \setminus I_{n-h},\ i \in I_h\}.$$

Then, $S_n[X] \cong S_h$. Let T be the neighbor-set of X in $S_n - X$. By the definition of S_n, for a vertex of X, since it has $h-1$ neighbors in X, it exactly has $(n-h)$ neighbors in T, and every vertex of T exactly has one neighbor in X. It follows that

$$|T| = h\,!\,(n-h).$$

Next we show T is an h-embedded vertex-cut of S_n. It suffices to show each vertex on $S_n-(X \cup T)$ is in some subgraph S_h of $S_n-(X \cup T)$.

Assume that $u = p'_1 p'_2 \cdots p'_n$ is a vertex in $S_n-(X \cup T)$, and let

$$J = \{j \in I_{n-h} : \ p'_{h+j} \neq j\} \text{ and } J' = \{j \in J : \ p'_1 \neq j\}.$$

Since $u \notin X$, we have $J \neq \emptyset$. We claim $J' \neq \emptyset$. Suppose to the contrary $J' = \emptyset$. Then $p'_1 = j$ for each $j \in J$, and so $|J| = 1$. Assume $J = \{j\}$. Note that $p'_1 = j$ and $p'_{h+j} \neq j$ $(1 \le j \le n-h)$. Thus u is a neighbor of some vertex in X, that is, $u \in T$, which contradicts to $u \notin T$.

Thus, $J' \neq \emptyset$. Let $j_0 \in J'$ and $p'_{i_0} = j_0$ $(1 \le i_0 \le n)$. Then $h + j_0 \neq i_0$ and $p'_1 \neq j_0$. We partition S_n by fixing digit j_0. Then $X \subseteq S_n^{(h+j_0):j_0}$ and $u \in S_n^{i_0:j_0}$. By Lemma 3.1, $S_n^{(h+j_0):j_0}$ and $S_n^{i_0:j_0}$ are both isomorphic to S_{n-1}, and there is no edge between $S_n^{(h+j_0):j_0}$ and $S_n^{i_0:j_0}$, and so u is in some S_{n-1} of $S_n \setminus (X \cup T)$, which implies that u is in some S_h of $S_n \setminus (X \cup T)$. Thus, T is an h-embedded vertex-cut of S_n, and so

$$\zeta^{(h)}(S_n) \le |T| = h\,!(n-h).$$

The lemma follows. □

Theorem 3.6. $\zeta_h(S_n) = \eta_h(S_n) = h\,!\,(n-h)$ *for any h with* $1 \le h \le n-1$.

Proof. For $1 \le h \le n-1$, combining Corollary 3.3 with Lemma 3.4, we have $\eta_h(S_n) = h!(n-h)$, and combining Corollary 3.3 with Lemma 3.5, we have $\zeta_h(S_n) = h\,!(n-h)$. □

4. Bubble-sort graphs

The n-dimensional bubble-sort graph B_n has vertex-set $P(n)$ and has an edge between any two vertices if and only if one can be obtained from the other by swapping the i-th digit and the $(i+1)$-th digit where $1 \le i \le n-1$. The bubble-sort graphs B_2, B_3 and B_4 are shown in Fig. 3.

It is shown that the bubble-sort graph B_n is a Cayley graph with degree $(n-1)$ (see Akers and Krishnamurthy [1]). More specifically, B_n is a bipartite graph with the girth 4 and contains n disjoint sub-graph B_{n-1}'s by fixing the 1-th digit or n-th digit (see Fig. 3), and $\lambda^2(B_n) = 4(n-3)$ for $n \ge 4$ (see [11]).

For each $t \in \{1, n\}$, $i \in I_n$, let $B_n^{t:i}$ denote a subgraph of B_n induced by all vertices whose the t-th digit is i. Clearly, $B_n^{t:i} \cong B_{n-1}$ for each $t \in \{1, n\}$, $i \in I_n$.

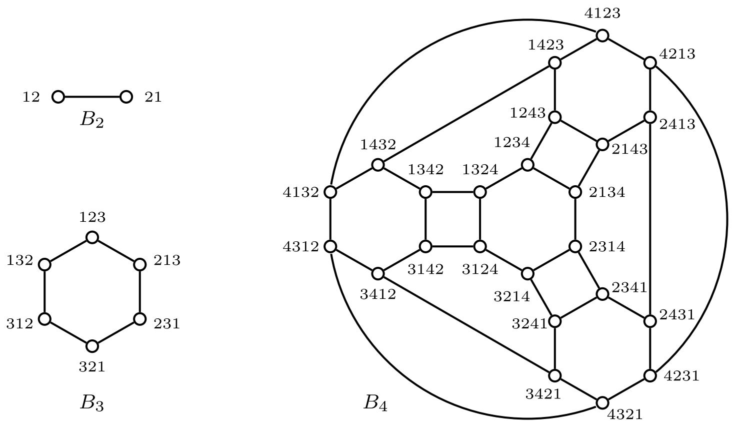


Fig. 3. The bubble-sort graphs B_2, B_3 and B_4.

