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Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Many-to-many disjoint paths in hypercubes with faulty vertices[☆]

Xiang-Jun Li[a], Bin Liu[b], Meijie Ma[c], Jun-Ming Xu[d,*]

[a] School of Information and Mathematics, Yangtze University, Jingzhou, Hubei, 434023, China
[b] School of Management and Economics, The Chinese University of Hong Kong, Shenzhen, Shenzhen 518172, China
[c] Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China
[d] School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, China

ARTICLE INFO

Article history:
Received 25 August 2015
Received in revised form 6 July 2016
Accepted 4 September 2016
Available online xxxx

Keywords:
Hypercube
Disjoint paths
Fault-tolerance
Parallel computing

ABSTRACT

This paper considers the problem of many-to-many disjoint paths in the hypercube Q_n with f faulty vertices and obtains the following result. For any integer k with $1 \leq k \leq n-1$ and any two sets S and T of k fault-free vertices in different partite sets of Q_n $(n \geq 2)$, if $f \leq 2n-2k-2$ and each fault-free vertex has at least two fault-free neighbors, then there exist k fully disjoint fault-free paths linking S and T which contain at least $2^n - 2f$ vertices. A linear algorithm for finding such disjoint paths is also given. This result improves some known results in a sense.

1. Introduction

The n-dimensional hypercube Q_n is a graph whose vertex-set consists of all binary vectors of length n, with two vertices being adjacent whenever the corresponding vectors differ in exactly one coordinate. There is a vast literature on graph-theoretic properties of hypercubes (e.g., see the comprehensive survey papers on early results [15] and recent results [27]) and their applications in parallel computing (e.g., see [21]).

One of the most central issues in various high-performance communication networks or parallel computing systems is to find a cycle or a path of some given length in the networks (see [4,11,14,17,23] and the survey paper [29]). It is well known that there are n internally disjoint paths connecting any two vertices u and v in Q_n. However, these paths do not always contain all vertices. In 2004, Chang et al. [5] showed that for any integer k with $1 \leq k \leq n$ and any two vertices u and v from different partite sets of Q_n, there are k internally disjoint uv-paths in Q_n which contain all vertices. In 2007, Chang et al. [6] further showed that for any integer k with $1 \leq k \leq n-4$ and any two vertices u and v from different partite sets of Q_n $(n \geq 5)$, there are k internally disjoint uv-paths in Q_n which contain all vertices and the difference of lengths of any two paths is at most two.

Since edge and/or vertex failures are inevitable when a large parallel computer system is put in use, finding disjoint paths with fault-tolerant routings among vertices has received great research attention recently. In disjoint-path problems, one

☆ The work was supported by NNSF of China (11571044), NSF of Hubei (2014CFB248), ZSDZZZZXK08, Young talent fund from Hubei Department of Education (Q20151311) and ORFIAM, Yangtze University (KF1505, KF1506).

* Corresponding author.
E-mail addresses: franklxj@mail.ustc.edu.cn (X.-J. Li), binliu1009@gmail.com (B. Liu), mameij@zjnu.cn (M. Ma), xujm@ustc.edu.cn (J.-M. Xu).

http://dx.doi.org/10.1016/j.dam.2016.09.013
0166-218X/

or more source vertices and one or more sink vertices are given to find the paths between them. According to the number of sources or sinks, there are one-to-one [16], one-to-many [20], and many-to-many [3,8,18,22,24–26,30] disjoint-path problems.

A path in Q_n with f faulty vertices is long if it contains at least $2^n - 2f - 1$ vertices. Note that every long path between vertices u and v has a length of at least $2^n - 2f$ if $d(u, v)$ is odd, where $d(u, v)$ is the distance between u and v. If all faulty vertices are in the same bipartite of Q_n, then every long fault-free path is the longest one.

Fu [12] showed that there is a long fault-free path in Q_n between every two fault-free vertices if $f \le n - 2$. Under the condition that every vertex has at least two fault-free neighbors, Kueng et al. [19] improved the bound in Fu to $2n - 5$. Recently, Fink et al. [10] improved the bound to $2n - 4$ with a weaker condition, and Dvořák et al. [9] further improved it to $\frac{n^2}{10} + \frac{n}{2} + 1$ when $n \ge 15$.

In 2009, Chen [1] considered the problem of many-to-many disjoint paths in the hypercube with faulty vertices and edges showing that for any integer k with $1 \le k \le n - 1$, and any two sets S and T of k fault-free vertices in different partite sets in Q_n with f faulty vertices and h faulty edges, if $f + h \le n - k - 1$, then there exist k disjoint fault-free paths linking S and T in Q_n which contain at least $2^n - 2f$ vertices.

Gregor et al. [13] and Chen [2] investigated the many-to-many disjoint paths in hypercubes with only faulty edges. In this paper, we consider Q_n with only faulty vertices and obtain the following result.

Theorem 1.1. *Let Q_n $(n \ge 2)$ be an n-dimensional hypercube with f faulty vertices and let k be an integer with $1 \le k \le n - 1$. If $f \le 2n - 2k - 2$ and each fault-free vertex has at least two fault-free neighbors, then for any two sets S and T of k fault-free vertices in different partite sets, there exist k disjoint fault-free paths linking S and T in Q_n which contain at least $2^n - 2f$ vertices.*

Under the condition that each fault-free vertex has at least two fault-free neighbors, our theorem generalizes the result of Fink et al. Since $k \ge 1$, our result also improves the bound obtained by Chen [1] provided that Q_n does not contain faulty edges.

When $f \le 2n - 2k - 2$, a centralized $O(2^n)$-algorithm to find the many-to-many disjoint paths is given, whose running time is a linear function of the number of the vertices in Q_n.

The rest of this paper is organized as follows. Section 2 introduces some notations and lemmas. An overview of the main proof is provided in Section 3. The proof of our result and constructive algorithms for finding disjoint paths are in Sections 4 and 5. Section 6 concludes.

2. Notations and lemmas

We follow [28] for graph-theoretical terminologies and notations not defined here. Let $G = (V, E)$ be a connected simple graph, where $V = V(G)$ is the vertex-set and $E = E(G)$ is the edge-set of G. If $uv \in E(G)$, u (resp. v) is called a neighbor of v (resp. u). A uv-path is a sequence of distinct vertices, written as $\langle v_0, v_1, v_2, \dots, v_m\rangle$, where $v_0 = u$, $v_m = v$ and $v_i v_{i+1} \in E(G)$ for each $i = 1, 2, \dots, m - 1$, (m is the length of P). The distance between u and v is the length of the shortest uv-path in G. For a path $P = \langle v_0, v_1, \dots, v_i, v_{i+1}, \dots, v_m\rangle$, it can also be expressed as $P = P(v_0, v_i) + v_i v_{i+1} + P(v_{i+1}, v_m)$, and the notation $P - v_i v_{i+1}$ denotes the subgraph obtained from P by deleting the edge $v_i v_{i+1}$. Two paths are disjoint if they have no vertices in common. Given two disjoint sets S and T of k vertices, if there exist k disjoint paths linking S and T, we call these paths k *disjoint ST-paths*.

The n-dimensional hypercube Q_n is a graph with 2^n vertices, with each vertex denoted by an n-bit binary string $x = x_1 x_2 \dots x_{n-1} x_n$. Two vertices are adjacent if and only if their strings differ in exactly one bit position. It has been proved that Q_n is a vertex- and edge-transitive bipartite graph.

By definition, for any $i \in \{1, 2, \dots, n\}$, Q_n can be expressed as $Q_n = L_i \odot R_i$, where L_i and R_i are two subgraphs of Q_n induced by the vertices with the ith bit position being 0 and 1, respectively, which are isomorphic to Q_{n-1}. We use E_i to denote the set of edges between L_i and R_i. The two end-vertices of an edge in E_i differ in exactly the ith bit position. Without loss of generality, we write $Q_n = L \odot R$. For convenience, for a vertex u_L in L, we use u_R to denote its unique neighbor in R. Similarly, for a vertex v_R in R, we use v_L to denote its unique neighbor in L. Clearly, for any two vertices u_L and v_L in L, $d_L(u_L, v_L) = d_R(u_R, v_R)$.

Let F denote the set of faulty vertices in Q_n, and $f = |F|$. When $Q_n = L \odot R$, we denote $f_L = |F \cap L|$ and $f_R = |F \cap R|$. A subgraph of Q_n is *fault-free* if it contains no vertices from F. For two subsets A and B of $V(Q_n)$, let $d(A, B) = \min\{d(x, y) : x \in A, y \in B\}$. For clarity, we list the notations that will be used frequently in Table 1.

We first show the following lemma which ensures that Q_n has a desired partition.

Lemma 2.1. *If $f \le 2n - 6$ and each fault-free vertex of $Q_n (n \ge 3)$ has at least two fault-free neighbors, then there exists some $j \in \{1, 2, \dots, n\}$ such that each fault-free vertex in L_j (or R_j) has at least two fault-free neighbors in L_j (or R_j).*

Proof. Let u be a fault-free vertex in Q_n. Denote $d_F(u)$ as the number of faulty vertices in $N_{Q_n}(u)$, that is, $d_F(u) = |N_{Q_n}(u) \cap F|$, then $d_F(u) \le n - 2$.

Table 1
Notations used in this paper.

