



Many-to-many disjoint paths in hypercubes with faulty vertices[☆]



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ABSTRACT

This paper considers the problem of many-to-many disjoint paths in the hypercube Q_n with f faulty vertices and obtains the following result. For any integer k with $1 \leq k \leq n-1$ and any two sets S and T of k fault-free vertices in different partite sets of Q_n ($n \geq 2$), if $f \leq 2n - 2k - 2$ and each fault-free vertex has at least two fault-free neighbors, then there exist k fully disjoint fault-free paths linking S and T which contain at least $2^n - 2f$ vertices. A linear algorithm for finding such disjoint paths is also given. This result improves some known results in a sense.

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1. Introduction

The n -dimensional hypercube Q_n is a graph whose vertex-set consists of all binary vectors of length n , with two vertices being adjacent whenever the corresponding vectors differ in exactly one coordinate. There is a vast literature on graph-theoretic properties of hypercubes (e.g., see the comprehensive survey papers on early results [15] and recent results [27]) and their applications in parallel computing (e.g., see [21]).

One of the most central issues in various high-performance communication networks or parallel computing systems is to find a cycle or a path of some given length in the networks (see [4,11,14,17,23] and the survey paper [29]). It is well known that there are n internally disjoint paths connecting any two vertices u and v in Q_n . However, these paths do not always contain all vertices. In 2004, Chang et al. [5] showed that for any integer k with $1 \leq k \leq n$ and any two vertices u and v from different partite sets of Q_n , there are k internally disjoint uv -paths in Q_n which contain all vertices. In 2007, Chang et al. [6] further showed that for any integer k with $1 \leq k \leq n-4$ and any two vertices u and v from different partite sets of Q_n ($n \geq 5$), there are k internally disjoint uv -paths in Q_n which contain all vertices and the difference of lengths of any two paths is at most two.

Since edge and/or vertex failures are inevitable when a large parallel computer system is put in use, finding disjoint paths with fault-tolerant routings among vertices has received great research attention recently. In disjoint-path problems, one

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or more source vertices and one or more sink vertices are given to find the paths between them. According to the number of sources or sinks, there are one-to-one [16], one-to-many [20], and many-to-many [3,8,18,22,24–26,30] disjoint-path problems.

A path in Q_n with f faulty vertices is long if it contains at least $2^n - 2f - 1$ vertices. Note that every long path between vertices u and v has a length of at least $2^n - 2f$ if $d(u, v)$ is odd, where $d(u, v)$ is the distance between u and v . If all faulty vertices are in the same bipartite of Q_n , then every long fault-free path is the longest one.

Fu [12] showed that there is a long fault-free path in Q_n between every two fault-free vertices if $f \leq n - 2$. Under the condition that every vertex has at least two fault-free neighbors, Kueng et al. [19] improved the bound in Fu to $2n - 5$. Recently, Fink et al. [10] improved the bound to $2n - 4$ with a weaker condition, and Dvořák et al. [9] further improved it to $\frac{n^2}{10} + \frac{n}{2} + 1$ when $n \geq 15$.

In 2009, Chen [1] considered the problem of many-to-many disjoint paths in the hypercube with faulty vertices and edges showing that for any integer k with $1 \leq k \leq n - 1$, and any two sets S and T of k fault-free vertices in different partite sets in Q_n with f faulty vertices and h faulty edges, if $f + h \leq n - k - 1$, then there exist k disjoint fault-free paths linking S and T in Q_n which contain at least $2^n - 2f$ vertices.

Gregor et al. [13] and Chen [2] investigated the many-to-many disjoint paths in hypercubes with only faulty edges. In this paper, we consider Q_n with only faulty vertices and obtain the following result.

Theorem 1.1. *Let Q_n ($n \geq 2$) be an n -dimensional hypercube with f faulty vertices and let k be an integer with $1 \leq k \leq n - 1$. If $f \leq 2n - 2k - 2$ and each fault-free vertex has at least two fault-free neighbors, then for any two sets S and T of k fault-free vertices in different partite sets, there exist k disjoint fault-free paths linking S and T in Q_n which contain at least $2^n - 2f$ vertices.*

Under the condition that each fault-free vertex has at least two fault-free neighbors, our theorem generalizes the result of Fink et al. Since $k \geq 1$, our result also improves the bound obtained by Chen [1] provided that Q_n does not contain faulty edges.

When $f \leq 2n - 2k - 2$, a centralized $O(2^n)$ -algorithm to find the many-to-many disjoint paths is given, whose running time is a linear function of the number of the vertices in Q_n .

The rest of this paper is organized as follows. Section 2 introduces some notations and lemmas. An overview of the main proof is provided in Section 3. The proof of our result and constructive algorithms for finding disjoint paths are in Sections 4 and 5. Section 6 concludes.

2. Notations and lemmas

We follow [28] for graph-theoretical terminologies and notations not defined here. Let $G = (V, E)$ be a connected simple graph, where $V = V(G)$ is the vertex-set and $E = E(G)$ is the edge-set of G . If $uv \in E(G)$, u (resp. v) is called a neighbor of v (resp. u). A uv -path is a sequence of distinct vertices, written as $\langle v_0, v_1, v_2, \dots, v_m \rangle$, where $v_0 = u$, $v_m = v$ and $v_i v_{i+1} \in E(G)$ for each $i = 1, 2, \dots, m - 1$, (m is the length of P). The distance between u and v is the length of the shortest uv -path in G . For a path $P = \langle v_0, v_1, \dots, v_i, v_{i+1}, \dots, v_m \rangle$, it can also be expressed as $P = P(v_0, v_i) + v_i v_{i+1} + P(v_{i+1}, v_m)$, and the notation $P - v_i v_{i+1}$ denotes the subgraph obtained from P by deleting the edge $v_i v_{i+1}$. Two paths are disjoint if they have no vertices in common. Given two disjoint sets S and T of k vertices, if there exist k disjoint paths linking S and T , we call these paths k disjoint ST -paths.

The n -dimensional hypercube Q_n is a graph with 2^n vertices, with each vertex denoted by an n -bit binary string $x = x_1 x_2 \dots x_{n-1} x_n$. Two vertices are adjacent if and only if their strings differ in exactly one bit position. It has been proved that Q_n is a vertex- and edge-transitive bipartite graph.

By definition, for any $i \in \{1, 2, \dots, n\}$, Q_n can be expressed as $Q_n = L_i \odot R_i$, where L_i and R_i are two subgraphs of Q_n induced by the vertices with the i th bit position being 0 and 1, respectively, which are isomorphic to Q_{n-1} . We use E_i to denote the set of edges between L_i and R_i . The two end-vertices of an edge in E_i differ in exactly the i th bit position. Without loss of generality, we write $Q_n = L \odot R$. For convenience, for a vertex u_L in L , we use u_R to denote its unique neighbor in R . Similarly, for a vertex v_R in R , we use v_L to denote its unique neighbor in L . Clearly, for any two vertices u_L and v_L in L , $d_L(u_L, v_L) = d_R(u_R, v_R)$.

Let F denote the set of faulty vertices in Q_n , and $f = |F|$. When $Q_n = L \odot R$, we denote $f_L = |F \cap L|$ and $f_R = |F \cap R|$. A subgraph of Q_n is *fault-free* if it contains no vertices from F . For two subsets A and B of $V(Q_n)$, let $d(A, B) = \min\{d(x, y) : x \in A, y \in B\}$. For clarity, we list the notations that will be used frequently in Table 1.

We first show the following lemma which ensures that Q_n has a desired partition.

Lemma 2.1. *If $f \leq 2n - 6$ and each fault-free vertex of Q_n ($n \geq 3$) has at least two fault-free neighbors, then there exists some $j \in \{1, 2, \dots, n\}$ such that each fault-free vertex in L_j (or R_j) has at least two fault-free neighbors in L_j (or R_j).*

Proof. Let u be a fault-free vertex in Q_n . Denote $d_F(u)$ as the number of faulty vertices in $N_{Q_n}(u)$, that is, $d_F(u) = |N_{Q_n}(u) \cap F|$, then $d_F(u) \leq n - 2$.

Table 1
Notations used in this paper.

| Symbols | Significations |
|--------------|--|
| F | The faulty vertices in Q_n |
| S, T | Two sets of k fault-free vertices |
| L, R | Two different subcubes Q_{n-1} of Q_n |
| F_L, F_R | The faulty vertices in L, R respectively |
| S_L, S_R | The vertices of S in L, R respectively |
| T_L, T_R | The vertices of T in L, R respectively |
| f_L, f_R | The cardinality of F_L, F_R respectively |
| p, q | The cardinality of S_L, T_L respectively |
| X, Y | the different partite sets of Q_n |
| $N_{Q_n}(u)$ | The neighbors of u in Q_n |
| $d_F(u)$ | The cardinality of the neighbor sets of u in F |
| W_L | The set of neighbors in L of vertices of S_R |
| U_L | $(p - q)$ fault-free vertices in $Y \cap L$ |
| U_R | The neighbors of U_L in R |
| B | The vertices in P_2, \dots, P_n |

It is easy to see that two vertices in Q_n ($n \geq 3$) have exactly two common neighbors if any. If there are two vertices u and v such that $d_F(u) = n - 2$ and $d_F(v) = n - 2$, then

$$\begin{aligned} f &\geq |(N_{Q_n}(u) \cup N_{Q_n}(v)) \cap F| \\ &= d_F(u) + d_F(v) - |N_{Q_n}(u) \cap N_{Q_n}(v) \cap F| \\ &\geq 2n - 4 - 2 \\ &= 2n - 6. \end{aligned}$$

Note that $f \leq 2n - 6$. Thus, we have $f = 2n - 6$ and $|N_{Q_n}(u) \cap N_{Q_n}(v) \cap F| = 2$. Furthermore, there are exactly two vertices u and v such that $d_F(u) = n - 2$ and $d_F(v) = n - 2$. Assume $N_{Q_n}(u) \cap N_{Q_n}(v) \cap F = \{x, y\}$. Then $uxvy$ is a 4-cycle in Q_n . Choose j such that $ux, vy \in E_j$.