Table 1
The disjoint paths between two disjoint B_3's in B_4.

From \ To	$B_4^{4,4}$	$B_4^{1,1}$
$B_4^{4,1}$	324**1** – 321**4**	324**1** – 3214 – 3124 – **1**324
	234**1** – 231**4**	234**1** – 2314 – 2134 – **1**234
	342**1** – 3412 – 3142 – 312**4**	342**1** – 3412 – 3142 – **1**342
	243**1** – 2413 – 2143 – 213**4**	243**1** – 2413 – 2143 – **1**243
	432**1** – 4312 – 4132 – 1432 – 1342 – 132**4**	432**1** – 4312 – 4132 – **1**432
	423**1** – 4213 – 4123 – 1423 – 1243 – 123**4**	423**1** – 4213 – 4123 – **1**423

Lemma 4.1. (Akers and Krishnamurthy [1], 1989.) *For a fixed $t \in \{1, n\}$, B_n can be partitioned into n subgraphs $B_n^{t:j}$ isomorphic to B_{n-1} for each $j \in I_n$, moreover, there are $(n-2)!$ independent edges between $B_n^{t:j_1}$ and $B_n^{t:j_2}$ for any $j_1, j_2 \in I_n$ with $j_1 \neq j_2$.*

Lemma 4.2. (Yang et al. [13], 2014.) *$\eta_h(B_n) \leq h!(n-h)$ for any h with $1 \leq h \leq n-1$ and $n \geq 2$.*

It was showed by Xu [9] in 2000 that for a vertex-transitive connected graph G with order n $(\geqslant 4)$ and with degree d $(\geqslant 2)$, $\lambda^1(G) = 2d - 2$ if n is odd or G contains no triangles. Since B_n is vertex-transitive and bipartite, we have the following result.

Lemma 4.3. *$\lambda^1(B_n) = 2n - 4$ for $n \geq 3$.*

Since $\delta_h = \delta(B_h) = h - 1$, by Lemma 1.1 and Lemma 4.3 $\eta_2(B_n) \geq \lambda^1(B_n) = 2n - 4$, and by Lemma 4.2 the following corollary holds.

Corollary 4.4. *$\eta_2(B_n) = 2n - 4$ if $n \geq 3$.*

In this section, we will determine $\eta_3(B_n) = 6(n-3)$ for $n \geq 4$. By Lemma 4.2, we only need to prove $\eta_3(B_n) \geq 6(n-3)$. The proof proceeds by induction on $n (\geq 4)$. And so we first consider the case of $n = 4$.

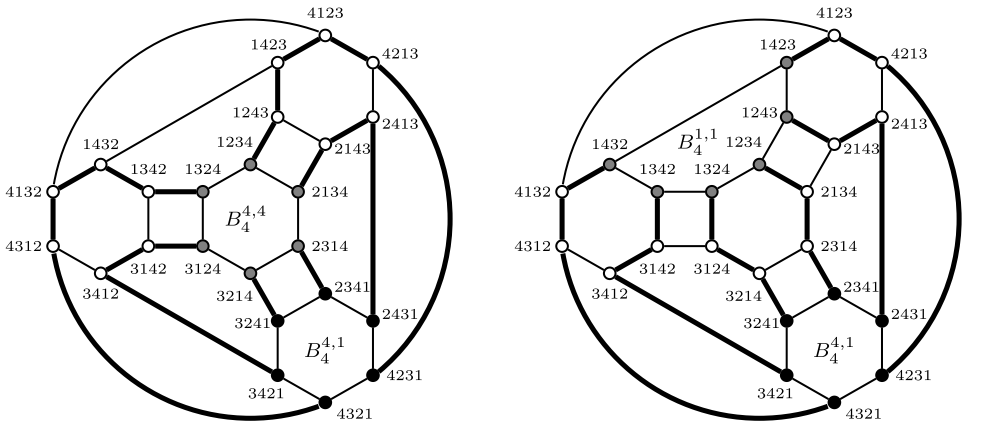
An edge-cut F of B_n is called a B_h-edge-cut if every component of $B_n - F$ contains a B_h as subgraph. By definition, every h-embedded edge-cut of B_n is certainly a B_h-edge-cut.