Symbols	Significations
F	The faulty vertices in Q_n
S, T	Two sets of k fault-free vertices
L, R	Two different subcubes Q_{n-1} of Q_n
F_L, F_R	The faulty vertices in L, R respectively
S_L, S_R	The vertices of S in L, R respectively
T_L, T_R	The vertices of T in L, R respectively
f_L, f_R	The cardinality of F_L, F_R respectively
p, q	The cardinality of S_L, T_L respectively
X, Y	the different partite sets of Q_n
$N_{Q_n}(u)$	The neighbors of u in Q_n
$d_F(u)$	The cardinality of the neighbor sets of u in F
W_L	The set of neighbors in L of vertices of S_R
U_L	$(p-q)$ fault-free vertices in $Y \cap L$
U_R	The neighbors of U_L in R
B	The vertices in $P_2, \ldots, P_n$

It is easy to see that two vertices in $Q_n (n \geq 3)$ have exactly two common neighbors if any. If there are two vertices u and v such that $d_F(u) = n-2$ and $d_F(v) = n-2$, then

$$\begin{aligned} f &\geq |(N_{Q_n}(u) \cup N_{Q_n}(v)) \cap F| \\ &= d_F(u) + d_F(v) - |N_{Q_n}(u) \cap N_{Q_n}(v) \cap F| \\ &\geq 2n-4-2 \\ &= 2n-6. \end{aligned}$$

Note that $f \leq 2n-6$. Thus, we have $f = 2n-6$ and $|N_{Q_n}(u) \cap N_{Q_n}(v) \cap F| = 2$. Furthermore, there are exactly two vertices u and v such that $d_F(u) = n-2$ and $d_F(v) = n-2$. Assume $N_{Q_n}(u) \cap N_{Q_n}(v) \cap F = \{x, y\}$. Then $uxvy$ is a 4-cycle in Q_n. Choose j such that $ux, vy \in E_j$.

If there is at most one vertex u satisfying $d_F(u) = n-2$, let $x \in N_{Q_n}(u) \cap F$. Also choose j such that $ux \in E_j$.

Therefore, the two fault-free vertices of u (or v) are both in L_j or R_j. For other fault-free vertex x, we have $d_F(x) \leq n-3$. Thus x has at least two fault-free neighbors in L_j or R_j to which x belongs. The result follows. ■

In $Q_n (n \geq 2)$ with at most $2n-4$ faulty vertices, Fink et al. [10] showed that for any two fault-free vertices u and v, there exists a long fault-free path between u and v if and only if the exceptional configurations do not occur. In fact, in the exceptional configurations, there are some vertices with at most one fault-free neighbor. So under the condition that every fault-free vertex has at least two fault-free neighbors, the exceptional configurations do not occur. The following three lemmas will be referred to in our paper.

Lemma 2.2 (*Fink et al. [10], 2009*). *If $f \leq 2n-4$ and each fault-free vertex of $Q_n (n \geq 2)$ has at least two fault-free neighbors then, for any two distinct fault-free vertices x and y with distance d, there is a fault-free xy-path containing at least $2^n - 2f$ vertices if d is odd and $2^n - 2f - 1$ vertices if d is even.*

Lemma 2.3 (*Chen [1], 2009*). *For any integer k with $1 \leq k \leq n-1$, if $f \leq n-k-1$ then, for any two sets S and T of k fault-free vertices in different partite sets in Q_n $(n \geq 2)$, there exist k disjoint fault-free ST-paths which contain at least $2^n - 2f$ vertices.*

Lemma 2.4 (*Dvořák [7], 2005*). *Let (x, y) and (u, v) be two disjoint pairs of vertices with odd distance in Q_n. If x and y are adjacent in Q_n with $n \geq 3$, then there exists a uv-path containing all vertices in $Q_n - \{x, y\}$ unless $n = 3$, $d(u, v) = 1$ and $d(\{x, y\}, \{u, v\}) = 2$.*

In the proof of Theorem 1.1, we need to construct many-to-many disjoint paths with much vertices in some cases, which are ensured by the following lemma.

Lemma 2.5. *Let x and y be two adjacent vertices of Q_n $(n \geq 3)$. Then, for any integer k with $1 \leq k \leq n-2$, and any two sets S and T of k vertices in different partite sets in $Q_n - \{x, y\}$, there exist k disjoint ST-paths containing all vertices in $Q_n - \{x, y\}$.*

Proof. We proceed by induction on k. If $k = 1$, the result is true by Lemma 2.4. We assume that the result holds for any integer less than k, and consider the case of k $(k \geq 2)$ and n (≥ 4). Let $\{X, Y\}$ be the bipartition of Q_n and let $S = \{s_1, s_2, \ldots, s_k\}$ and $T = \{t_1, t_2, \ldots, t_k\}$ be two sets of k vertices in different partite sets in $Q_n - \{x, y\}$. Since Q_n is vertex-transitive and edge-transitive, without loss of generality, we may assume $x = 00\cdots0$, $y = 10\cdots0$, $S \cup \{x\} \subseteq X$, $T \cup \{y\} \subseteq Y$. Let $Q_n = L_j \odot R_j$, then the edge xy belongs to $E(L_j)$ for each dimension j $(2 \leq j \leq n)$.

We first note that there exists some j $(2 \leq j \leq n)$ such that $(S \cup T) \cap L_j \neq \emptyset$ and $(S \cup T) \cap R_j \neq \emptyset$. Otherwise, for each $j \in \{2, \ldots, n\}$, the jth bit of all vertices in $S \cup T$ are the same so that $|S \cup T| \leq 2$, contradicting the assumption that $|S \cup T| = 2k \geq 4$ since $k \geq 2$. Therefore, $xy \in E(L)$, $(S \cup T) \cap L \neq \emptyset$ and $(S \cup T) \cap R \neq \emptyset$.

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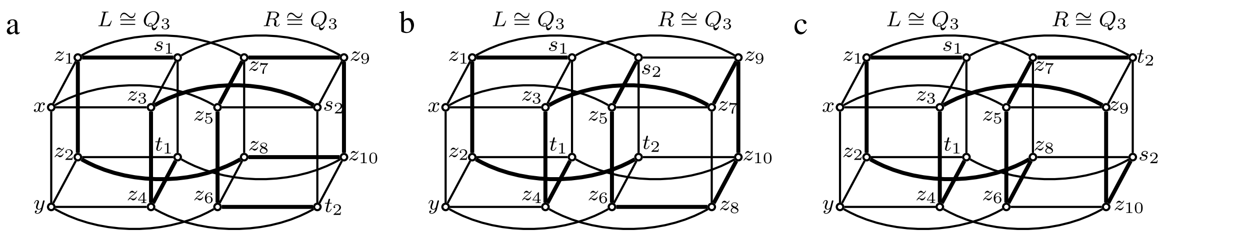

Fig. 1. Illustrations for the proof of Subcase 1.1 of Lemma 2.5.

Let

$$\begin{aligned} &S_L = S \cap L, \qquad T_L = T \cap L \quad \text{and} \quad p = |S_L|, \qquad q = |T_L|, \\ &S_R = S \cap R, \qquad T_R = T \cap R. \end{aligned}$$

By the symmetry of S and T, we assume $p \geq q$. Then $p \geq 1$ and $q \leq k-1$.

Let W_L be the set of neighbors in L of vertices of S_R. Then $|W_L| = |S_R| = |S| - |S_L| = k - p$, and $W_L \subset Y$ since $S \subset X$. If $n = 4$, then $L \cong R \cong Q_3$. In Q_3, we have to consider some exceptional cases by Lemma 2.4. It is easy to see that there exists a unique vertex $u_0 \in L \cap Y$ such that $d_L(u_0, \{x, y\}) = 2$. Set

$$F = \begin{cases} \{u_0\} & \text{if } n = 4; \\ \emptyset & \text{if } n \geq 5. \end{cases}$$

To avoid the exceptional case, we may assume that F is the faulty vertex.

Note that $|Y \cap V(L)| = 2^{n-2}$. For $n \geq 4$ and $k \leq n-2$, we have

$$\begin{aligned} |Y \cap V(L)| - |T_L \cup W_L \cup \{y\} \cup F| &\geq 2^{n-2} - q - (k-p) - 2 \\ &= (2^{n-2} - k - 2) + (p - q) \\ &\geq p - q. \end{aligned}$$

This implies that there exists a set U_L of $(p-q)$ fault-free vertices in $Y \cap V(L - T_L - \{y\})$ such that its neighbor-set U_R in R is not in S_R.

If $p > q$, let

$$\begin{aligned} &U_L = \{u_1, \ldots, u_{p-q}\} \subseteq Y \cap V(L - T_L - \{y\}) \quad \text{and} \\ &U_R = \{v_1, \ldots, v_{p-q}\} \subseteq X \cap V(R - S_R) \quad \text{with} \\ &u_i v_i \in E(Q_n) \quad \text{for } 1 \leq i \leq p - q. \end{aligned}$$

If $p = q$, let $U_L = U_R = \emptyset$.

We consider two cases: $1 \leq p \leq k-1$ and $p = k$.

Case 1. $1 \leq p \leq k-1$.

Let $T'_L = T_L \cup U_L$ and $S'_R = S_R \cup U_R$. Then $|T'_L| = q + (p-q) = p = |S_L|$ and $|S'_R| = (k-p) + (p-q) = k - q = |T_R|$.

Subcase 1.1. $n = 4$.

In this case, $k = 2$ and $p = 1$. First, we consider the case $q = 1$. Without loss of generality, assume $s_1, t_1 \in L$. Let $M = \{v : v \in (S \cup T) \cap L, d(v, \{x, y\}) = 2\}$. We choose a dimension j to partition Q_4 such that $xy \in E(L)$, $(S \cup T) \cap L \neq \emptyset$, $(S \cup T) \cap R \neq \emptyset$ and $|M|$ is as small as possible.