If there is at most one vertex u satisfying $d_F(u) = n - 2$, let $x \in N_{Q_n}(u) \cap F$. Also choose j such that $ux \in E_j$.

Therefore, the two fault-free vertices of u (or v) are both in L_j or R_j . For other fault-free vertex x , we have $d_F(x) \leq n - 3$. Thus x has at least two fault-free neighbors in L_j or R_j to which x belongs to. The result follows. ■

In Q_n ($n \geq 2$) with at most $2n - 4$ faulty vertices, Fink et al. [10] showed that for any two fault-free vertices u and v , there exists a long fault-free path between u and v if and only if the exceptional configurations do not occur. In fact, in the exceptional configurations, there are some vertices with at most one fault-free neighbor. So under the condition that every fault-free vertex has at least two fault-free neighbors, the exceptional configurations do not occur. The following three lemmas will be referred to in our paper.

Lemma 2.2 (Fink et al. [10], 2009). *If $f \leq 2n - 4$ and each fault-free vertex of Q_n ($n \geq 2$) has at least two fault-free neighbors then, for any two distinct fault-free vertices x and y with distance d , there is a fault-free xy -path containing at least $2^n - 2f$ vertices if d is odd and $2^n - 2f - 1$ vertices if d is even.*

Lemma 2.3 (Chen [1], 2009). *For any integer k with $1 \leq k \leq n - 1$, if $f \leq n - k - 1$ then, for any two sets S and T of k fault-free vertices in different partite sets in Q_n ($n \geq 2$), there exist k disjoint fault-free ST -paths which contain at least $2^n - 2f$ vertices.*

Lemma 2.4 (Dvořák [7], 2005). *Let (x, y) and (u, v) be two disjoint pairs of vertices with odd distance in Q_n . If x and y are adjacent in Q_n with $n \geq 3$, then there exists a uv -path containing all vertices in $Q_n - \{x, y\}$ unless $n = 3$, $d(u, v) = 1$ and $d(\{x, y\}, \{u, v\}) = 2$.*

In the proof of Theorem 1.1, we need to construct many-to-many disjoint paths with much vertices in some cases, which are ensured by the following lemma.

Lemma 2.5. *Let x and y be two adjacent vertices of Q_n ($n \geq 3$). Then, for any integer k with $1 \leq k \leq n - 2$, and any two sets S and T of k vertices in different partite sets in $Q_n - \{x, y\}$, there exist k disjoint ST -paths containing all vertices in $Q_n - \{x, y\}$.*

Proof. We proceed by induction on k . If $k = 1$, the result is true by Lemma 2.4. We assume that the result holds for any integer less than k , and consider the case of k ($k \geq 2$) and n ($n \geq 4$). Let $\{X, Y\}$ be the bipartition of Q_n and let $S = \{s_1, s_2, \dots, s_k\}$ and $T = \{t_1, t_2, \dots, t_k\}$ be two sets of k vertices in different partite sets in $Q_n - \{x, y\}$. Since Q_n is vertex-transitive and edge-transitive, without loss of generality, we may assume $x = 00 \dots 0$, $y = 10 \dots 0$, $S \cup \{x\} \subseteq X$, $T \cup \{y\} \subseteq Y$. Let $Q_n = L_j \odot R_j$, then the edge xy belongs to $E(L_j)$ for each dimension j ($2 \leq j \leq n$).

We first note that there exists some j ($2 \leq j \leq n$) such that $(S \cup T) \cap L_j \neq \emptyset$ and $(S \cup T) \cap R_j \neq \emptyset$. Otherwise, for each $j \in \{2, \dots, n\}$, the j th bit of all vertices in $S \cup T$ are the same so that $|S \cup T| \leq 2$, contradicting the assumption that $|S \cup T| = 2k \geq 4$ since $k \geq 2$. Therefore, $xy \in E(L)$, $(S \cup T) \cap L \neq \emptyset$ and $(S \cup T) \cap R \neq \emptyset$.

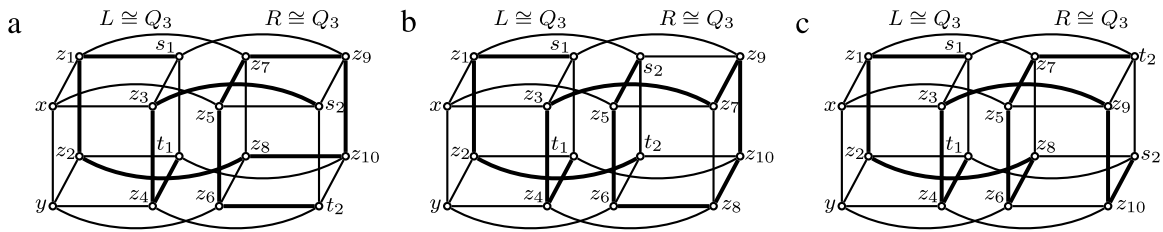


Fig. 1. Illustrations for the proof of Subcase 1.1 of Lemma 2.5.

Let

$$\begin{aligned} S_L &= S \cap L, & T_L &= T \cap L & \text{and} & & p &= |S_L|, & q &= |T_L|, \\ S_R &= S \cap R, & T_R &= T \cap R. \end{aligned}$$

By the symmetry of S and T , we assume $p \geq q$. Then $p \geq 1$ and $q \leq k - 1$.

Let W_L be the set of neighbors in L of vertices of S_R . Then $|W_L| = |S_R| = |S| - |S_L| = k - p$, and $W_L \subset Y$ since $S \subset X$. If $n = 4$, then $L \cong R \cong Q_3$. In Q_3 , we have to consider some exceptional cases by Lemma 2.4. It is easy to see that there exists a unique vertex $u_0 \in L \cap Y$ such that $d_L(u_0, \{x, y\}) = 2$. Set

$$F = \begin{cases} \{u_0\} & \text{if } n = 4; \\ \emptyset & \text{if } n \geq 5. \end{cases}$$

To avoid the exceptional case, we may assume that F is the faulty vertex.

Note that $|Y \cap V(L)| = 2^{n-2}$. For $n \geq 4$ and $k \leq n - 2$, we have

$$\begin{aligned} |Y \cap V(L)| - |T_L \cup W_L \cup \{y\} \cup F| &\geq 2^{n-2} - q - (k - p) - 2 \\ &= (2^{n-2} - k - 2) + (p - q) \\ &\geq p - q. \end{aligned}$$

This implies that there exists a set U_L of $(p - q)$ fault-free vertices in $Y \cap V(L - T_L - \{y\})$ such that its neighbor-set U_R in R is not in S_R .

If $p > q$, let

$$\begin{aligned} U_L &= \{u_1, \dots, u_{p-q}\} \subseteq Y \cap V(L - T_L - \{y\}) \quad \text{and} \\ U_R &= \{v_1, \dots, v_{p-q}\} \subseteq X \cap V(R - S_R) \quad \text{with} \\ u_i v_i &\in E(Q_n) \quad \text{for } 1 \leq i \leq p - q. \end{aligned}$$

If $p = q$, let $U_L = U_R = \emptyset$.

We consider two cases: $1 \leq p \leq k - 1$ and $p = k$.

Case 1. $1 \leq p \leq k - 1$.

Let $T'_L = T_L \cup U_L$ and $S'_R = S_R \cup U_R$. Then $|T'_L| = q + (p - q) = p = |S_L|$ and $|S'_R| = (k - p) + (p - q) = k - q = |T_R|$.

Subcase 1.1. $n = 4$.

In this case, $k = 2$ and $p = 1$. First, we consider the case $q = 1$. Without loss of generality, assume $s_1, t_1 \in L$. Let $M = \{v : v \in (S \cup T) \cap L, d(v, \{x, y\}) = 2\}$. We choose a dimension j to partition Q_4 such that $xy \in E(L)$, $(S \cup T) \cap L \neq \emptyset$, $(S \cup T) \cap R \neq \emptyset$ and $|M|$ is as small as possible.

If $|M| \leq 1$, by Lemma 2.4, there is an $s_1 t_1$ path P_1 in L containing all the vertices in $L - \{x, y\}$. By Lemma 2.3, there is an $s_2 t_2$ path P_2 containing all vertices in R . Then paths P_1 and P_2 are disjoint and contain all vertices in $Q_4 - \{x, y\}$.

If $|M| = 2$, then $d(\{s_2\}, \{x, y\}) = d(\{t_2\}, \{x, y\}) \geq 2$, and s_2 and t_2 must be adjacent in R . Otherwise, we can choose another dimension j to partition Q_4 such that $xy \in E(L)$, $(S \cup T) \cap L \neq \emptyset$, $(S \cup T) \cap R \neq \emptyset$ and $|M| \leq 1$, which contradicts the minimality of $|M|$ and $|M| = 2$. Thus, there are only three configurations of S and T (see Fig. 1).