Lemma 4.5. *If F is a B_3-edge-cut of B_4, then $|F| \geq 6$.*

Proof. We first prove that there are 6 vertex-disjoint paths between each two disjoint B_3s in B_4. By symmetry, we only need to consider the paths from $B_4^{4,1}$ to $B_4^{4,4}$ and $B_4^{1,1}$. Such paths are illustrated in the Table 1 (also see Fig. 4 by heavy edges). If F is a B_3-edge-cut of B_4, then F contains at least one edge of every path in the 6 vertex-disjoint paths between the two disjoint B_3s. It follows that $|F| \geq 6$. □

Theorem 4.6. *$\eta_3(B_n) = 6(n-3)$ if $n \geq 4$.*

Proof. By Lemma 4.2, we only need to prove $\eta_3(B_n) \geq 6(n-3)$ for $n \geq 4$. Let F be a B_3-edge-cut of B_n. Since any 3-embedded edge-cut is certainly a B_3-edge-cut of B_n, we have that $\eta_3(B_n) \geq |F|$. Thus it suffices to show



(a) 6 vertex-disjoint paths between $B_4^{4,1}$ and $B_4^{4,4}$ (b) 6 vertex-disjoint paths between $B_4^{4,1}$ and $B_4^{1,1}$

Fig. 4. The disjoint paths between two disjoint B_3's in B_4 by heavy edges.

$$|F| \geq 6(n-3). \tag{4.1}$$

The proof proceeds by induction on $n\,(\geq 4)$. When $n=4$, the result follows by Lemma 4.5. Assume the induction hypothesis for any m with $4 \leq m \leq n-1$ with $n \geq 5$, that is

$$|F'| \geq 6(m-3) \text{ for any } B_3\text{-edge-cut } F' \text{ of } B_m. \tag{4.2}$$

Let F be a minimum B_3-edge-cut of B_n. Clearly, $B_n - F$ has exactly two connected components, denoted by X and Y, respectively, and assume $|X| \leq |Y|$ without loss of generality.

Use notations $\delta(X)$ and $\delta(Y)$ to denote the minimum degrees of X and Y, respectively. We assert that $\delta(X) \geq 2$ and $\delta(Y) \geq 2$.

Assume to the contrary that there exists a vertex $x \in X$ such that $d_X(x) = 1$. Then x has at least 3 neighbors in Y since B_n is $(n-1)$-regular and $n \geq 5$. Let $X' = X \setminus \{x\}$, $Y' = Y \cup \{x\}$ and F' be the set of edges between X' and Y'. Then F' is also a B_3-edge-cut of B_n and $|F'| \leq |F| - 3 + 1 = |F| - 2$, which contradicts to the minimality of F.

To complete our proof, for a fixed $t \in \{1, n\}$ and any $i \in I_n$, let

$$X_i = X \cap V(B_n^{t:i}), \quad Y_i = Y \cap V(B_n^{t:i}),$$
$$F_i = F \cap E(B_n^{t:i}), \quad F_{ij} = F \cap E(B_n^{t:i}, B_n^{t:j}),$$

where $E(B_n^{t:i}, B_n^{t:j})$ denotes the set of edges between $B_n^{t:i}$ and $B_n^{t:j}$ for $i \neq j$. Let

$$J_X = \{i \in I_n : X_i \neq \emptyset\}, J_Y = \{i \in I_n : Y_i \neq \emptyset\} \text{ and } J_0 = J_X \cap J_Y.$$

Since $X \neq \emptyset$ and $Y \neq \emptyset$, we have $|J_X| \geq 1$ and $|J_Y| \geq 1$. We choose such $t \in \{1, n\}$ that $|J_X|$ is as large as possible, say $t = 1$.

First assume $|J_0| \geq 3$. For $i \in J_0$, we have $\delta(X_i) \geq 1$ and $\delta(Y_i) \geq 1$ since $\delta(X) \geq 2$ and $\delta(Y) \geq 2$, and each vertex in $B_n^{1:i}$ only has one neighbor not in $B_n^{1:i}$ by Lemma 4.1. It follows that F_i is a 1-edge-cut of $B_n^{1:i}$ for $i \in J_0$. By Lemma 4.3, we have that

$$|F_i| \geq \lambda^1(B_{n-1}) = 2(n-3) \text{ for } i \in J_0, \tag{4.3}$$

and so

$$|F| \geq \sum_{i \in J_0} |F_i| \geq |J_0| 2(n-3) \geq 6(n-3).$$

The inequality (4.1) follows.