If $|M| \leq 1$, by Lemma 2.4, there is an s_1t_1 path P_1 in L containing all the vertices in $L - \{x, y\}$. By Lemma 2.3, there is an s_2t_2-path P_2 containing all vertices in R. Then paths P_1 and P_2 are disjoint and contain all vertices in $Q_4 - \{x, y\}$.

If $|M| = 2$, then $d(\{s_2\}, \{x, y\}) = d(\{t_2\}, \{x, y\}) \geq 2$, and s_2 and t_2 must be adjacent in R. Otherwise, we can choose another dimension j to partition Q_4 such that $xy \in E(L)$, $(S \cup T) \cap L \neq \emptyset$, $(S \cup T) \cap R \neq \emptyset$ and $|M| \leq 1$, which contradicts the minimality of $|M|$ and $|M| = 2$. Thus, there are only three configurations of S and T (see Fig. 1).

In configuration (a), $P_1 = \langle s_1, z_1, z_2, z_8, z_{10}, z_9, z_7, z_5, z_6, t_2 \rangle$ and $P_2 = \langle s_2, z_3, z_4, t_1 \rangle$ are two disjoint paths satisfying the requirements (see Fig. 1(a)). In configuration (b), $P_1 = \langle s_1, z_1, z_2, t_2 \rangle$ and $P_2 = \langle s_2, z_5, z_6, z_8, z_{10}, z_9, z_7, z_3, z_4, t_1 \rangle$ are two disjoint paths satisfying the requirements (see Fig. 1(b)). In configuration (c), $P_1 = \langle s_1, z_1, z_2, z_8, z_6, z_5, z_7, t_2 \rangle$ and $P_2 = \langle s_2, z_{10}, z_9, z_3, z_4, t_1 \rangle$ are two disjoint paths satisfying the requirements (see Fig. 1(c)).

We now consider $q = 0$. Note that $u_1 \neq u_0$ and u_1 is fault-free. By Lemma 2.4, we obtain s_1u_1-path P_L containing all vertices in $L - \{x, y\}$. By Lemma 2.3, there are two disjoint $S'_R T_R$-paths P_R and P_2 containing all vertices in R, where P_R is a v_1t_1-path and P_2 is an s_2t_2-path. Let $P_1 = P_L + u_1v_1 + P_R$. Then P_1 and P_2 are two disjoint ST-paths containing all vertices in $Q_4 - \{x, y\}$.

Subcase 1.2. $n \geq 5$.

Since $p \leq k-1$, by the induction hypothesis, in L there are p disjoint $S_L T'_L$-paths containing all vertices in $L - \{x, y\}$, in which let $P_1, P_2, \ldots, P_q$ be $S_L T_L$-paths, and $P'_1, P'_2, \ldots, P'_{p-q}$ be $S_L U_L$-paths, where P'_i connects one vertex in S_L to u_i. By

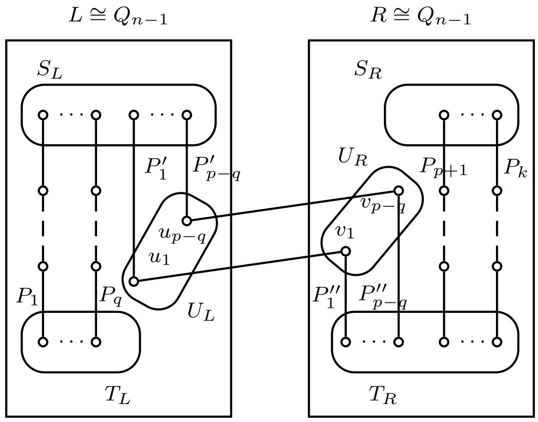

Fig. 2. An illustration for the proof of Subcase 1.2 of Lemma 2.5.

Lemma 2.3, in R there are $k-q$ disjoint S'_RT_R-paths containing all vertices in R, in which let $P''_1, P''_2, \ldots, P''_{p-q}$ be U_RT_R-paths, where P''_i connects v_i to one vertex in T_R, and $P_{p+1}, P_{p+2}, \ldots, P_k$ be S_RT_R-paths. Let (see Fig. 2)

$$P_{q+i} = P'_i + u_iv_i + P''_i \quad \text{for each } i = 1, 2, \ldots, p-q.$$

Then $P_1, P_2, \ldots, P_k$ are k disjoint ST-paths containing all vertices in $Q_n - \{x, y\}$.

Case 2. $p = k$.

By the hypothesis of $q \le k-1$, $T_R \ne \emptyset$, say $t_k \in T_R$. Let $S' = S - \{s_k\}$, $U' = U_L - \{u_{k-q}\}$ and $T' = T_L \cup U'$. If $n = 4$, then $p = k = 2$. We can choose s_2 such that $d(\{s_1, T'\}, \{x, y\}) \ne 2$. By Lemma 2.4, in L there exists an $S'T'$-path P_1 containing all vertices in $L - \{x, y\}$. If $n \ge 5$, by the induction hypothesis, there exist $k-1$ disjoint $S'T'$-paths containing all vertices in $L - \{x, y\}$. Without loss of generality, assume that $P_1, P_2, \ldots, P_q$ are $S'T_L$-paths and that P'_i is an $S'U'$-path connecting one vertex of S' to u_i for each $i \in \{1, 2, \ldots, k-1-q\}$.

Assume that s_k is in some path connecting s_i to some vertex t' in T'. Let u_L be the neighbor of s_k in the path closer to s_i and u_R be the neighbor of u_L in R, and let u_R be v_{k-q}. By Lemma 2.3, there are $k-q$ disjoint U_RT_R-paths $P''_1, P''_2, \ldots, P''_{k-1-q}, P''_{k-q}$ that contain all vertices in R, where P''_i connects v_i and t_{q+i} for each $i \in \{1, 2, \ldots, k-q\}$. Let

$$P_{q+i} = P'_i + u_iv_i + P''_i \quad \text{for each } i \in \{1, 2, \ldots, k-1-q\}.$$

If $t' \in T_L$ (say $t' = t_1$) and s_k in P_1 connects s_1 to t_1 (see Fig. 3(a)), let

$$P^*_1 = P_1(s_k, t_1),$$
$$P_k = P_1(s_1, u_L) + u_Lu_R + P''_{k-q}.$$

Then $P^*_1, P_2, \ldots, P_{k-1}, P_k$ are k disjoint ST-paths containing all vertices in $Q_n - \{x, y\}$.

If $t' \in U'$ (say $t' = u_1$) and s_k in P'_{q+1} connects s_{q+1} to u_1 (see Fig. 3(b)), let

$$P^*_{q+1} = P'_{q+1}(s_k, u_1) + u_1v_1 + P''_{q+1},$$
$$P_k = P'_{q+1}(s_{q+1}, u_L) + u_Lu_R + P''_{k-q}.$$

Then $P_1, \ldots, P_q, P^*_{q+1}, P_{q+2}, \ldots, P_{k-1}, P_k$ are k disjoint ST-paths containing all vertices in $Q_n - \{x, y\}$.

Summing up the above two cases, we complete the proof of the lemma. ■

3. Overview of the proofs

The proof proceeds by induction on the dimension n (≥ 2). The base of induction is already known. The case of $n = 2$ follows from Fink et al. [10], and the case of $k = n-1$ follows from Chen [1].

In induction step, we partition Q_n to two subcubes L and R with dimension $n-1$, such that every fault-free vertex in L(or R) has at least two fault-free neighbors in the subcube L (or R). With such a partition, the vertices of $S \cup T$ will distribute in the two subcubes, so do the faulty vertices. In the proof, we fully utilize the symmetry of L and R and the symmetry of S and T. Using vertex sets U_L and U_R as the transitive vertices, we construct the required many-to-many disjoint paths.

In Q_{n-1}, we first use induction to obtain many-to-many disjoint paths, and then we merge the disjoint paths in the two Q_{n-1}s. Unfortunately, sometimes we cannot use induction directly since there can be one or two excessive faulty vertices in Q_{n-1}. We have two methods to tackle this problem. One of them is to use the induction by putting the excessive faulty vertices as temporarily fault-free vertices. After obtaining the disjoint paths, if they contain temporarily fault-free vertices, we should replace the vertices by fault-free vertices, meanwhile we still preserve the paths containing many vertices. The other method is to put one or two vertices of $S \cup T$ as temporarily fault-free, using the induction to obtain $k-1$ disjoint paths and then add a new path connecting the temporarily fault-free vertices in $S \cup T$ to get the required k disjoint paths.

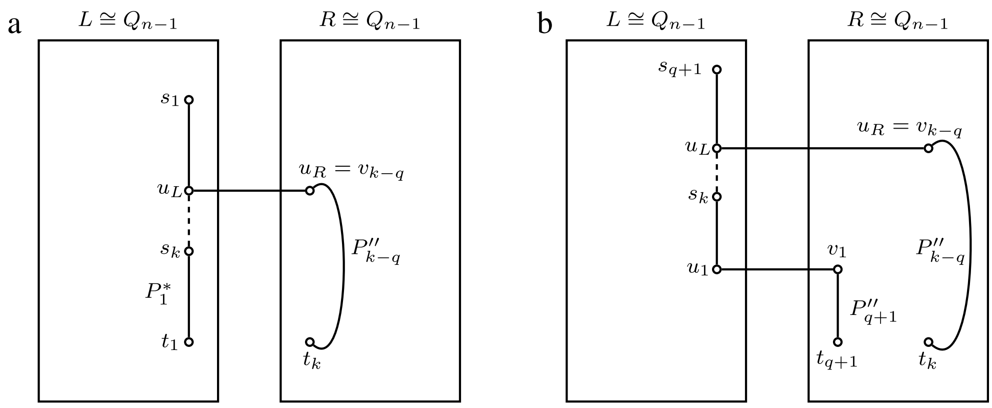

Fig. 3. Illustrations for the proof of Case 2 of Lemma 2.5.