In configuration (a), $P_1 = \langle s_1, z_1, z_2, z_8, z_{10}, z_9, z_7, z_5, z_6, t_2 \rangle$ and $P_2 = \langle s_2, z_3, z_4, t_1 \rangle$ are two disjoint paths satisfying the requirements (see Fig. 1(a)). In configuration (b), $P_1 = \langle s_1, z_1, z_2, t_2 \rangle$ and $P_2 = \langle s_2, z_5, z_6, z_8, z_{10}, z_9, z_7, z_3, z_4, t_1 \rangle$ are two disjoint paths satisfying the requirements (see Fig. 1(b)). In configuration (c), $P_1 = \langle s_1, z_1, z_2, z_8, z_6, z_5, z_7, t_2 \rangle$ and $P_2 = \langle s_2, z_{10}, z_9, z_3, z_4, t_1 \rangle$ are two disjoint paths satisfying the requirements (see Fig. 1(c)).

We now consider $q = 0$. Note that $u_1 \neq u_0$ and u_1 is fault-free. By Lemma 2.4, we obtain $s_1 u_1$ -path P_L containing all vertices in $L - \{x, y\}$. By Lemma 2.3, there are two disjoint $S'_R T_R$ -paths P_R and P_2 containing all vertices in R , where P_R is a $v_1 t_1$ -path and P_2 is an $s_2 t_2$ -path. Let $P_1 = P_L + u_1 v_1 + P_R$. Then P_1 and P_2 are two disjoint ST -paths containing all vertices in $Q_4 - \{x, y\}$.

Subcase 1.2. $n \geq 5$.

Since $p \leq k - 1$, by the induction hypothesis, in L there are p disjoint $S_L T'_L$ -paths containing all vertices in $L - \{x, y\}$, in which let P_1, P_2, \dots, P_p be $S_L T'_L$ -paths, and $P'_1, P'_2, \dots, P'_{p-q}$ be $S_L U_L$ -paths, where P'_i connects one vertex in S_L to u_i . By

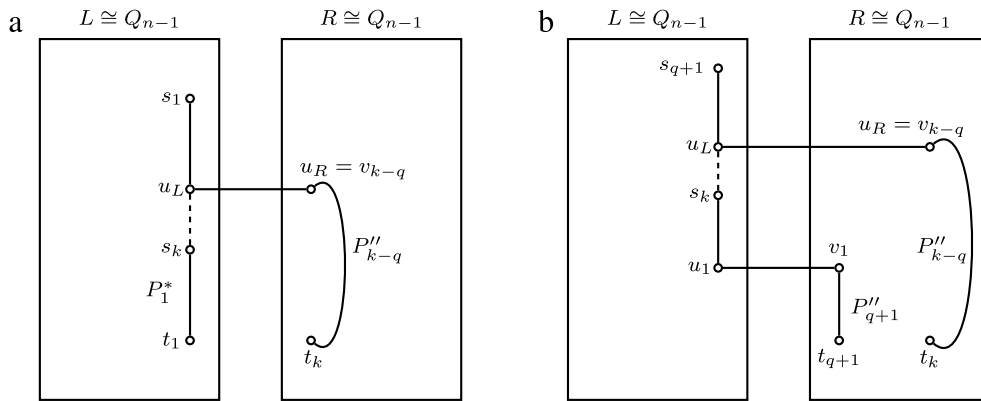


Fig. 3. Illustrations for the proof of Case 2 of Lemma 2.5.

4. Proof of theorem

The proof proceeds by induction on $n \geq 2$. If $k = 1$, then $f \leq 2n - 4$. The theorem follows from Lemma 2.2. If $k = n - 1$, then $f = 0$. The theorem follows from Lemma 2.3. Thus, the theorem holds for $2 \leq n \leq 3$. In the following discussion, we assume $n \geq 4$ and $2 \leq k \leq n - 2$.

Let Q_n be an n -dimensional hypercube, $\{X, Y\}$ be a bipartition of $V(Q_n)$, $S = \{s_1, s_2, \dots, s_k\} \subset X$ and $T = \{t_1, t_2, \dots, t_k\} \subset Y$ be any two sets of k fault-free vertices. Our aim is to construct k disjoint fault-free ST -paths containing at least $2^n - 2f$ vertices.

Since $k \geq 2$, we have $f \leq 2n - 6$. By Lemma 2.1, there exists some $j \in \{1, 2, \dots, n\}$ such that each fault-free vertex in L_j (or R_j) has at least two fault-free neighbors in L_j (or R_j). When $k = n - 2$, $f \leq 2$, we choose some $j \in \{1, 2, \dots, n\}$ such that all faulty vertices belong to either L_j or R_j . Let $Q_n = L \odot R$, where $L = L_j$ and $R = R_j$.

We construct the required k disjoint ST -paths by considering two cases according to whether $S \cup T$ is in L (or R) or not, respectively.

Case 1. $S \cup T \subseteq L$ or $S \cup T \subseteq R$.

In this case, both of the two sets S and T are in L or R . By the symmetry of L and R , we may assume that both S and T are in L . There are two subcases.

Subcase 1.1. $f_L \leq 2n - 2k - 4$.

Since $f_L \leq 2n - 2k - 4 = 2(n - 1) - 2k - 2$ and $k \leq n - 2 = (n - 1) - 1$, by the induction hypothesis, in L there are k disjoint ST -paths

$$P'_1, P'_2, \dots, P'_k \quad (4.1)$$

containing at least $2^{n-1} - 2f_L$ vertices. Note that, when $n \geq 4$ and $f \leq 2n - 6$,

$$\begin{aligned} \sum_{i=1}^k |E(P'_i)| &= \sum_{i=1}^k |V(P'_i)| - k \\ &\geq 2^{n-1} - 2f_L - k \\ &\geq 2^{n-1} - 2(2n - 2k - 4) - k \\ &> 2(2n - 6) \\ &\geq 2f. \end{aligned}$$

There is an edge $u_L v_L$ in some path, say P'_1 , such that both u_R and v_R are fault-free. Since $f_R \leq f \leq 2(n - 1) - 4$, by Lemma 2.2, in R there is a $u_R v_R$ -path P_R containing at least $2^{n-1} - 2f_R$ vertices. Let

$$\begin{aligned} P_1 &= P'_1 - u_L v_L + u_L u_R + P_R + v_R v_L, \\ P_i &= P'_i \text{ for each } i \in \{2, 3, \dots, k\}. \end{aligned} \quad (4.2)$$

Then P_1, P_2, \dots, P_k are k disjoint ST -paths containing at least $2^n - 2f$ vertices in Q_n .

Subcase 1.2. $f_L = 2n - 2k - 3$.

In this case, $f_R \leq 1$ and all paths in (4.1) contain at most one faulty vertex since $f \leq 2n - 2k - 2$. If they contain no faulty vertex, then paths defined in (4.2) are as required. Assume that some path, say P'_1 , contains a faulty vertex w and connects s_1 and t_1 . Note that, in this case, paths in (4.1) contain at least $2^{n-1} - 2(f_L - 1)$ vertices in L . Let u_L and v_L be two neighbors of w in P'_1 , where u_L is in $P'_1(s_1, w)$ and v_L is in $P'_1(w, t_1)$. Then $d_L(u_L, v_L) = 2$ since Q_n contains no triangles, so $d_R(u_R, v_R) = 2$.

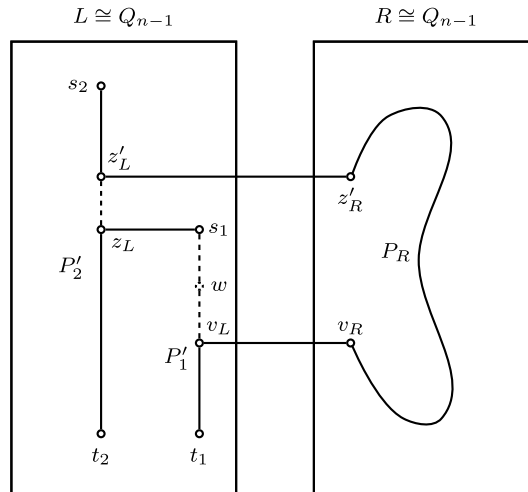


Fig. 4. An illustration for the proof of Subcase 1.2(c) of the theorem.

If both u_R and v_R are fault-free then, by Lemma 2.2, there is a fault-free $u_R v_R$ -path P_R containing at least $2^{n-1} - 2f_R - 1$ vertices in R . Let

$$P_1^* = P'_1(s_1, u_L) + u_L u_R + P_R + v_R v_L + P'_1(v_L, t_1).$$

Then, replacing P_1 in (4.2) by P_1^* yields k disjoint fault-free ST -paths with at least $2^n - 2f$ vertices in Q_n .

Now assume that one of u_R and v_R is faulty vertex, say u_R . Then u_R is the only faulty vertex in R since $f_R \leq 1$.

If $u_L \neq s_1$, let z_L be the fault-free neighbor of u_L in $P'_1(s_1, u_L)$ (maybe $z_L = s_1$), then z_R is fault-free in R , and the distance between z_R and v_R is odd. By Lemma 2.2, there is a fault-free $z_R v_R$ -path P_R containing at least $2^{n-1} - 2f_R$ vertices in R . Let

$$P_1^* = P'_1(s_1, z_L) + z_L z_R + P_R + v_R v_L + P'_1(v_L, t_1).$$

Replacing P_1 in (4.2) by P_1^* yields k disjoint fault-free ST -paths with vertices at least

$$(2^{n-1} - 2(f_L - 1) - |\{w, u_L\}|) + (2^{n-1} - 2f_R) = 2^n - 2f.$$

We now assume $u_L = s_1$ and that u_R is the only faulty vertex in R . Note that u_L has at least two fault-free neighbors in L .