Now assume $|J_0| \leq 2$. Let $a = |J_X \setminus J_0|$ and $b = |J_Y \setminus J_0|$. Then $2a \leq n$ since $|X| \leq |Y|$. Since any $i \in I_n$ not in J_X is certainly in J_Y, $a + b = n - |J_0| \geq n - 2$.

Assume $a \geq 1$ and $j_1 \in J_X \setminus J_0$, $j_2 \in J_Y \setminus J_0$. By Lemma 4.1 there are $(n-2)!$ independent edges between $B_n^{1:j_1}$ and $B_n^{1:j_2}$, and so there are $(ab(n-2)!)$ independent edges between $\cup_{j_1 \in J_X \setminus J_0} B_n^{1:j_1}$ and $\cup_{j_2 \in J_Y \setminus J_0} B_n^{1:j_2}$. It follows that for $n \geq 5$,

$$\begin{aligned} |F| &\geq ab(n-2)! \geq a(n-2-a)(n-2)! \\ &\geq (n-2-1)(n-2)! \geq 6(n-3). \end{aligned}$$

The inequality (4.1) follows.

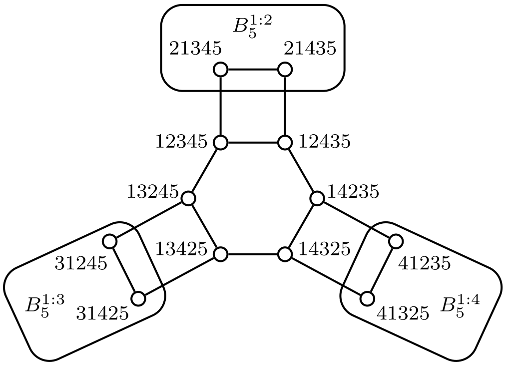


Fig. 5. The neighbors of a B_3 in a B_5.

Next assume $a = 0$. Then $1 \le |J_X| = |J_0| \le 2$ and $|J_Y| = n$. We first show that

$$Y_i \text{ contains a } B_3 \text{ as a subgraph for each } i \in I_n. \tag{4.4}$$

To this end, for $i \in I_n$, let K_i be the set of the n-th digits of vertices in X_i, that is, $K_i = \{p_n : ix_2 \cdots x_{n-1}p_n \in X_i,\ p_n \in I_n \setminus \{i\}, x_j \in I_n \setminus \{i, p_n\}, 2 \le j \le n-1\}$. Let $K'_i = I_n \setminus (K_i \cup \{i\})$. By choice of t and $|J_X| \le 2$, we have $|K_i| \le 2$ and $|K'_i| = (n-1) - |K_i| \ge n-3 \ge 2$. Thus, for fixed $i \in I_n$ and $p_n \in K'_i$, the subgraph induced by $\{ix_2 \cdots x_{n-1}p_n :\ x_j \in I_n \setminus \{i, p_n\}, 2 \le j \le n-1\}$ is isomorphic to a B_{n-2} and is contained in Y_i, which implies that Y_i contains a B_3 since $n \ge 5$, and so the assertion (4.4) holds. Thus, for any $i \in J_X$, if X_i contains a B_3, then F_i is a B_3-edge-cut of $B_n^{1:i}$ by the assertion (4.4). By the induction hypothesis (4.2), we have that

$$|F_i| \ge 6(n-4) \text{ if } X_i \text{ contains a } B_3 \text{ as a subgraph for each } i \in J_X. \tag{4.5}$$

We consider two cases according as $|J_X| = 1$ and $|J_X| = 2$.

Case 1. $|J_X| = 1$.

Without loss of generality, assume that $J_0 = J_X = \{1\}$, and so $X_1 = X \cap V(B_n^{1:1})$. By the hypothesis of F, X_1 contains a B_3, $|X_1| \ge 6$, and each vertex in X_1 has one neighbor not in $B_n^{1:1}$. Combining these facts with the assumption (4.5), we have that

$$|F| \ge |F_1| + |X_1| \ge 6(n-4) + 6 = 6(n-3).$$

Case 2. $|J_X| = 2$.