4. Proof of theorem

The proof proceeds by induction on $n \geq 2$. If $k = 1$, then $f \leq 2n - 4$. The theorem follows from Lemma 2.2. If $k = n - 1$, then $f = 0$. The theorem follows from Lemma 2.3. Thus, the theorem holds for $2 \leq n \leq 3$. In the following discussion, we assume $n \geq 4$ and $2 \leq k \leq n - 2$.

Let Q_n be an n-dimensional hypercube, $\{X, Y\}$ be a bipartition of $V(Q_n)$, $S = \{s_1, s_2, \ldots, s_k\} \subset X$ and $T = \{t_1, t_2, \ldots, t_k\} \subset Y$ be any two sets of k fault-free vertices. Our aim is to construct k disjoint fault-free ST-paths containing at least $2^n - 2f$ vertices.

Since $k \geq 2$, we have $f \leq 2n - 6$. By Lemma 2.1, there exists some $j \in \{1, 2, \ldots, n\}$ such that each fault-free vertex in L_j (or R_j) has at least two fault-free neighbors in L_j (or R_j). When $k = n - 2$, $f \leq 2$, we choose some $j \in \{1, 2, \ldots, n\}$ such that all faulty vertices belong to either L_j or R_j. Let $Q_n = L \odot R$, where $L = L_j$ and $R = R_j$.

We construct the required k disjoint ST-paths by considering two cases according to whether $S \cup T$ is in L (or R) or not, respectively.

Case 1. $S \cup T \subseteq L$ or $S \cup T \subseteq R$.

In this case, both of the two sets S and T are in L or R. By the symmetry of L and R, we may assume that both S and T are in L. There are two subcases.

Subcase 1.1. $f_L \leq 2n - 2k - 4$.

Since $f_L \leq 2n - 2k - 4 = 2(n-1) - 2k - 2$ and $k \leq n - 2 = (n-1) - 1$, by the induction hypothesis, in L there are k disjoint ST-paths

$$P'_1, P'_2, \ldots, P'_k \tag{4.1}$$

containing at least $2^{n-1} - 2f_L$ vertices. Note that, when $n \geq 4$ and $f \leq 2n - 6$,

$$\begin{aligned}
\sum_{i=1}^{k} |E(P'_i)| &= \sum_{i=1}^{k} |V(P'_i)| - k \\
&\geq 2^{n-1} - 2f_L - k \\
&\geq 2^{n-1} - 2(2n - 2k - 4) - k \\
&> 2(2n - 6) \\
&\geq 2f.
\end{aligned}$$

There is an edge $u_L v_L$ in some path, say P'_1, such that both u_R and v_R are fault-free. Since $f_R \leq f \leq 2(n-1) - 4$, by Lemma 2.2, in R there is a $u_R v_R$-path P_R containing at least $2^{n-1} - 2f_R$ vertices. Let

$$\begin{aligned}
&P_1 = P'_1 - u_L v_L + u_L u_R + P_R + v_R v_L, \\
&P_i = P'_i \quad \text{for each } i \in \{2, 3, \ldots, k\}.
\end{aligned} \tag{4.2}$$

Then $P_1, P_2, \ldots, P_k$ are k disjoint ST-paths containing at least $2^n - 2f$ vertices in Q_n.

Subcase 1.2. $f_L = 2n - 2k - 3$.

In this case, $f_R \leq 1$ and all paths in (4.1) contain at most one faulty vertex since $f \leq 2n - 2k - 2$. If they contain no faulty vertex, then paths defined in (4.2) are as required. Assume that some path, say P'_1, contains a faulty vertex w and connects s_1 and t_1. Note that, in this case, paths in (4.1) contain at least $2^{n-1} - 2(f_L - 1)$ vertices in L. Let u_L and v_L be two neighbors of w in P'_1, where u_L is in $P'_1(s_1, w)$ and v_L is in $P'_1(w, t_1)$. Then $d_L(u_L, v_L) = 2$ since Q_n contains no triangles, so $d_R(u_R, v_R) = 2$.

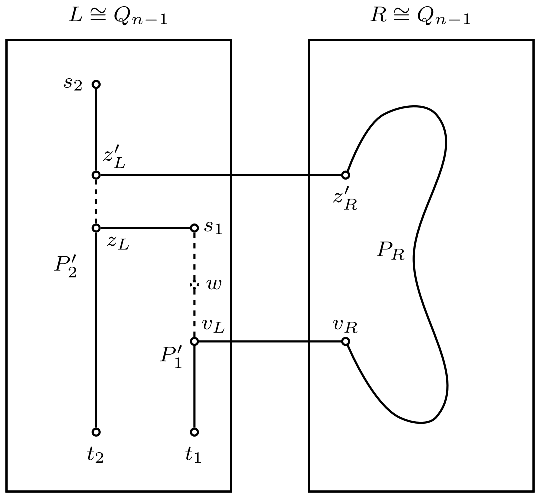

Fig. 4. An illustration for the proof of Subcase 1.2(c) of the theorem.

If both u_R and v_R are fault-free then, by Lemma 2.2, there is a fault-free u_Rv_R-path P_R containing at least $2^{n-1} - 2f_R - 1$ vertices in R. Let

$$P_1^* = P'_1(s_1, u_L) + u_Lu_R + P_R + v_Rv_L + P'_1(v_L, t_1).$$

Then, replacing P_1 in (4.2) by P_1^* yields k disjoint fault-free ST-paths with at least $2^n - 2f$ vertices in Q_n.

Now assume that one of u_R and v_R is faulty vertex, say u_R. Then u_R is the only faulty vertex in R since $f_R \le 1$.

If $u_L \ne s_1$, let z_L be the fault-free neighbor of u_L in $P'_1(s_1, u_L)$ (maybe $z_L = s_1$), then z_R is fault-free in R, and the distance between z_R and v_R is odd. By Lemma 2.2, there is a fault-free z_Rv_R-path P_R containing at least $2^{n-1} - 2f_R$ vertices in R. Let

$$P_1^* = P'_1(s_1, z_L) + z_Lz_R + P_R + v_Rv_L + P'_1(v_L, t_1).$$

Replacing P_1 in (4.2) by P_1^* yields k disjoint fault-free ST-paths with vertices at least

$$(2^{n-1} - 2(f_L - 1) - |\{w, u_L\}|) + (2^{n-1} - 2f_R) = 2^n - 2f.$$

We now assume $u_L = s_1$ and that u_R is the only faulty vertex in R. Note that u_L has at least two fault-free neighbors in L.

(a) If there is a fault-free neighbor z_L of $u_L(= s_1)$ not in P'_i for each $i = 1, 2, \ldots, k$, we consider two vertices z_R and v_R in R. They are fault-free in R since u_R is the only faulty vertex in R. Since the distance between z_L and v_L is odd, so is the distance between z_R and v_R. By Lemma 2.2, P_R is a z_Rv_R-path with at least $2^{n-1} - 2f_R$ vertices in R. Let

$$P_1^* = s_1z_L + z_Lz_R + P_R + v_Rv_L + P'_1(v_L, t_1).$$

Replacing P_1 in (4.2) by P_1^* yields k disjoint fault-free ST-paths with vertices at least

$$(2^{n-1} - 2(f_L - 1) - |\{w\}|) + (2^{n-1} - 2f_R) > 2^n - 2f.$$

(b) If all the fault-free neighbors of $u_L(= s_1)$ are in P'_1, then one of them, say z_L, is not t_1 since $u_L(= s_1)$ has at least two fault-free neighbors in L. Clearly, $z_L \ne v_L$ since Q_n contains no triangles. Let z'_L be the neighbor of z_L in $P'_1(z_L, t_1)$, then z'_R is fault-free in R. Since both v_L and z'_L are in X, the distance between v_R and z'_R is even. Let P_R be a $v_Rz'_R$-path with at least $2^{n-1} - 2f_R - 1$ vertices in R, and let

$$P_1^* = s_1z_L + P'_1(z_L, v_L) + v_Lv_R + P_R + z'_Rz'_L + P'_1(z'_L, t_1).$$

Replacing P_1 in (4.2) by P_1^* yields k disjoint fault-free ST-paths with vertices at least

$$(2^{n-1} - 2(f_L - 1) - |\{w\}|) + (2^{n-1} - 2f_R - 1) = 2^n - 2f.$$

(c) If there is a fault-free neighbor z_L of $u_L(= s_1)$ that is in some $P'_i(i \ne 1)$, without loss of generality, assume that z_L is in P'_2 and that P'_2 connects s_2 and t_2 (see Fig. 4). Then $z_L \ne s_2$ since $z_L \in Y$ and $s_2 \in X$. Let z'_L be the neighbor of z_L in $P'_2(s_2, z_L)$ (maybe $z'_L = s_2$), then z'_R is fault-free in R. Since both v_L and z'_L are in X, both v_R and z'_R are in Y, and so the distance between z'_R and v_R is even. Let P_R be a z'_Rv_R-path with at least $2^{n-1} - 2f_R - 1$ vertices in R, and let

$$P_1^* = s_1z_L + P'_2(z_L, t_2),\ P_2^* = P'_2(s_2, z'_L) + z'_Lz'_R + P_R + v_Rv_L + P'_1(v_L, t_1).$$

Replacing P_1 and P_2 in (4.2) by P_1^* and P_2^* yields k disjoint fault-free ST-paths with vertices at least

$$(2^{n-1} - 2(f_L - 1) - |\{w\}|) + (2^{n-1} - 2f_R - 1) = 2^n - 2f.$$

Summing up the above discussion, the theorem holds for $f_L = 2n - 2k - 4$.