- (a) If there is a fault-free neighbor z_L of $u_L (= s_1)$ not in P'_i for each $i = 1, 2, \dots, k$, we consider two vertices z_R and v_R in R . They are fault-free in R since u_R is the only faulty vertex in R . Since the distance between z_L and v_L is odd, so is the distance between z_R and v_R . By Lemma 2.2, P_R is a $z_R v_R$ -path with at least $2^{n-1} - 2f_R$ vertices in R . Let

$$P_1^* = s_1 z_L + z_L z_R + P_R + v_R v_L + P'_1(v_L, t_1).$$

Replacing P_1 in (4.2) by P_1^* yields k disjoint fault-free ST -paths with vertices at least

$$(2^{n-1} - 2(f_L - 1) - |\{w\}|) + (2^{n-1} - 2f_R) > 2^n - 2f.$$

- (b) If all the fault-free neighbors of $u_L (= s_1)$ are in P'_i , then one of them, say z_L , is not t_1 since $u_L (= s_1)$ has at least two fault-free neighbors in L . Clearly, $z_L \neq v_L$ since Q_n contains no triangles. Let z'_L be the neighbor of z_L in $P'_1(z_L, t_1)$, then z'_R is fault-free in R . Since both v_L and z'_L are in X , the distance between v_R and z'_R is even. Let P_R be a $v_R z'_R$ -path with at least $2^{n-1} - 2f_R - 1$ vertices in R , and let

$$P_1^* = s_1 z_L + P'_1(z_L, v_L) + v_L v_R + P_R + z'_R z'_L + P'_1(z'_L, t_1).$$

Replacing P_1 in (4.2) by P_1^* yields k disjoint fault-free ST -paths with vertices at least

$$(2^{n-1} - 2(f_L - 1) - |\{w\}|) + (2^{n-1} - 2f_R - 1) = 2^n - 2f.$$

- (c) If there is a fault-free neighbor z_L of $u_L (= s_1)$ that is in some P'_i ($i \neq 1$), without loss of generality, assume that z_L is in P'_2 and that P'_2 connects s_2 and t_2 (see Fig. 4). Then $z_L \neq s_2$ since $z_L \in Y$ and $s_2 \in X$. Let z'_L be the neighbor of z_L in $P'_2(s_2, z_L)$ (maybe $z'_L = s_2$), then z'_R is fault-free in R . Since both v_L and z'_L are in X , both v_R and z'_R are in Y , and so the distance between z'_R and v_R is even. Let P_R be a $z'_R v_R$ -path with at least $2^{n-1} - 2f_R - 1$ vertices in R , and let

$$P_1^* = s_1 z_L + P'_2(z_L, t_2), \quad P_2^* = P'_2(s_2, z'_L) + z'_R z'_L + P_R + v_R v_L + P'_1(v_L, t_1).$$

Replacing P_1 and P_2 in (4.2) by P_1^* and P_2^* yields k disjoint fault-free ST -paths with vertices at least

$$(2^{n-1} - 2(f_L - 1) - |\{w\}|) + (2^{n-1} - 2f_R - 1) = 2^n - 2f.$$

Summing up the above discussion, the theorem holds for $f_L = 2n - 2k - 4$.

Subcase 1.3. $f_L = 2n - 2k - 2$.

In this case, $f_R = 0$ since $f \leq 2n - 2k - 2$. Let $S' = S - \{s_k\}$ and $T' = T - \{t_k\}$. Since $f_L = f = 2n - 2k - 2 = 2(n - 1) - 2(k - 1) - 2$, by the induction hypothesis, in L there are $(k - 1)$ disjoint fault-free $S'T'$ -paths

$$P'_1, P'_2, \dots, P'_{k-1} \quad (4.3)$$

containing at least $2^{n-1} - 2f$ vertices. Let

$$K = V(P'_1 \cup P'_2 \cup \dots \cup P'_{k-1}),$$

and let s_R and t_R be neighbors of s_k and t_k in R , respectively. Since s_k and t_k are in different partite sets in Q_n , $d_L(s_k, t_k)$ is odd, and so is $d_R(s_R, t_R)$.

If $|\{s_k, t_k\} \cap K| = 0$, since $d_R(s_R, t_R)$ is odd, by Lemma 2.2, there is an $s_R t_R$ -path P_R containing 2^{n-1} vertices in R . Let

$$\begin{aligned} P_i &= P'_i, \quad i = 1, 2, 3, \dots, k-1, \\ P_k &= s_k s_R + P_R + t_R t_k. \end{aligned} \quad (4.4)$$

Then P_1, \dots, P_{k-1}, P_k are k disjoint fault-free ST -paths containing at least $2^n - 2f$ vertices, as required.

If $|\{s_k, t_k\} \cap K| = 1$, without loss of generality, assume $s_k \in K$ and $t_k \notin K$. We can further assume that s_k is in P'_1 with two end-vertices s_1 and t_1 . Let x_L be the neighbor of s_k in $P'_1(s_1, s_k)$, and t_R be the neighbor of t_k in R . Since both x_R and t_R are in X and $f_R = 0$, there is a fault-free $x_R t_R$ -path P_R containing $2^{n-1} - 1$ vertices. Let

$$\begin{aligned} P_1^* &= P'_1(s_1, x_L) + x_L x_R + P_R + t_R t_k \\ P_k^* &= P'_1(s_k, t_1). \end{aligned}$$

Replacing P_1 and P_k in (4.4) by P_1^* and P_k^* yields k disjoint fault-free ST -paths containing vertices at least

$$(2^{n-1} - 2f + |\{t_k\}|) + (2^{n-1} - 1) = 2^n - 2f.$$

The remaining case is $|\{s_k, t_k\} \cap K| = 2$, which contains two cases.

- (a) Both s_k and t_k are in the same path in (4.3), say P'_1 . Let x_L and y_L be two neighbors of s_k and t_k in P'_1 but not in the subpath $P'_1(s_1, t_k)$, respectively. Since $d_L(s_k, t_k)$ is odd, both $d_L(x_L, y_L)$ and $d_R(x_R, y_R)$ are odd. By Lemma 2.2, there is an $x_R y_R$ -path P_R containing 2^{n-1} vertices in R . Let

$$\begin{aligned} P_1^* &= P'_1(s_1, x_L) + x_L x_R + P_R + y_R y_L + P'_1(y_L, t_1), \\ P_k^* &= P'_1(s_k, t_k). \end{aligned}$$

Replacing P_1 and P_k in (4.4) by P_1^* and P_k^* yields k disjoint fault-free ST -paths containing at least $2^n - 2f$ vertices.

- (b) Both s_k and t_k are in different paths in (4.3). Without loss of generality, suppose that s_k is in P'_1 , t_k is in P'_2 and the two end-vertices of P'_1 are s_1 and t_1 for $i = 1, 2$. Let x_L be the neighbor of s_k in $P'_1(s_1, s_k)$, and y_L be the neighbor of t_k in $P'_2(t_k, t_2)$. Since $d_L(s_k, t_k)$ is odd, both $d_L(x_L, y_L)$ and $d_R(x_R, y_R)$ are odd. By Lemma 2.2, there is an $x_R y_R$ -path P_R containing 2^{n-1} vertices in R . Let

$$\begin{aligned} P_1^* &= P'_1(s_1, x_L) + x_L x_R + P_R + y_R y_L + P'_2(y_L, t_2), \\ P_2^* &= P'_2(s_2, t_k), \\ P_k^* &= P'_1(s_k, t_1). \end{aligned}$$

Replacing P_1, P_2 and P_k in (4.4) by P_1^*, P_2^* and P_k^* yields k disjoint fault-free ST -paths containing at least $2^n - 2f$ vertices.

Summing up the above discussion, the theorem holds when $S \cup T \subseteq L$ or $S \cup T \subseteq R$.

Case 2. $(S \cup T) \cap L \neq \emptyset$ and $(S \cup T) \cap R \neq \emptyset$.

Let (see Fig. 5)

$$\begin{aligned} S_L &= S \cap L, & T_L &= T \cap L & \text{and} & & p &= |S_L|, & q &= |T_L|, \\ S_R &= S \cap R, & T_R &= T \cap R. \end{aligned}$$

By the symmetry of L and R , we assume $f_L \leq f_R$. By the symmetry of S and T , assume that $p \geq q$.

Let W_L be the set of neighbors of vertices of S_R in L . Then $W_L \subset Y$ since $S \subset X$. For $n \geq 4$ and $k \geq 2$, we have

$$\begin{aligned} |Y \cap V(L)| - |T_L \cup W_L \cup F| &\geq 2^{n-2} - q - (k - p) - f \\ &\geq 2^{n-2} - q - (k - p) - (2n - 2k - 3) \\ &= (2^{n-2} - 2n + k + 3) + (p - q) \\ &> p - q. \end{aligned}$$

This implies that there is a set U_L of $(p - q)$ fault-free vertices in $Y \cap V(L - T_L)$ such that its neighbor-set U_R in R is in $X \cap V(R - S_R)$ and is fault-free. If $p > q$, let

$$U_L = \{u_1, u_2, \dots, u_{p-q}\} \quad \text{and} \quad U_R = \{v_1, v_2, \dots, v_{p-q}\}, \quad (4.5)$$

be any two such vertex-sets, where $u_i v_i \in E(Q_n)$ for each $i = 1, 2, \dots, p - q$. If $p = q$, let $U_L = U_R = \emptyset$. There are two cases. One is $1 \leq q \leq k - 1$ or $q = 0$ and $f_R \leq 2n - 2k - 4$, and the other one is $q = 0$ and $f_R \geq 2n - 2k - 3$.