Without loss of generality, assume $J_X = \{1, 2\}$, and so $X_1 = X \cap V(B_n^{1:1})$ and $X_2 = X \cap V(B_n^{1:2})$. By the hypothesis of F, $X = X_1 \cup X_2$ contains a B_3. By the structure of 6-cycles in B_n, if neither X_1 nor X_2 contains a B_3, then there is a vertex in the B_3 whose the first digit is different from 1 and 2, which implies $|J_X| \ge 3$, a contradiction. Thus, without loss of generality, assume that X_1 contains a B_3. By the assumptions (4.5) and (4.3), for $n \ge 6$ we have that

$$|F| \ge |F_1| + |F_2| \ge 6(n-4) + 2(n-3) \ge 6(n-3). \tag{4.6}$$

Now assume $n = 5$. Then the vertices of B_3 in X_1 must have forms $1x_2x_3x_4p_5$ or $1p_2x_3x_4x_5$, where p_2 and p_5 are fixed. Since $|K_i| \le 2$ for each $i \in I_n$, the vertices of B_3 in X_1 have the former form. Without loss of generality, say $p_5 = 5$. Then the vertex-set of B_3 and the neighbors of X_1 not in $B_5^{1:1}$ are shown in Fig. 5, where four neighbors are in $B_5^{1:3} \cup B_5^{1:4}$. This shows that no matter how $x_2x_3x_4p_5$ is chosen, we always have $\sum_{j=3}^{5} F_{1j} \ge 2 \times 2 = 4$. It follows that

$$|F| \ge |F_1| + |F_2| + \sum_{j=3}^{n} F_{1j} \ge 6(n-4) + 2(n-3) + 4 > 6(n-3).$$

The inequality (4.1) follows. By induction principles, the theorem follows. $\square$

5. Conclusions

In this paper, we investigate the h-embedded connectivity ζ_h and h-embedded edge-connectivity η_h in the hypercube Q_n, the star graph S_n and the bubble-sort graph B_n. We determine that $\zeta_h(Q_n) = 2^h(n-h)$ for $h \le n-2$, $\eta_h(Q_n) = 2^h(n-h)$ for $h \le n-1$, $\zeta_h(S_n) = \eta_h(S_n) = h!(n-h)$ for $1 \le h \le n-1$, and $\eta_3(B_n) = 6(n-3)$ for $n \ge 4$. These results can provide more accurate measurements for fault tolerance of the system when the graphs are used to model the topological structure of large-scale parallel processing systems. The value of $\zeta_h(B_n)$ for $h \geq 3$ and the value of $\eta_h(B_n)$ for $h \geq 4$ deserve further research.

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References

[1] S.B. Akers, B. Krishnamurthy, A group theoretic model for symmetric interconnection networks, IEEE Trans. Comput. 38 (4) (1989) 555–566.
[2] F. Harary, Conditional connectivity, Networks 13 (1983) 347–357.
[3] S. Latifi, M. Hegde, M. Naraghi-Pour, Conditional connectivity measures for large multiprocessor systems, IEEE Trans. Comput. 43 (2) (1994) 218–222.
[4] X.-J. Li, J.-M. Xu, Generalized measures for fault tolerance of star networks, Networks 63 (3) (2014) 225–230.
[5] A.D. Oh, H. Choi, Generalized measures of fault tolerance in n-cube networks, IEEE Trans. Parallel Distrib. Syst. 4 (1993) 702–703.
[6] W. Shi, F. Luo, P.K. Srimani, A new hierarchical structure of star graphs and applications, in: Lecture Notes in Computer Science, vol. 7154, 2012, pp. 267–268.
[7] J. Wu, G. Guo, Fault tolerance measures for m-ary n-dimensional hypercubes based on forbidden faulty sets, IEEE Trans. Comput. 47 (1998) 888–893.
[8] J.-M. Xu, On conditional edge-connectivity of graphs, Acta Math. Appl. Sin. 16 (4) (2000) 414–419.
[9] J.-M. Xu, Restricted edge-connectivity of vertex-transitive graphs, Chinese J. Contemp. Math. 21 (4) (2000) 369–374.
[10] J.-M. Xu, Topological Structure and Analysis of Interconnection Networks, Kluwer Academic Publishers, Dordrecht/Boston/London, 2001.
[11] W.-H. Yang, H.-Z. Li, J.-X. Meng, Conditional connectivity of Cayley graphs generated by transposition trees, Inform. Process. Lett. 110 (2010) 1027–1030.
[12] Y. Yang, S. Wang, Conditional connectivity of star graph networks under embedded restriction, Inform. Sci. 199 (15) (2012) 187–192.
[13] Y. Yang, S. Wang, J. Li, Conditional connectivity of recursive interconnection networks respect to embedding restriction, Inform. Sci. 279 (2014) 273–279.