Please cite this article in press as: X.-J. Li, et al., Many-to-many disjoint paths in hypercubes with faulty vertices, Discrete Applied Mathematics (2016), http://dx.doi.org/10.1016/j.dam.2016.09.013

Subcase 1.3. $f_L = 2n - 2k - 2$.

In this case, $f_R = 0$ since $f \le 2n - 2k - 2$. Let $S' = S - \{s_k\}$ and $T' = T - \{t_k\}$. Since $f_L = f = 2n - 2k - 2 = 2(n-1) - 2(k-1) - 2$, by the induction hypothesis, in L there are $(k-1)$ disjoint fault-free $S'T'$-paths

$$P'_1, P'_2, \ldots, P'_{k-1} \tag{4.3}$$

containing at least $2^{n-1} - 2f$ vertices. Let

$$K = V(P'_1 \cup P'_2 \cup \cdots \cup P'_{k-1}),$$

and let s_R and t_R be neighbors of s_k and t_k in R, respectively. Since s_k and t_k are in different partite sets in Q_n, $d_L(s_k, t_k)$ is odd, and so is $d_R(s_R, t_R)$.

If $|\{s_k, t_k\} \cap K| = 0$, since $d_R(s_R, t_R)$ is odd, by Lemma 2.2, there is an s_Rt_R-path P_R containing 2^{n-1} vertices in R. Let

$$\begin{aligned} &P_i = P'_i, \quad i = 1, 2, 3, \ldots, k-1, \\ &P_k = s_k s_R + P_R + t_R t_k. \end{aligned} \tag{4.4}$$

Then $P_1, \ldots, P_{k-1}, P_k$ are k disjoint fault-free ST-paths containing at least $2^n - 2f$ vertices, as required.

If $|\{s_k, t_k\} \cap K| = 1$, without loss of generality, assume $s_k \in K$ and $t_k \notin K$. We can further assume that s_k is in P'_1 with two end-vertices s_1 and t_1. Let x_L be the neighbor of s_k in $P'_1(s_1, s_k)$, and t_R be the neighbor of t_k in R. Since both x_R and t_R are in X and $f_R = 0$, there is a fault-free x_Rt_R-path P_R containing $2^{n-1} - 1$ vertices. Let

$$\begin{aligned} &P^*_1 = P'_1(s_1, x_L) + x_L x_R + P_R + t_R t_k \\ &P^*_k = P'_1(s_k, t_1). \end{aligned}$$

Replacing P_1 and P_k in (4.4) by P^*_1 and P^*_k yields k disjoint fault-free ST-paths containing vertices at least

$$(2^{n-1} - 2f + |\{t_k\}|) + (2^{n-1} - 1) = 2^n - 2f.$$

The remaining case is $|\{s_k, t_k\} \cap K| = 2$, which contains two cases.

(a) Both s_k and t_k are in the same path in (4.3), say P'_1. Let x_L and y_L be two neighbors of s_k and t_k in P'_1 but not in the subpath $P'_1(s_k, t_k)$, respectively. Since $d_L(s_k, t_k)$ is odd, both $d_L(x_L, y_L)$ and $d_R(x_R, y_R)$ are odd. By Lemma 2.2, there is an x_Ry_R-path P_R containing 2^{n-1} vertices in R. Let

$$\begin{aligned} &P^*_1 = P'_1(s_1, x_L) + x_L x_R + P_R + y_R y_L + P'_1(y_L, t_1), \\ &P^*_k = P'_1(s_k, t_k). \end{aligned}$$

Replacing P_1 and P_k in (4.4) by P^*_1 and P^*_k yields k disjoint fault-free ST-paths containing at least $2^n - 2f$ vertices.

(b) Both s_k and t_k are in different paths in (4.3). Without loss of generality, suppose that s_k is in P'_1, t_k is in P'_2 and the two end-vertices of P'_i are s_i and t_i for $i = 1, 2$. Let x_L be the neighbor of s_k in $P'_1(s_1, s_k)$, and y_L be the neighbor of t_k in $P'_2(t_k, t_2)$. Since $d_L(s_k, t_k)$ is odd, both $d_L(x_L, y_L)$ and $d_R(x_R, y_R)$ are odd. By Lemma 2.2, there is an x_Ry_R-path P_R containing 2^{n-1} vertices in R. Let

$$\begin{aligned} &P^*_1 = P'_1(s_1, x_L) + x_L x_R + P_R + y_R y_L + P'_2(y_L, t_2), \\ &P^*_2 = P'_2(s_2, t_k), \\ &P^*_k = P'_1(s_k, t_1). \end{aligned}$$

Replacing P_1, P_2 and P_k in (4.4) by P^*_1, P^*_2 and P^*_k yields k disjoint fault-free ST-paths containing at least $2^n - 2f$ vertices.

Summing up the above discussion, the theorem holds when $S \cup T \subseteq L$ or $S \cup T \subseteq R$.

Case 2. $(S \cup T) \cap L \ne \emptyset$ and $(S \cup T) \cap R \ne \emptyset$.

Let (see Fig. 5)

$$\begin{aligned} &S_L = S \cap L, \qquad T_L = T \cap L \quad \text{and} \quad p = |S_L|, \qquad q = |T_L|, \\ &S_R = S \cap R, \qquad T_R = T \cap R. \end{aligned}$$

By the symmetry of L and R, we assume $f_L \le f_R$. By the symmetry of S and T, assume that $p \ge q$.

Let W_L be the set of neighbors of vertices of S_R in L. Then $W_L \subset Y$ since $S \subset X$. For $n \ge 4$ and $k \ge 2$, we have

$$\begin{aligned} |Y \cap V(L)| - |T_L \cup W_L \cup F| &\ge 2^{n-2} - q - (k-p) - f \\ &\ge 2^{n-2} - q - (k-p) - (2n - 2k - 3) \\ &= (2^{n-2} - 2n + k + 3) + (p - q) \\ &> p - q. \end{aligned}$$

This implies that there is a set U_L of $(p-q)$ fault-free vertices in $Y \cap V(L - T_L)$ such that its neighbor-set U_R in R is in $X \cap V(R - S_R)$ and is fault-free. If $p > q$, let

$$U_L = \{u_1, u_2, \ldots, u_{p-q}\} \quad \text{and} \quad U_R = \{v_1, v_2, \ldots, v_{p-q}\}, \tag{4.5}$$

be any two such vertex-sets, where $u_iv_i \in E(Q_n)$ for each $i = 1, 2, \ldots, p-q$. If $p = q$, let $U_L = U_R = \emptyset$. There are two cases. One is $1 \le q \le k-1$ or $q = 0$ and $f_R \le 2n - 2k - 4$, and the other one is $q = 0$ and $f_R \ge 2n - 2k - 3$.

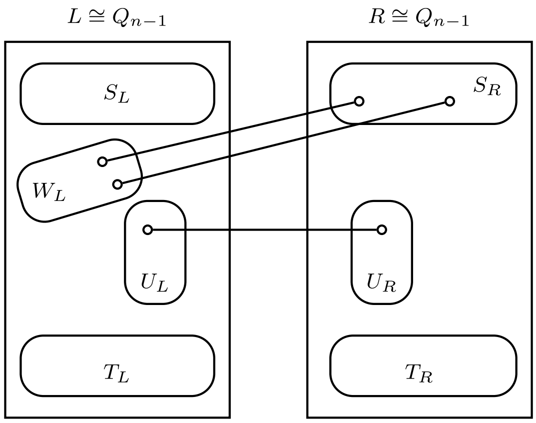

Fig. 5. An illustration for the proof of Case 2 of the theorem.

Subcase 2.1. $1 \le q \le k-1$ or $q = 0$ and $f_R \le 2n-2k-4$.

Since $f_L \le f_R$, then $f_L \le \frac{1}{2}f \le \frac{1}{2}(2n-2k-2) \le 2(n-1)-2k-2$ for $k \le n-3$ and $f_L \le f_R = 0$ for $k = n-2$. By the induction hypothesis, in L there are $p(\le k)$ disjoint fault-free paths

$$P'_1, \ldots, P'_{p-q}, P_{p-q+1}, \ldots, P_p$$

connecting S_L and $T_L \cup U_L$ and containing at least $2^{n-1} - 2f_L$ vertices, where P'_i connects some vertex in S_L to the vertex u_i in U_L for each $i = 1, 2, \ldots, p-q$, provided that $p > q$.

If $q \ge 1$, then $f_R \le f \le 2n-2k-2 \le 2(n-1)-2(k-q)-2$ and $k-q \le k-1 \le n-2$. If $q = 0$ and $f_R \le 2n-2k-4$, then $f_R \le 2(n-1)-2k-2$, $k \le n-2$. In any case, by the induction hypothesis, in R there are $k-q$ disjoint fault-free paths

$$P''_1, \ldots, P''_{p-q}, P_{p+1}, \ldots, P_k$$

connecting $S_R \cup U_R$ and T_R and containing at least $2^{n-1} - 2f_R$ vertices, where P''_i connects the vertex v_i in U_R to some vertex in T_R for each $i = 1, 2, \ldots, p-q$, provided that $p > q$. Let (see Fig. 2)

$$P_i = P'_i + u_iv_i + P''_i, \quad i = 1, 2, \ldots, p-q.$$

Then the k paths $P_1, P_2, \ldots, P_{p-q}, P_{p-q+1}, \ldots, P_p, P_{p+1}, \ldots, P_k$ satisfy our requirements.

Subcase 2.2. $q = 0$ and $f_R \ge 2n-2k-3$.