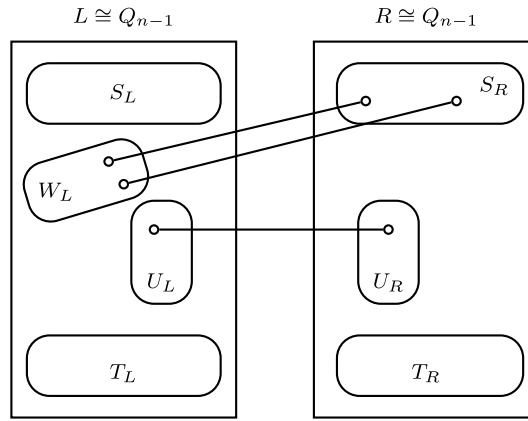


Fig. 5. An illustration for the proof of Case 2 of the theorem.

Subcase 2.1. $1 \leq q \leq k-1$ or $q=0$ and $f_R \leq 2n-2k-4$.

Since $f_L \leq f_R$, then $f_L \leq \frac{1}{2}f \leq \frac{1}{2}(2n-2k-2) \leq 2(n-1)-2k-2$ for $k \leq n-3$ and $f_L \leq f_R = 0$ for $k = n-2$. By the induction hypothesis, in L there are $p(\leq k)$ disjoint fault-free paths

$$P'_1, \dots, P'_{p-q}, P_{p-q+1}, \dots, P_p$$

connecting S_L and $T_L \cup U_L$ and containing at least $2^{n-1} - 2f_L$ vertices, where P'_i connects some vertex in S_L to the vertex u_i in U_L for each $i = 1, 2, \dots, p-q$, provided that $p > q$.

If $q \geq 1$, then $f_R \leq f \leq 2n-2k-2 \leq 2(n-1)-2(k-q)-2$ and $k-q \leq k-1 \leq n-2$. If $q=0$ and $f_R \leq 2n-2k-4$, then $f_R \leq 2(n-1)-2k-2$, $k \leq n-2$. In any case, by the induction hypothesis, in R there are $k-q$ disjoint fault-free paths

$$P''_1, \dots, P''_{p-q}, P_{p+1}, \dots, P_k$$

connecting $S_R \cup U_R$ and T_R and containing at least $2^{n-1} - 2f_R$ vertices, where P''_i connects the vertex v_i in U_R to some vertex in T_R for each $i = 1, 2, \dots, p-q$, provided that $p > q$. Let (see Fig. 2)

$$P_i = P'_i + u_i v_i + P''_i, \quad i = 1, 2, \dots, p-q.$$

Then the k paths $P_1, P_2, \dots, P_{p-q}, P_{p-q+1}, \dots, P_p, P_{p+1}, \dots, P_k$ satisfy our requirements.

Subcase 2.2. $q=0$ and $f_R \geq 2n-2k-3$.

In this case, $T_R = T$ and $f_L \leq 1$. We can write (4.5) as

$$U_L = \{u_1, u_2, \dots, u_p\} \quad \text{and} \quad U_R = \{v_1, v_2, \dots, v_p\}. \quad (4.6)$$

Since $f_L \leq 1 < 2(n-1)-2(n-3)-2$ and $f_L = 0$ for $k = n-2$, by the induction hypothesis, in L there are $p(\leq k)$ disjoint fault-free $S_L U_L$ -paths

$$P'_1, P'_2, \dots, P'_p \quad (4.7)$$

containing at least $2^{n-1} - 2f_L$ vertices. Without loss of generality, assume that P'_i connects s_i to u_i for each $i = 1, 2, \dots, p$ (see Fig. 6).

Since $f_R \leq 2n-2k-2 = 2(n-1)-2(k-1)-2$, by the induction hypothesis, in R there are $k-1$ disjoint fault-free paths

$$P''_2, \dots, P''_p, P_{p+1}, \dots, P_k \quad (4.8)$$

connecting $S_R \cup U_R - \{v_1\}$ and $T - \{t_1\}$ and containing at least $2^{n-1} - 2f_R$ vertices in R , where P''_i connects the vertex v_i in U_R and some vertex in $T - \{t_1\}$ for each $i = 2, 3, \dots, p$ (see Fig. 6). Then

$$\begin{aligned} P_1 &= P'_1, \\ P_i &= P'_i + u_i v_i + P''_i, \quad i = 2, 3, \dots, p \\ P_i &, \quad i = p+1, \dots, k \end{aligned} \quad (4.9)$$

are k disjoint fault-free paths between S and $(T - \{t_1\}) \cup \{u_1\}$ containing at least $2^n - 2f$ vertices in Q_n . Without loss of generality, assume that P_i connects s_i to t_i for each $i = 2, 3, \dots, k$ and P_1 connects s_1 to u_1 (see Fig. 6). Let

$$B = V(P_2 \cup \dots \cup P_k).$$

Note if $p = 1$, P'_i in (4.7) and P''_i in (4.8) are empty for $i \geq 2$. Since each vertex in T has a unique neighbor in L , we can choose $t_1 \in T$ such that $s_1 t_1 \notin E(Q_n)$ when $p = 1$.

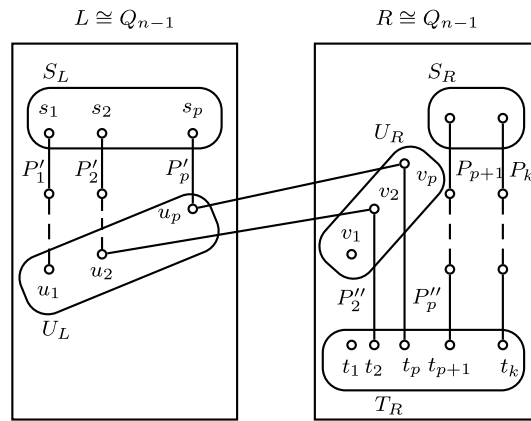


Fig. 6. An illustration for the proof of Subcase 2.2 of the theorem.

We consider two subcases according to whether t_1 is in B or not.

Subcase 2.2a. $t_1 \notin B$.

Let t_L be the neighbor of t_1 in L . Suppose that t_L is fault-free.

Assume $t_L \in S$, say $t_L = s_1$. By our assumption $s_1 t_1 \notin E(Q_n)$ if $p = 1$, and so $p \geq 2$. We put t_L as a temporarily faulty vertex. Let $S' = S_L - \{s_1\}$ and $U' = U_L - \{u_1\}$. Then L contains at most two faulty vertices; $p - 1$ disjoint fault-free $S'U'$ -paths P'_2, \dots, P'_p shown in (4.7) contain at least $2^{n-1} - 2(f_L + 1)$ vertices; P_2, P_3, \dots, P_k in (4.9) are $k - 1$ disjoint paths containing at least $2^n - 2f - 2$ vertices. Together with the path $P_1 = s_1 t_1$, we obtain k paths satisfying the requirements.

Assume now $t_L \notin S$. Since $f_L \leq 1$, we have $|F_L \cup (U_L - \{u_1\})| \leq k \leq n - 3$. Thus, t_L has at least a fault-free neighbor $z \in V(L) - (U_L - \{u_1\})$. Since $z \in Y$, we can choose u_1 of U_L as z in (4.6). We put t_L as a temporarily faulty vertex. Then L contains at most two faulty vertices. The p disjoint fault-free $S_L U_L$ -paths shown in (4.7) contain at least $2^{n-1} - 2(f_L + 1)$ vertices. Let

$$P_1^* = P_1 + u_1 t_L + t_L t_1.$$

Replacing P_1 in (4.9) by P_1^* yields k disjoint fault-free ST -paths in Q_n . Note that t_1 and t_L are two new vertices. Then the k paths P_1^*, P_2, \dots, P_k satisfy our requirements.

Now suppose that t_L is a faulty vertex. Then t_L is the only faulty vertex in L . Take a fault-free neighbor, say w_R of t_1 in R . Then its neighbor w_L in L is fault-free and in Y . If $w_R \notin B$, then choose $u_1 = w_L \in U_L$ in (4.6). Let

$$P_1^* = P_1 + u_1 w_R + w_R t_1.$$

Replacing P_1 in (4.9) by P_1^* yields k disjoint fault-free ST -paths in Q_n , as required. Assume that w_R is in some P_i ($2 \leq i \leq k$), say in $P_2(s_2, t_2)$. Then $w_R \neq t_2$ since w_R and t_2 are in different partite sets. Let u_R be the neighbor of w_R in $P_2(w_R, t_2)$ and u_L be the neighbor of u_R in L . Then $u_L \in X$.

Assume $u_L \in S$, say $u_L = s_1$. If $p = 1$, we can choose another w_R such that $u_L \neq s_1$. So assume $p \geq 2$. We put s_1 as a temporarily faulty vertex. Let $S' = S_L - \{s_1\}$ and $U' = U_L - \{u_1\}$. Then L contains at most two faulty vertices; there are $p - 1$ disjoint fault-free $S'U'$ -paths P'_2, \dots, P'_p containing at least $2^{n-1} - 2(f_L + 1)$ vertices; P_2, P_3, \dots, P_k in (4.9) are $k - 1$ disjoint paths containing at least $2^n - 2f - 2$ vertices. Let

$$\begin{aligned} P_1 &= s_1 u_R + P_2(u_R, t_2), \\ P_2^* &= P_2(s_2, w_R) + w_R t_1. \end{aligned}$$

Thus $P_1, P_2^*, P_3, \dots, P_k$ are k disjoint fault-free ST -paths in Q_n , where t_1 and s_1 are two new vertices, as required.

Assume now $u_L \notin S$. Choose a fault-free neighbor z_L of u_L in L such that it is not any u_i in U_L for $2 \leq i \leq p$. Then $z_L \in Y$. So choose $u_1 = z_L \in U_L$ in (4.6). We put u_L as a temporarily faulty vertex. Then L contains at most two faulty vertices. The k disjoint fault-free SU_L -paths shown in (4.7) contain at least $2^{n-1} - 2(f_L + 1)$ vertices. Let

$$\begin{aligned} P_1^* &= P_1 + z_L u_L + u_L u_R + P_2(u_R, t_2), \\ P_2^* &= P_2(s_2, w_R) + w_R t_1. \end{aligned}$$

Replacing P_1 and P_2 in (4.9) by P_1^* and P_2^* yields k disjoint fault-free ST -paths in Q_n , where t_1 and u_L are two new vertices, as required.

Subcase 2.2b. $t_1 \in B$.

In this case, without loss of generality, we assume t_1 is in P_2 that connects s_2 to t_2 . Let u_R and v_R be two neighbors of t_1 in P_2 and let u_L and v_L be two neighbors of u_R and v_R in L , respectively. Then both u_L and v_L are in Y . Without loss of generality, assume that v_R is in $P_2(s_2, t_1)$ and u_R is in $P_2(t_1, t_2)$.

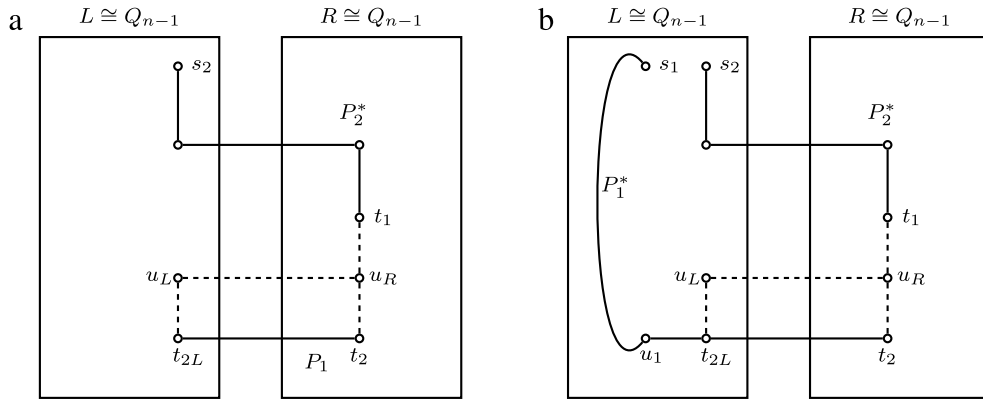


Fig. 7. Illustrations for the proof of Subcase 2.2b of the theorem.

If u_L is fault-free, then choose $u_1 = u_L$ in U_L in (4.6). Let

$$\begin{aligned} P_1^* &= P_2(s_2, t_1), \\ P_2^* &= P_1 + u_L u_R + P_2(u_R, t_2). \end{aligned}$$

Replacing P_1 and P_2 in (4.9) by P_1^* and P_2^* yields k disjoint fault-free ST -paths in Q_n , as required.

Assume now that u_L is faulty vertex. Since $f \leq 2n - 2k - 2$, $f_R = 2n - 2k - 3$. If $d(u_R, t_2) > 1$, we can choose t_2 instead of t_1 , which contradicts the assumption that u_L is a faulty vertex. Thus we have $d(u_R, t_2) = 1$.

Let t_{2L} be the neighbor of t_2 in L . Then we have $u_L t_{2L} \in E(Q_n)$.

Assume $t_{2L} \in S_L$, say $t_{2L} = s_1$ (see Fig. 7(a)). By the assumption $s_1 t_1 \notin E(Q_n)$ if $p = 1$. Thus assume $p > 1$. Let $S' = S - \{s_1\}$ and $U' = U_L - \{u_1\}$. Applying Lemma 2.5 and choosing xy to be $u_L t_{2L}$, there are $p - 1 (\leq k)$ disjoint $S'U'$ -paths

$$P_2', \dots, P_p' \quad (4.10)$$

that contain all vertices in $L - \{u_L, t_{2L}\}$, say P_i' connects s_i to u_i for each $i = 2, 3, \dots, p$. Combining (4.8) with (4.10), we obtain $k - 1$ disjoint fault-free paths between S' and $T - \{t_1\}$

$$\begin{aligned} P_i &= P_i' + u_i v_i + P_i'', \quad i = 2, 3, \dots, p \\ P_i, \quad i &= p + 1, \dots, k \end{aligned} \quad (4.11)$$

that contain at least $(2^{n-1} - 2) + (2^{n-1} - 2f_R) = 2^n - 2f$ vertices in Q_n . Let

$$P_1 = s_1 t_2 \quad \text{and} \quad P_2^* = P_2(s_2, t_1).$$

Then the paths $P_1, P_2^*, P_3, \dots, P_k$ satisfy the requirements.

Assume now $t_{2L} \notin S_L$. Let $z \in N_{Q_n}(t_{2L}) \cap L - \{u_L\} - U'$ (see Fig. 7(b)). Choose $u_1 = z$ in U_L in (4.6). Applying Lemma 2.5 and choosing xy to be $u_L t_{2L}$, there are $p (\leq k)$ disjoint $S_L U_L$ -paths

$$P_1', P_2', \dots, P_p' \quad (4.12)$$

that contain all vertices in $L - \{u_L, t_{2L}\}$, say P_i' connects s_i to u_i for each $i = 1, 2, \dots, p$. Combining (4.8) with (4.12), we obtain k disjoint fault-free paths between S and $(T - \{t_1\}) \cup \{u_1\}$

$$\begin{aligned} P_1 &= P_1', \\ P_i &= P_i' + u_i v_i + P_i'', \quad i = 2, 3, \dots, p \\ P_i, \quad i &= p + 1, \dots, k \quad (\text{defined in (4.8)}) \end{aligned} \quad (4.13)$$

that contain at least $(2^{n-1} - 2) + (2^{n-1} - 2f_R) = 2^n - 2f$ vertices in Q_n . Let

$$P_1^* = P_1 + u_1 t_{2L} + t_{2L} t_2 \quad \text{and} \quad P_2^* = P_2(s_2, t_1).$$

Replacing P_1 and P_2 in (4.13) by P_1^* and P_2^* yields k disjoint fault-free ST -paths in Q_n . We remove one vertex u_R from paths in (4.11) and add one vertex t_{2L} to obtain new paths. Thus, these paths still contain at least $2^n - 2f$ vertices in Q_n , as required.

The theorem follows.

5. A constructive algorithm for disjoint paths

In this section, we propose an effective algorithm to find the many-to-many disjoint paths in faulty hypercubes. Before offering our algorithm, we first recall several existing algorithms.

Theorem 5.1 (See Fink et al. [10], Dvořák et al. [9]). Let Q_n ($n \geq 2$) be an n -dimensional hypercube with faulty vertex set F of size at most $2n - 4$ and every vertex in Q_n has at least two fault-free neighbors, then there exists an $O(2^n)$ -algorithm $\text{LongP}(u, v, F, n)$ to find a long fault-free path connecting u and v .

Theorem 5.2. For any integer k with $1 \leq k \leq n - 1$ and for any two sets S and T of k fault-free vertices in different partite sets in Q_n ($n \geq 2$), there exists an $O(2^n)$ algorithm $\text{MDPChen}(S, T, n)$ to find k disjoint fault-free ST -paths which contain all vertices in Q_n .

Proof. Let $T(n)$ be the maximum time to implement $\text{MDPChen}(S, T, n)$. We have $T(n) \leq T(n - 1) + O(2^{n-1})$ in Case 1, and $T(n) \leq 2T(n - 1) + O(1)$ in other Cases (See Chen [1]). Note that $T(2) = O(1)$. Thus $T(n) \leq O(2^n)$. ■

As Lemma 2.5 establishes the existence of disjoint paths with much vertices, the procedure MDPE is the method to find such paths. This is the following theorem.

Theorem 5.3. Let x and y be two adjacent vertices of Q_n ($n \geq 3$). Then, for any integer k with $1 \leq k \leq n - 2$, and any two sets S and T of k vertices in different partite sets in $Q_n - \{x, y\}$, there exists an $O(2^n)$ -algorithm $\text{MDPE}(S, T, xy, n)$ to find k disjoint ST -paths containing all vertices in $Q_n - \{x, y\}$.