In this case, $T_R = T$ and $f_L \le 1$. We can write (4.5) as

$$U_L = \{u_1, u_2, \ldots, u_p\} \quad \text{and} \quad U_R = \{v_1, v_2, \ldots, v_p\}. \tag{4.6}$$

Since $f_L \le 1 < 2(n-1)-2(n-3)-2$ and $f_L = 0$ for $k = n-2$, by the induction hypothesis, in L there are $p(\le k)$ disjoint fault-free S_LU_L-paths

$$P'_1, P'_2, \ldots, P'_p \tag{4.7}$$

containing at least $2^{n-1} - 2f_L$ vertices. Without loss of generality, assume that P'_i connects s_i to u_i for each $i = 1, 2, \ldots, p$ (see Fig. 6).

Since $f_R \le 2n-2k-2 = 2(n-1)-2(k-1)-2$, by the induction hypothesis, in R there are $k-1$ disjoint fault-free paths

$$P''_2, \ldots, P''_p, P_{p+1}, \ldots, P_k \tag{4.8}$$

connecting $S_R \cup U_R - \{v_1\}$ and $T - \{t_1\}$ and containing at least $2^{n-1} - 2f_R$ vertices in R, where P''_i connects the vertex v_i in U_R and some vertex in $T - \{t_1\}$ for each $i = 2, 3, \ldots, p$ (see Fig. 6). Then

$$\begin{aligned} &P_1 = P'_1, \\ &P_i = P'_i + u_iv_i + P''_i, \quad i = 2, 3, \ldots, p \\ &P_i, \quad i = p+1, \ldots, k \end{aligned} \tag{4.9}$$

are k disjoint fault-free paths between S and $(T - \{t_1\}) \cup \{u_1\}$ containing at least $2^n - 2f$ vertices in Q_n. Without loss of generality, assume that P_i connects s_i to t_i for each $i = 2, 3, \ldots, k$ and P_1 connects s_1 to u_1 (see Fig. 6). Let

$$B = V(P_2 \cup \cdots \cup P_k).$$

Note if $p = 1$, P'_i in (4.7) and P''_i in (4.8) are empty for $i \ge 2$. Since each vertex in T has a unique neighbor in L, we can choose $t_1 \in T$ such that $s_1t_1 \notin E(Q_n)$ when $p = 1$.

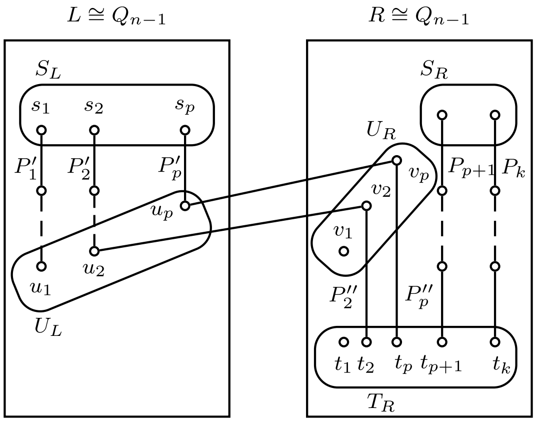

Fig. 6. An illustration for the proof of Subcase 2.2 of the theorem.

We consider two subcases according to whether t_1 is in B or not.

Subcase 2.2a. $t_1 \notin B$.

Let t_L be the neighbor of t_1 in L. Suppose that t_L is fault-free.

Assume $t_L \in S$, say $t_L = s_1$. By our assumption $s_1t_1 \notin E(Q_n)$ if $p = 1$, and so $p \geq 2$. We put t_L as a temporarily faulty vertex. Let $S' = S_L - \{s_1\}$ and $U' = U_L - \{u_1\}$. Then L contains at most two faulty vertices; $p-1$ disjoint fault-free $S'U'$-paths $P'_2, \dots, P'_p$ shown in (4.7) contain at least $2^{n-1} - 2(f_L + 1)$ vertices; $P_2, P_3, \dots, P_k$ in (4.9) are $k-1$ disjoint paths containing at least $2^n - 2f - 2$ vertices. Together with the path $P_1 = s_1t_1$, we obtain k paths satisfying the requirements.

Assume now $t_L \notin S$. Since $f_L \leq 1$, we have $|F_L \cup (U_L - \{u_1\})| \leq k \leq n-3$. Thus, t_L has at least a fault-free neighbor $z \in V(L) - (U_L - \{u_1\})$. Since $z \in Y$, we can choose u_1 of U_L as z in (4.6). We put t_L as a temporarily faulty vertex. Then L contains at most two faulty vertices. The p disjoint fault-free S_LU_L-paths shown in (4.7) contain at least $2^{n-1} - 2(f_L + 1)$ vertices. Let

$$P_1^* = P_1 + u_1t_L + t_Lt_1.$$

Replacing P_1 in (4.9) by P_1^* yields k disjoint fault-free ST-paths in Q_n. Note that t_1 and t_L are two new vertices. Then the k paths $P_1^*, P_2, \dots, P_k$ satisfy our requirements.

Now suppose that t_L is a faulty vertex. Then t_L is the only faulty vertex in L. Take a fault-free neighbor, say w_R of t_1 in R. Then its neighbor w_L in L is fault-free and in Y. If $w_R \notin B$, then choose $u_1 = w_L \in U_L$ in (4.6). Let

$$P_1^* = P_1 + u_1w_R + w_Rt_1.$$

Replacing P_1 in (4.9) by P_1^* yields k disjoint fault-free ST-paths in Q_n, as required. Assume that w_R is in some $P_i (2 \leq i \leq k)$, say in $P_2(s_2, t_2)$. Then $w_R \neq t_2$ since w_R and t_2 are in different partite sets. Let u_R be the neighbor of w_R in $P_2(w_R, t_2)$ and u_L be the neighbor of u_R in L. Then $u_L \in X$.

Assume $u_L \in S$, say $u_L = s_1$. If $p = 1$, we can choose another w_R such that $u_L \neq s_1$. So assume $p \geq 2$. We put s_1 as a temporarily faulty vertex. Let $S' = S_L - \{s_1\}$ and $U' = U_L - \{u_1\}$. Then L contains at most two faulty vertices; there are $p-1$ disjoint fault-free $S'U'$-paths $P'_2, \dots, P'_p$ containing at least $2^{n-1} - 2(f_L + 1)$ vertices; $P_2, P_3, \dots, P_k$ in (4.9) are $k-1$ disjoint paths containing at least $2^n - 2f - 2$ vertices. Let

$$P_1 = s_1u_R + P_2(u_R, t_2),$$
$$P_2^* = P_2(s_2, w_R) + w_Rt_1.$$

Thus $P_1, P_2^*, P_3, \dots, P_k$ are k disjoint fault-free ST-paths in Q_n, where t_1 and s_1 are two new vertices, as required.

Assume now $u_L \notin S$. Choose a fault-free neighbor z_L of u_L in L such that it is not any u_i in U_L for $2 \leq i \leq p$. Then $z_L \in Y$. So choose $u_1 = z_L \in U_L$ in (4.6). We put u_L as a temporarily faulty vertex. Then L contains at most two faulty vertices. The k disjoint fault-free SU_L-paths shown in (4.7) contain at least $2^{n-1} - 2(f_L + 1)$ vertices. Let

$$P_1^* = P_1 + z_Lu_L + u_Lu_R + P_2(u_R, t_2),$$
$$P_2^* = P_2(s_2, w_R) + w_Rt_1.$$

Replacing P_1 and P_2 in (4.9) by P_1^* and P_2^* yields k disjoint fault-free ST-paths in Q_n, where t_1 and u_L are two new vertices, as required.

Subcase 2.2b. $t_1 \in B$.

In this case, without loss of generality, we assume t_1 is in P_2 that connects s_2 to t_2. Let u_R and v_R be two neighbors of t_1 in P_2 and let u_L and v_L be two neighbors of u_R and v_R in L, respectively. Then both u_L and v_L are in Y. Without loss of generality, assume that v_R is in $P_2(s_2, t_1)$ and u_R is in $P_2(t_1, t_2)$.

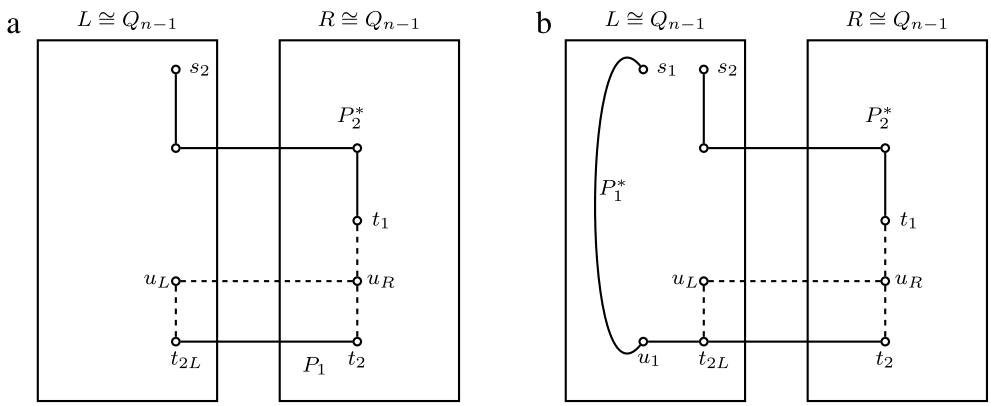

Fig. 7. Illustrations for the proof of Subcase 2.2b of the theorem.

If u_L is fault-free, then choose $u_1 = u_L$ in U_L in (4.6). Let

$$P_1^* = P_2(s_2, t_1),$$
$$P_2^* = P_1 + u_L u_R + P_2(u_R, t_2).$$

Replacing P_1 and P_2 in (4.9) by P_1^* and P_2^* yields k disjoint fault-free ST-paths in Q_n, as required.

Assume now that u_L is faulty vertex. Since $f \le 2n - 2k - 2$, $f_R = 2n - 2k - 3$. If $d(u_R, t_2) > 1$, we can choose t_2 instead of t_1, which contradicts the assumption that u_L is a faulty vertex. Thus we have $d(u_R, t_2) = 1$.

Let t_{2L} be the neighbor of t_2 in L. Then we have $u_L t_{2L} \in E(Q_n)$.

Assume $t_{2L} \in S_L$, say $t_{2L} = s_1$ (see Fig. 7(a)). By the assumption $s_1 t_1 \notin E(Q_n)$ if $p = 1$. Thus assume $p > 1$. Let $S' = S - \{s_1\}$ and $U' = U_L - \{u_1\}$. Applying Lemma 2.5 and choosing xy to be $u_L t_{2L}$, there are $p - 1$ $(\le k)$ disjoint $S'U'$-paths

$$P_2', \ldots, P_p' \tag{4.10}$$

that contain all vertices in $L - \{u_L, t_{2L}\}$, say P_i' connects s_i to u_i for each $i = 2, 3, \ldots, p$. Combining (4.8) with (4.10), we obtain $k - 1$ disjoint fault-free paths between S' and $T - \{t_1\}$

$$\begin{array}{l} P_i = P_i' + u_i v_i + P_i'', \quad i = 2, 3, \ldots, p \\ P_i, \quad i = p+1, \ldots, k \end{array} \tag{4.11}$$

that contain at least $(2^{n-1} - 2) + (2^{n-1} - 2f_R) = 2^n - 2f$ vertices in Q_n. Let

$$P_1 = s_1 t_2 \quad \text{and} \quad P_2^* = P_2(s_2, t_1).$$

Then the paths $P_1, P_2^*, P_3, \ldots, P_k$ satisfy the requirements.

Assume now $t_{2L} \notin S_L$. Let $z \in N_{Q_n}(t_{2L}) \cap L - \{u_L\} - U'$ (see Fig. 7(b)). Choose $u_1 = z$ in U_L in (4.6). Applying Lemma 2.5 and choosing xy to be $u_L t_{2L}$, there are p $(\le k)$ disjoint $S_L U_L$-paths

$$P_1', P_2', \ldots, P_p' \tag{4.12}$$

that contain all vertices in $L - \{u_L, t_{2L}\}$, say P_i' connects s_i to u_i for each $i = 1, 2, \ldots, p$. Combining (4.8) with (4.12), we obtain k disjoint fault-free paths between S and $(T - \{t_1\}) \cup \{u_1\}$

$$\begin{array}{l} P_1 = P_1', \\ P_i = P_i' + u_i v_i + P_i'', \quad i = 2, 3, \ldots, p \\ P_i, \quad i = p+1, \ldots, k \text{ (defined in (4.8))} \end{array} \tag{4.13}$$

that contain at least $(2^{n-1} - 2) + (2^{n-1} - 2f_R) = 2^n - 2f$ vertices in Q_n. Let

$$P_1^* = P_1 + u_1 t_{2L} + t_{2L} t_2 \quad \text{and} \quad P_2^* = P_2(s_2, t_1).$$

Replacing P_1 and P_2 in (4.13) by P_1^* and P_2^* yields k disjoint fault-free ST-paths in Q_n. We remove one vertex u_R from paths in (4.11) and add one vertex t_{2L} to obtain new paths. Thus, these paths still contain at least $2^n - 2f$ vertices in Q_n, as required.

The theorem follows.

5. A constructive algorithm for disjoint paths

In this section, we propose an effective algorithm to find the many-to-many disjoint paths in faulty hypercubes. Before offering our algorithm, we first recall several existing algorithms.

Theorem 5.1 *(See Fink et al. [10], Dvořák et al. [9]). Let Q_n $(n \geq 2)$ be an n-dimensional hypercube with faulty vertex set F of size at most $2n-4$ and every vertex in Q_n has at least two fault-free neighbors, then there exists an $O(2^n)$-algorithm* $\mathtt{LongP}(u, v, F, n)$ *to find a long fault-free path connecting u and v.*

Theorem 5.2. *For any integer k with $1 \leq k \leq n-1$ and for any two sets S and T of k fault-free vertices in different partite sets in Q_n $(n \geq 2)$, there exists an $O(2^n)$ algorithm* $\mathtt{MDPChen}(S, T, n)$ *to find k disjoint fault-free ST-paths which contain all vertices in Q_n.*

Proof. Let $T(n)$ be the maximum time to implement $\mathtt{MDPChen}(S, T, n)$. We have $T(n) \leq T(n-1) + O(2^{n-1})$ in Case 1, and $T(n) \leq 2T(n-1) + O(1)$ in other Cases (See Chen [1]). Note that $T(2) = O(1)$. Thus $T(n) \leq O(2^n)$. ■

As Lemma 2.5 establishes the existence of disjoint paths with much vertices, the procedure MDPE is the method to find such paths. This is the following theorem.

Theorem 5.3. *Let x and y be two adjacent vertices of Q_n $(n \geq 3)$, Then, for any integer k with $1 \leq k \leq n-2$, and any two sets S and T of k vertices in different partite sets in $Q_n - \{x, y\}$, there exists an $O(2^n)$-algorithm* $\mathtt{MDPE}(S, T, xy, n)$ *to find k disjoint ST-paths containing all vertices in $Q_n - \{x, y\}$.*

Proof. Let $T(n)$ be the maximum time to run procedure $\mathtt{MDPE}(S, T, xy, n)$. In the constructive proof of Lemma 2.5, we implement MDPE in the subcube L, and call procedure MDPChen in R. By Theorem 5.2, we have $T(n) \leq T(n-1) + O(2^{n-1})$. Note that $T(3) = O(1)$. Thus $T(n) \leq O(2^n)$. ■

Algorithm 1 $\mathtt{MDPMain}(S, T, F, n)$

Input: $S = \{s_1, s_2, \ldots, s_k\}$, $T = \{t_1, t_2, \ldots, t_k\}$, F, n.
Output: $(P_1, P_2, \ldots, P_k)$
if $k = n-1$ **then**
  $(P_1, P_2, \ldots, P_k) \leftarrow \mathtt{MDPChen}(S, T, n)$;
end if
Choose some proper $j \in \{1, 2, \ldots, n\}$ to partition Q_n as $L_j \odot R_j$, $L \leftarrow L_j$, $R \leftarrow R_j$;
if $S \cup T \subseteq L$ or $S \cup T \subseteq R$, say the former one **then**
  $(P_1, P_2, \ldots, P_k) \leftarrow \mathtt{MDP1}(S, T, F, n)$;
else // $(S \cup T) \cap L \neq \emptyset$ and $(S \cup T) \cap R \neq \emptyset$
  $(P_1, P_2, \ldots, P_k) \leftarrow \mathtt{MDP2}(S, T, F, n)$;
end if

Algorithm 2 $\mathtt{MDP1}(S, T, F, n)$

Input: $S = \{s_1, s_2, \ldots, s_k\}$, $T = \{t_1, t_2, \ldots, t_k\}$ such that $S \cup T \subseteq L$, F, n.
Output: $(P_1, P_2, \ldots, P_k)$
if $f_L \leq 2n - 2k - 4$ **then**
  $(P'_1, P'_2, \ldots, P'_k) \leftarrow \mathtt{MDPMain}(S, T, F_L, n-1)$;
  Choose z_R and v_R in R, $P_R \leftarrow \mathtt{LongP}(z_R, v_R, F_R, n-1)$, $\mathtt{Merge}$;
else if $f_L = 2n - 2k - 3$ **then**
  Choose a faulty vertex w to be temporarily fault-free, $(P'_1, P'_2, \ldots, P'_k) \leftarrow \mathtt{MDPMain}(S, T, F_L, n-1)$;
  Choose z_R and v_R in R, $P_R \leftarrow \mathtt{LongP}(z_R, v_R, F_R, n-1)$, $\mathtt{Merge}$;
else // $f_L = 2n - 2k - 2$
  $(P'_1, P'_2, \ldots, P'_{k-1}) \leftarrow \mathtt{MDPMain}(S - \{s_k\}, T - \{t_k\}, F_L, n-1)$;
  Choose x_R and y_R in R, $P_R \leftarrow \mathtt{LongP}(x_R, y_R, F_R, n-1)$, $\mathtt{Merge}$;
end if

Theorem 5.4. *Let Q_n $(n \geq 2)$ be an n-dimensional hypercube with faulty vertex set F of size f, and let k be an integer with $1 \leq k \leq n-1$. If $f \leq 2n - 2k - 2$ and each fault-free vertex has at least two fault-free neighbors then, for any two sets S and T of k fault-free vertices in different partite sets, there exists an $O(2^n)$-algorithm* $\mathtt{MDPMain}(S, T, F, n)$ *to construct the k disjoint fault-free paths linking S and T in Q_n which contain at least $2^n - 2f$ vertices.*

Proof. We offer an efficient algorithm, MDPMain(Algorithm 1), to find many-to-many disjoint paths in hypercubes with faulty vertices. In implementing MDPMain, it may be called procedure MDPChen (Theorem 5.2), MDP1 (Algorithm 2) or MDP2 (Algorithm 3).

After obtaining the disjoint paths in L and R, one needs to incorporate the paths and avoid some temporarily fault-free vertices at the same time. We call this procedure Merge.

Now we analyze the time complexity of algorithm MDPMain. Let $T(n)$ be the maximum time to run procedure $\mathtt{MDPMain}(S, T, F, n)$. Then $T(2) = O(1)$.