Proof. Let $T(n)$ be the maximum time to run procedure $\text{MDPE}(S, T, xy, n)$. In the constructive proof of Lemma 2.5, we implement MDPE in the subcube L , and call procedure MDPChen in R . By Theorem 5.2, we have $T(n) \leq T(n - 1) + O(2^{n-1})$. Note that $T(3) = O(1)$. Thus $T(n) \leq O(2^n)$. ■

Algorithm 1 $\text{MDPMain}(S, T, F, n)$

Input: $S = \{s_1, s_2, \dots, s_k\}, T = \{t_1, t_2, \dots, t_k\}, F, n$.
Output: (P_1, P_2, \dots, P_k)
1: **if** $k = n - 1$ **then**
2: $(P_1, P_2, \dots, P_k) \leftarrow \text{MDPChen}(S, T, n)$;
3: **end if**
4: Choose some proper $j \in \{1, 2, \dots, n\}$ to partition Q_n as $L_j \odot R_j, L \leftarrow L_j, R \leftarrow R_j$;
5: **if** $S \cup T \subseteq L$ or $S \cup T \subseteq R$, say the former one **then**
6: $(P_1, P_2, \dots, P_k) \leftarrow \text{MDP1}(S, T, F, n)$;
7: **else** // $(S \cup T) \cap L \neq \emptyset$ and $(S \cup T) \cap R \neq \emptyset$
8: $(P_1, P_2, \dots, P_k) \leftarrow \text{MDP2}(S, T, F, n)$;
9: **end if**

Algorithm 2 $\text{MDP1}(S, T, F, n)$

Input: $S = \{s_1, s_2, \dots, s_k\}, T = \{t_1, t_2, \dots, t_k\}$ such that $S \cup T \subseteq L, F, n$.
Output: (P_1, P_2, \dots, P_k)
1: **if** $f_L \leq 2n - 2k - 4$ **then**
2: $(P'_1, P'_2, \dots, P'_k) \leftarrow \text{MDPMain}(S, T, F_L, n - 1)$;
3: Choose z_R and v_R in $R, P_R \leftarrow \text{LongP}(z_R, v_R, F_R, n - 1), \text{Merge}$;
4: **else if** $f_L = 2n - 2k - 3$ **then**
5: Choose a faulty vertex w to be temporarily fault-free, $(P'_1, P'_2, \dots, P'_k) \leftarrow \text{MDPMain}(S, T, F_L, n - 1)$;
6: Choose z_R and v_R in $R, P_R \leftarrow \text{LongP}(z_R, v_R, F_R, n - 1), \text{Merge}$;
7: **else** // $f_L = 2n - 2k - 2$
8: $(P'_1, P'_2, \dots, P'_{k-1}) \leftarrow \text{MDPMain}(S - \{s_k\}, T - \{t_k\}, F_L, n - 1)$;
9: Choose x_R and y_R in $R, P_R \leftarrow \text{LongP}(x_R, y_R, F_R, n - 1), \text{Merge}$;
10: **end if**

Theorem 5.4. Let Q_n ($n \geq 2$) be an n -dimensional hypercube with faulty vertex set F of size f , and let k be an integer with $1 \leq k \leq n - 1$. If $f \leq 2n - 2k - 2$ and each fault-free vertex has at least two fault-free neighbors then, for any two sets S and T of k fault-free vertices in different partite sets, there exists an $O(2^n)$ -algorithm $\text{MDPMain}(S, T, F, n)$ to construct the k disjoint fault-free paths linking S and T in Q_n which contain at least $2^n - 2f$ vertices.

Proof. We offer an efficient algorithm, MDPMain (Algorithm 1), to find many-to-many disjoint paths in hypercubes with faulty vertices. In implementing MDPMain , it may be called procedure MDPChen (Theorem 5.2), MDP1 (Algorithm 2) or MDP2 (Algorithm 3).

After obtaining the disjoint paths in L and R , one needs to incorporate the paths and avoid some temporarily fault-free vertices at the same time. We call this procedure Merge .

Now we analyze the time complexity of algorithm MDPMain . Let $T(n)$ be the maximum time to run procedure $\text{MDPMain}(S, T, F, n)$. Then $T(2) = O(1)$.

If $k = n - 1$, implement MDPChen in Q_n . By Theorem 5.2, we have

$$T(n) \leq O(2^n). \quad (5.14)$$

Algorithm 3 MDP2(S, T, F, n)

Input: $S = \{s_1, s_2, \dots, s_k\}, T = \{t_1, t_2, \dots, t_k\}$ such that $(S \cup T) \cap L \neq \emptyset$ and $(S \cup T) \cap R \neq \emptyset, F, n$.

Output: (P_1, P_2, \dots, P_k)

```

1:  $S_L \leftarrow S \cap L, T_L \leftarrow T \cap L, p \leftarrow |S_L|, q \leftarrow |T_L|, S_R \leftarrow S \cap R, T_R \leftarrow T \cap R$ ;
2: Choose  $L$  and  $R$  such that  $f_L \leq f_R$ , Choose  $S$  and  $T$  such that  $p \geq q$ , Find proper vertex set  $U_L$  and  $U_R$ ;
3: if  $1 \leq q \leq k-1$  or  $q = 0$  and  $f_R \leq 2n - 2k - 4$  then
4:    $(P'_1, \dots, P'_{p-q}, P_{p-q+1}, \dots, P_p) \leftarrow \text{MDPMain}(S_L, T_L \cup U_L, F_L, n-1)$ ;
5:    $(P''_1, \dots, P''_{p-q}, P_{p+1}, \dots, P_k) \leftarrow \text{MDPMain}(S_R \cup U_R, T_R), \text{Merge}$ ;
6: else //  $q = 0$  and  $f_R \geq 2n - 2k - 3$ 
7:   if  $p = 1$  then
8:     Choose  $t_1$  in  $T$  such that  $s_1 t_1 \notin E(Q_n)$ ;
9:   end if
10:   $(P'_2, \dots, P'_p, P_{p+1}, \dots, P_k) \leftarrow \text{MDPMain}(S_R \cup U_R - \{u_1\}, T - \{t_1\}, F_R, n-1)$ ;
11:   $B \leftarrow V(P'_2, \dots, P'_p, P_{p+1}, \dots, P_k)$ ;
12:  if  $t_1 \notin B$  then
13:     $t_L \leftarrow N(t_1) \cap L$ ;
14:    if  $t_L$  is fault-free then
15:      if  $t_L \in S$ , say  $t_L = s_1$ ; then
16:         $S' \leftarrow S_L - \{s_1\}, U' \leftarrow U_L - \{u_1\}, (P'_2, \dots, P'_p) \leftarrow \text{MDPMain}(S', U', F_L \cup \{t_L\}, n-1)$ ;
17:         $P_1 \leftarrow s_1 t_1, \text{Merge}$ ;
18:      else //  $t_L \notin S$ 
19:         $(P'_1, \dots, P'_p) \leftarrow \text{MDPMain}(S_L, U_L, F_L \cup \{t_L\}, n-1), P_1 \leftarrow P'_1 + u_1 t_L + t_L t_1, \text{Merge}$ ;
20:      end if
21:    else //  $t_L$  is a faulty vertex;
22:      Choose a fault-free vertex  $w_R$  in  $N(t_1) \cap R, w_L \leftarrow N(w_R) \cap L$ ;
23:      if  $w_R \notin B$  then
24:         $u_1 \leftarrow w_L, (P'_1, P'_2, \dots, P'_p) \leftarrow \text{MDPMain}(S_L, U_L, F_L, n-1), P_1 \leftarrow P'_1 + u_1 t_L + t_L t_1, \text{Merge}$ ;
25:      else //  $w_R \in B$ , say  $w_R \in P_2(s_2, t_2)$ 
26:         $u_R \leftarrow N(w_R) \cap P_2(w_R, t_2), u_L \leftarrow u_R \cap L$ ;
27:        if  $u_L \in S$ , say  $u_L = s_1$  then
28:           $S' \leftarrow S_L - \{s_1\}, U' \leftarrow U_L - \{u_1\}, (P'_1, P'_2, \dots, P'_p) \leftarrow \text{MDPMain}(S', U', F_L \cup \{s_1\}, n-1)$ ;
29:           $\text{Merge}, P_1 \leftarrow s_1 u_R + P_2(u_R, t_2), P_2 \leftarrow \bar{P}_2(s_2, w_R) + w_R t_1$ ;
30:        else //  $u_L \notin S$ ;
31:          Choose a fault-free vertex  $u_1$  in  $N(u_L) \cap L$ ;
32:           $(P'_1, P'_2, \dots, P'_p) \leftarrow \text{MDPMain}(S_L, U_L, F_L \cup \{u_L\}, n-1), \text{Merge}$ ;
33:           $P_1 \leftarrow P_1 + z_L u_L + u_L u_R + P_2(u_R, t_2), P_2 \leftarrow P_2(s_2, w_R) + w_R t_1$ ;
34:        end if
35:      end if
36:    end if
37:  else //  $t_1 \in B$ , Say  $t_1 \in P_2$ 
38:     $u_R \leftarrow N(t_1) \cap P_2(t_1, t_2), v_R \leftarrow N(t_1) \cap P_2(s_2, t_1)$ ;
39:     $u_L \leftarrow N(u_R) \cap L, v_L \leftarrow N(v_R) \cap L$ ;
40:    if  $u_L$  is fault-free then
41:       $u_1 \leftarrow u_L, (P'_1, P'_2, \dots, P'_p) \leftarrow \text{MDPMain}(S_L, U_L, F_L, n-1), \text{Merge}$ ;
42:       $P_1 \leftarrow P_2(s_2, t_1), P_2 \leftarrow P_1 + u_L u_R + P_2(u_R, t_2)$ ;
43:    else //  $u_L$  is fault vertex,  $d(u_R, t_2) = 1$ .
44:       $t_{2L} \leftarrow t_2 \cap L$ ;
45:      if  $t_{2L} \in S_L$  then
46:         $S' \leftarrow S - \{s_1\}, U' \leftarrow U_L - \{u_1\}$ ;
47:         $(P'_2, \dots, P'_p) \leftarrow \text{MDPE}(S', U', u_L t_{2L}, n-1)$ ; // Using Lemma2.5
48:         $\text{Merge}, P_1 \leftarrow s_1 t_1, P_2 \leftarrow P_2(s_2, t_1)$ ;
49:      else //  $t_{2L} \notin S_L$ 
50:        Choose  $u_1 \leftarrow N_{Q_n}(t_{2L}) \cap L - \{u_L\} - U'$ ;
51:         $(P'_1, P'_2, \dots, P'_p) \leftarrow \text{MDPE}(S', U', u_L t_{2L}, n-1)$ ; // Using Lemma2.5
52:         $\text{Merge}, P_1 \leftarrow P_1 + u_1 t_{2L} + t_{2L} t_2, P_2 \leftarrow P_2(s_2, t_1)$ ;
53:      end if
54:    end if
55:  end if
56: end if

```

We take $O(n)$ time to partition Q_n in a suitable way. If MDP1 (Algorithm 2) is implemented, then MDPMain in L takes time $T(n-1)$; choosing two proper vertices in the paths of L takes time $O(n)$; LongP in R takes time $O(2^{n-1})$. Thus

$$T(n) \leq O(n) + T(n-1) + O(n) + O(2^{n-1}). \quad (5.15)$$

If MDP2 (Algorithm 3) is implemented, then in Subcase 2.1 and Subcase 2.2a,

$$T(n) \leq 2T(n-1) + O(n); \quad (5.16)$$

in Subcase 2.2b, MDPMain in L takes time $T(n-1)$; MDPE in R takes time $O(2^{n-1})$. Thus

$$T(n) \leq T(n-1) + O(n) + O(2^{n-1}). \quad (5.17)$$

Then by (5.14)–(5.17), we have $T(n) \leq O(2^n)$. ■

We should point out that LongP cannot solve k disjoint-path problem with $k \geq 2$ and MDPChen does not work if $f \geq n-k$. Comparing to this, our algorithm is an effective procedure to find many-to-many disjoint paths in hypercubes with more faulty vertices.

When $f \leq 2n - 2k - 2$, our algorithm can solve k disjoint-path problem. The total running time is bounded by $O(2^n)$, which is a linear function of the number of the vertices in Q_n . Since our algorithm is a centralized one, when n is very large, our algorithm cannot find the many-to-many disjoint paths in a reasonable amount of time. It could be possible to modify our algorithm to a distributed one and find the many-to-many paths in a relatively short time.

When the number of faulty vertices exceeds $2n - 2k - 2$, our algorithm may not find the many-to-many disjoint paths. The total number and distribution of the faulty vertices will affect the running time of our algorithm. But under the condition that $f \leq 2n - 2k - 2$, this effect is not significant.

6. Conclusion

In this paper, we consider the problem of many-to-many disjoint paths in the hypercube Q_n with f faulty vertices. We prove that for any integer k with $1 \leq k \leq n-1$ and any two sets S and T of k fault-free vertices in different parts of Q_n ($n \geq 3$), if $f \leq 2n - 2k - 2$ and each fault-free vertex has at least two fault-free neighbors, then there exist k fully disjoint fault-free paths linking S and T which contain at least $2^n - 2f$ vertices. Weakening the condition that each fault-free vertex has at least two fault-free neighbors is an interesting future research question. We also propose a centralized $O(2^n)$ -algorithm for finding many-to-many disjoint paths in Q_n , whose running time is not greatly affected by the faulty vertices. When n is very large, our algorithm cannot find the paths required in a reasonable time. It could be possible to modify our algorithm to a distributed one and find the paths in a relatively less time.

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References

- [1] X.-B. Chen, Many-to-many disjoint paths in faulty hypercubes, *Inf. Sci.* 179 (2009) 3110–3115.
- [2] X.-B. Chen, Paired many-to-many disjoint path covers of hypercubes with faulty edges, *Inf. Process. Lett.* 112 (2012) 61–66.
- [3] D. Cheng, R.-X. Hao, Y.-Q. Feng, Two node-disjoint paths in balanced hypercubes, *Appl. Math. Comput.* 242 (2014) 127–142.
- [4] N.-W. Chang, S.-Y. Hsieh, Fault-tolerant bipancyclicity of faulty hypercubes under the generalized conditional-fault model, *IEEE Trans. Commun.* 59 (12) (2011) 3400–3409.
- [5] C.-H. Chang, C.-K. Lin, H.-M. Huang, L.-H. Hsu, The super laceability of the hypercubes, *Inform. Process. Lett.* 92 (2004) 15–21.
- [6] C.-H. Chang, C.-M. Sun, H.-M. Huang, L.-H. Hsu, On the equitable k^* -laceability of hypercubes, *J. Comb. Optim.* 14 (2007) 349–364.
- [7] T. Dvořák, Hamiltonian cycles with prescribed edges in hypercubes, *SIAM J. Discrete Math.* 19 (2005) 135–144.
- [8] T. Dvořák, P. Gregor, Partitions of faulty hypercubes into paths with prescribed endvertices, *SIAM J. Discrete Math.* 22 (4) (2008) 1448–1461.
- [9] T. Dvořák, V. Kouřek, Long paths in hypercubes with a quadratic number of faults, *Inf. Sci.* 179 (2009) 3763–3771.
- [10] J. Fink, P. Gregor, Long paths and cycles in hypercubes with faulty vertices, *Inf. Sci.* 179 (2009) 3634–3644.
- [11] J. Fink, P. Gregor, Long cycles in hypercubes with optimal number of faulty vertices, *J. Comb. Optim.* 24 (2012) 240–265.
- [12] J.-S. Fu, Longest fault-free paths in hypercubes with vertex faults, *Inf. Sci.* 176 (7) (2006) 759–771.
- [13] P. Gregor, T. Dvořák, Path partitions of hypercubes, *Inform. Process. Lett.* 108 (2008) 402–406.
- [14] P. Gregor, R. Škrekovski, Long cycles in hypercubes with distant faulty vertices, *Discrete Math. Theor. Comput. Sci.* 11 (1) (2009) 185–198.
- [15] F. Harary, J.P. Hayes, H.-J. Wu, A survey of the theory of hypercube graphs, *Comput. Math. Appl.* 15 (4) (1988) 277–289.
- [16] S.-Y. Hsieh, G.-H. Chen, C.-W. Ho, Longest fault-free paths in star graphs with vertex faults, *Theoret. Comput. Sci.* 262 (2001) 215–227.
- [17] S.-Y. Hsieh, Y.-F. Weng, Fault-tolerant embedding of pairwise independent Hamiltonian paths on a faulty hypercube with edge faults, *Theor. Comput. Syst.* 45 (2) (2009) 407–425.
- [18] S. Jo, J.-H. Park, K.-Y. Chwa, Paired many-to-many disjoint path covers in faulty hypercubes, *Theoret. Comput. Sci.* 513 (2013) 1–24.
- [19] T.-L. Kueng, T. Liang, L.-H. Hsu, J.J.M. Tan, Long paths in hypercubes with conditional node-faults, *Inf. Sci.* 179 (2009) 667–681.
- [20] C.-N. Lai, An efficient construction of one-to-many node-disjoint paths in folded hypercubes, *J. Parallel Distrib. Comput.* 74 (2014) 2310–2316.
- [21] F.T. Leighton, *Introduction to Parallel Algorithms and Architectures: Arrays, Trees, Hypercubes*, Morgan Kaufmann, San Mateo, CA, 1992.
- [22] D. Liu, J. Li, Many-to-many n -disjoint path covers in n -dimensional hypercubes, *Inform. Process. Lett.* 110 (14–15) (2010) 580–584.
- [23] M.-J. Ma, The spanning connectivity of folded hypercubes, *Inf. Sci.* 180 (2010) 3373–3379.
- [24] J.-H. Park, I. Ihm, Many-to-many two-disjoint path covers in cylindrical and toroidal grids, *Discrete Appl. Math.* 185 (2015) 168–191.
- [25] J.-H. Park, H.-C. Kim, H.-S. Lim, Many-to-many disjoint path covers in hypercube-like interconnection networks with faulty elements, *IEEE Trans. Parallel. Distrib. Syst.* 17 (3) (2006) 227–240.
- [26] J.-H. Park, H.-C. Kim, H.-S. Lim, Many-to-many disjoint path covers in the presence of faulty elements, *IEEE Trans. Comput.* 58 (2009) 528–540.
- [27] J.-M. Xu, *Topological Structure and Analysis of Interconnection Networks*, Kluwer Academic Publishers, Dordrecht/Boston/London, 2001.
- [28] J.-M. Xu, *Theory and Application of Graphs*, Kluwer Academic Publishers, Dordrecht/Boston/London, 2003.
- [29] J.-M. Xu, M.-J. Ma, A survey on cycle and path embedding in some networks, *Front. Math. China.* 4 (2) (2009) 217–252.
- [30] S. Zhang, S. W. Many-to-many disjoint path covers in k -ary n -cubes, *Theoret. Comput. Sci.* 491 (2013) 103–118.