If $k = n-1$, implement MDPChen in Q_n. By Theorem 5.2, we have

$$T(n) \leq O(2^n). \tag{5.14}$$

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Algorithm 3 $\mathtt{MDP2}(S, T, F, n)$

Input: $S = \{s_1, s_2, \ldots, s_k\}$, $T = \{t_1, t_2, \ldots, t_k\}$ such that $(S \cup T) \cap L \neq \emptyset$ and $(S \cup T) \cap R \neq \emptyset$, F, n.
Output: $(P_1, P_2, \ldots, P_k)$

$S_L \leftarrow S \cap L$, $T_L \leftarrow T \cap L$, $p \leftarrow |S_L|$, $q \leftarrow |T_L|$, $S_R \leftarrow S \cap R$, $T_R \leftarrow T \cap R$;
Choose L and R such that $f_L \leq f_R$, Choose S and T such that $p \geq q$, Find proper vertex set U_L and U_R;
if $1 \leq q \leq k-1$ or $q = 0$ and $f_R \leq 2n - 2k - 4$ **then**
  $(P'_1, \ldots, P'_{p-q}, P_{p-q+1}, \ldots, P_p) \leftarrow \mathtt{MDPMain}(S_L, T_L \cup U_L, F_L, n-1)$;
  $(P''_1, \ldots, P''_{p-q}, P_{p+1}, \ldots, P_k) \leftarrow \mathtt{MDPMain}(S_R \cup U_R, T_R)$, Merge;
else // $q = 0$ and $f_R \geq 2n - 2k - 3$
  if $p = 1$ **then**
    Choose t_1 in T such that $s_1 t_1 \notin E(Q_n)$;
  end if
  $(P''_2, \ldots, P''_p, P_{p+1}, \ldots, P_k) \leftarrow \mathtt{MDPMain}(S_R \cup U_R - \{v_1\}, T - \{t_1\}, F_R, n-1)$;
  $B \leftarrow V(P''_2, \ldots, P''_p, P_{p+1}, \ldots, P_k)$;
  if $t_1 \notin B$ **then**
    $t_L \leftarrow N(t_1) \cap L$;
    if t_L is fault-free **then**
      if $t_L \in S$, say $t_L = s_1$; **then**
        $S' \leftarrow S_L - \{s_1\}$, $U' \leftarrow U_L - \{u_1\}$, $(P'_2, \ldots, P'_p) \leftarrow \mathtt{MDPMain}(S', U', F_L \cup \{t_L\}, n-1)$,
        $P_1 \leftarrow s_1 t_1$, Merge;
      else // $t_L \notin S$
        $(P'_1, \ldots, P'_p) \leftarrow \mathtt{MDPMain}(S_L, U_L, F_L \cup \{t_L\}, n-1)$, $P_1 \leftarrow P'_1 + u_1 t_L + t_L t_1$, Merge;
      end if
    else // t_L is a faulty vertex;
      Choose a fault-free vertex w_R in $N(t_1) \cap R$, $w_L \leftarrow N(w_R) \cap L$;
      if $w_R \notin B$ **then**
        $u_1 \leftarrow w_L$, $(P'_1, P'_2, \ldots, P_p) \leftarrow \mathtt{MDPMain}(S_L, U_L, F_L, n-1)$, $P_1 \leftarrow P'_1 + u_1 t_L + t_L t_1$, Merge;
      else // $w_R \in B$, say $w_R \in P_2(s_2, t_2)$
        $u_R \leftarrow N(w_R) \cap P_2(w_R, t_2)$, $u_L \leftarrow u_R \cap L$;
        if $u_L \in S$, say $u_L = s_1$ **then**
          $S' \leftarrow S_L - \{s_1\}$, $U' \leftarrow U_L - \{u_1\}$, $(P'_1, P'_2, \ldots, P_p) \leftarrow \mathtt{MDPMain}(S', U', F_L \cup \{s_1\}, n-1)$;
          Merge, $P_1 \leftarrow s_1 u_R + P_2(u_R, t_2)$, $P_2 \leftarrow P_2(s_2, w_R) + w_R t_1$;
        else // $u_L \notin S$;
          Choose a fault-free vertex u_1 in $N(u_L) \cap L$,
          $(P'_1, P'_2, \ldots, P_p) \leftarrow \mathtt{MDPMain}(S_L, U_L, F_L \cup \{u_L\}, n-1)$, Merge;
          $P_1 \leftarrow P_1 + z_L u_L + u_L u_R + P_2(u_R, t_2)$, $P_2 \leftarrow P_2(s_2, w_R) + w_R t_1$;
        end if
      end if
    end if
  else // $t_1 \in B$, Say $t_1 \in P_2$
    $u_R \leftarrow N(t_1) \cap P_2(t_1, t_2)$, $v_R \leftarrow N(t_1) \cap P_2(s_2, t_1)$;
    $u_L \leftarrow N(u_R) \cap L$, $v_L \leftarrow N(v_R) \cap L$;
    if u_L is fault-free **then**
      $u_1 \leftarrow u_L$, $(P'_1, P'_2, \ldots, P_p) \leftarrow \mathtt{MDPMain}(S_L, U_L, F_L, n-1)$, Merge;
      $P_1 \leftarrow P_2(s_2, t_1)$, $P_2 \leftarrow P_1 + u_L u_R + P_2(u_R, t_2)$;
    else // u_L is fault vertex, $d(u_R, t_2) = 1$.
      $t_{2L} \leftarrow t_2 \cap L$;
      if $t_{2L} \in S_L$ **then**
        $S' \leftarrow S - \{s_1\}$, $U' \leftarrow U_L - \{u_1\}$;
        $(P'_2, \ldots, P'_p) \leftarrow \mathtt{MDPE}(S', U', u_L t_{2L}, n-1)$; // Using Lemma2.5
        Merge, $P_1 \leftarrow s_1 t_1$, $P_2 \leftarrow P_2(s_2, t_1)$;
      else // $t_{2L} \notin S_L$
        Choose $u_1 \leftarrow N_{Q_n}(t_{2L}) \cap L - \{u_L\} - U'$;
        $(P'_1, P'_2, \ldots, P'_p) \leftarrow \mathtt{MDPE}(S', U', u_L t_{2L}, n-1)$; // Using Lemma2.5
        Merge, $P_1 \leftarrow P_1 + u_1 t_{2L} + t_{2L} t_2$, $P_2 \leftarrow P_2(s_2, t_1)$;
      end if
    end if
  end if
end if

We take $O(n)$ time to partition Q_n in a suitable way. If $\mathtt{MDP1}$ (Algorithm 2) is implemented, then $\mathtt{MDPMain}$ in L takes time $T(n-1)$; choosing two proper vertices in the paths of L takes time $O(n)$; $\mathtt{LongP}$ in R takes time $O(2^{n-1})$. Thus

$$T(n) \leq O(n) + T(n-1) + O(n) + O(2^{n-1}). \tag{5.15}$$

If $\mathtt{MDP2}$ (Algorithm 3) is implemented, then in Subcase 2.1 and Subcase 2.2a,

$$T(n) \leq 2T(n-1) + O(n); \tag{5.16}$$

in Subcase 2.2b, $\mathtt{MDPMain}$ in L takes time $T(n-1)$; $\mathtt{MDPE}$ in R takes time $O(2^{n-1})$. Thus

$$T(n) \leq T(n-1) + O(n) + O(2^{n-1}). \tag{5.17}$$

Then by (5.14)–(5.17), we have $T(n) \leq O(2^n)$. ■

Please cite this article in press as: X.-J. Li, et al., Many-to-many disjoint paths in hypercubes with faulty vertices, Discrete Applied Mathematics (2016), http://dx.doi.org/10.1016/j.dam.2016.09.013

We should point out that LongP cannot solve k disjoint-path problem with $k \geq 2$ and MDPChen does not work if $f \geq n-k$. Comparing to this, our algorithm is an effective procedure to find many-to-many disjoint paths in hypercubes with more faulty vertices.

When $f \leq 2n-2k-2$, our algorithm can solve k disjoint-path problem. The total running time is bounded by $O(2^n)$, which is a linear function of the number of the vertices in Q_n. Since our algorithm is a centralized one, when n is very large, our algorithm cannot find the many-to-many disjoint paths in a reasonable amount of time. It could be possible to modify our algorithm to a distributed one and find the many-to-many paths in a relatively short time.

When the number of faulty vertices exceeds $2n-2k-2$, our algorithm may not find the many-to-many disjoint paths. The total number and distribution of the faulty vertices will affect the running time of our algorithm. But under the condition that $f \leq 2n-2k-2$, this effect is not significant.

6. Conclusion

In this paper, we consider the problem of many-to-many disjoint paths in the hypercube Q_n with f faulty vertices. We prove that for any integer k with $1 \leq k \leq n-1$ and any two sets S and T of k fault-free vertices in different parts of Q_n $(n \geq 3)$, if $f \leq 2n-2k-2$ and each fault-free vertex has at least two fault-free neighbors, then there exist k fully disjoint fault-free paths linking S and T which contain at least $2^n - 2f$ vertices. Weakening the condition that each fault-free vertex has at least two fault-free neighbors is an interesting future research question. We also purpose a centralized $O(2^n)$-algorithm for finding many-to-many disjoint paths in Q_n, whose running time is not greatly affected by the faulty vertices. When n is very large, our algorithm cannot find the paths required in a reasonable time. It could be possible to modify our algorithm to a distributed one and find the paths in a relatively less time.

Acknowledgments

The authors would like to express their gratitude to the anonymous referees for their kind suggestions and comments on the original manuscript, which greatly improve the quality of this paper.

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