

On restricted edge-connectivity of replacement product graphs

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Abstract This paper considers the edge-connectivity and the restricted edge-connectivity of replacement product graphs, gives some bounds on edge-connectivity and restricted edge-connectivity of replacement product graphs and determines the exact values for some special graphs. In particular, the authors further confirm that under certain conditions, the replacement product of two Cayley graphs is also a Cayley graph, and give a necessary and sufficient condition for such Cayley graphs to have maximum restricted edge-connectivity. Based on these results, we construct a Cayley graph with degree d whose restricted edge-connectivity is equal to $d + s$ for given odd integer d and integer s with $d \geq 5$ and $1 \leq s \leq d - 3$, which answers a problem proposed ten years ago.

Keywords graph theory, connectivity, restricted edge-connectivity, replacement product, Cayley graph

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1 Introduction

Throughout this paper, we follow [35] for graph-theoretical terminologies and notation not defined here. Specially, $G = (V, E)$ is a simple connected undirected graph, where $V = V(G)$ is the vertex-set of G and $E = E(G)$ is the edge-set of G ; $d_G(x)$ is the degree of a vertex x in G , the number of edges incident with x in G ; $\delta(G) = \min\{d_G(x) : x \in V(G)\}$ is the minimum degree of G ; $\xi(G) = \min\{d_G(x) + d_G(y) - 2 : xy \in E(G)\}$ is the minimum edge-degree of G .

The connectivity $\kappa(G)$ (resp. edge-connectivity $\lambda(G)$) of G is defined as the minimum number of vertices (resp. edges) whose removal results in disconnected. The well-known Whitney inequality states that $\kappa(G) \leq \lambda(G) \leq \delta(G)$ for any graph G . In this paper, we are interested in the edge-connectivity $\lambda(G)$.

It is well known that when the underlying topology of an interconnection network is modeled by a connected graph $G = (V, E)$, where V is the set of processors and E is the set of communication links in the network, the edge-connectivity $\lambda(G)$ of G is an important measurement for reliability and fault tolerance of the network since the larger $\lambda(G)$ is, the more reliable the network is. However, when computing $\lambda(G)$, one implicitly assumes that all edges incident with the same vertex may fail

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simultaneously. Consequently, this measurement is inaccurate for large-scale processing systems in which some subsets of system links cannot fail at the same time in real applications.

To overcome the shortcomings of edge-connectivity, Esfahanian and Hakimi [5] proposed the concept of the restricted edge-connectivity $\lambda'(G)$ of a graph G , which is the minimum number of edges whose removal results in disconnected and no isolated vertices, and gave the following result.

Theorem 1.1 (See [5]). $\lambda(G) \leq \lambda'(G) \leq \xi(G)$ for any graph G of order n (≥ 4) except for a star $K_{1,n-1}$.

A graph G is vertex-transitive if for any two vertices x and y in G , there is a $\sigma \in \text{Aut}(G)$ such that $y = \sigma(x)$, where $\text{Aut}(G)$ is the automorphism group of G . Clearly, $\xi(G) = 2d - 2$ for a vertex-transitive connected graph G with degree d . Xu et al. obtained the following results.

Theorem 1.2 (See [37]). Let G be a vertex-transitive connected graph with order n (≥ 4) and degree d (≥ 2). Then

- (a) $\lambda'(G) = \xi(G) = 2d - 2$ if n is odd or G contains no triangles;
- (b) there exists an integer m (≥ 2) such that $d \leq \lambda'(G) = \frac{n}{m} \leq 2d - 3$ otherwise.

Theorem 1.3 (See [18]). For any given integers d and s with $d \geq 3$ and $0 \leq s \leq d - 3$, there is a connected vertex-transitive graph G with degree d and $\lambda'(G) = d + s$ if and only if either d is odd or s is even.

In [18], for any odd integer d (≥ 3) and any integer s with $0 \leq s \leq d - 3$, Li and Xu constructed a vertex-transitive graph G with degree d and $\lambda'(G) = d + s = \frac{1}{2}n$. Note that the condition " $d \leq \lambda'(G) \leq 2d - 3$ " implies $\lambda'(G) = d$ if $d = 3$. By Theorem 1.2, if a vertex-transitive graph G is not λ' -optimal, then $d \leq \lambda'(G) \leq \frac{n}{2}$. Thus, a quite natural problem is proposed as follows (see [36, Conjecture 1]).

Problem 1.4. Given an odd integer d (≥ 5) and any integer s with $1 \leq s \leq d - 3$, whether or not there is a vertex-transitive graph G with order n and degree d such that $\lambda'(G) = d + s < \frac{1}{2}n$.

In this paper, we answer this question confirmedly by constructing a Cayley graph, which is the replacement product of two Cayley graphs.

We will discuss the restricted edge-connectivity of a replacement product graph in this paper. The rest of this paper is organized as follows. In Section 2, we give some definitions with related results. In Section 3, we establish the bounds on the edge-connectivity for a replacement product graph and determine exact values under some special conditions. In Section 4, we give the lower and upper bounds on restricted edge-connectivity for replacement product graphs and determine exact values under some given conditions. In Section 5, we focus on Cayley graphs and further confirm that under certain conditions, the replacement product of two Cayley graphs is still a Cayley graph, and give a necessary and sufficient condition for such Cayley graphs to have maximum restricted edge-connectivity. Based on these results, we construct a Cayley graph to answer Problem 1.4 confirmedly. A conclusion is presented in Section 6.

2 Preliminaries

We first introduce the concept of the restricted edge-connectivity, which was proposed by Esfahanian and Hakimi [5], and we state it here slightly different from theirs.

Let G be a non-trivial connected graph and $F \subset E(G)$. If $G - F$ is disconnected and contains no isolated vertices, then F is called a restricted edge-cut of G . The restricted edge-connectivity of G , denoted by $\lambda'(G)$, is defined as the minimum cardinality over all restricted edge-cuts of G . Esfahanian and Hakimi [5] proved $\lambda'(G)$ is well-defined for any connected graph G of order n (≥ 4) except for a star $K_{1,n-1}$. A graph G is λ' -connected if $\lambda'(G)$ exists, and a restricted edge-cut F is a λ' -cut if $|F| = \lambda'(G)$. A λ' -connected graph is λ' -optimal if $\lambda'(G) = \xi(G)$, and not λ' -optimal otherwise. It is clear that if G is a δ -regular and λ' -optimal graph of order n , then $\lambda(G) = \delta(G) = \delta$ and $n \geq 4$.

The restricted edge-connectivity provides a more accurate measure of fault-tolerance of networks than the edge-connectivity (see [4, 5]). Thus, determining the value of λ' for some special classes of graphs or

characterizing λ' -optimal graphs have received considerable attention in the literature (see, for example, [5, 11, 12, 20, 23, 24, 31–33]).

Let Γ be a finite group, and let S be a subset of Γ not containing the identity element of Γ . The *Cayley graph* $C_\Gamma(S)$ is the graph having vertex-set Γ and edge-set $\{xy : x^{-1}y \in S, x, y \in \Gamma\}$.

Generally speaking, $C_\Gamma(S)$ is a digraph. The following result is well-known (see, for example, [34]).

Lemma 2.1. *Cayley graphs are vertex-transitive and the Cartesian product of Cayley graphs is a Cayley graph.*

If $S = S^{-1}$, then $C_\Gamma(S)$ is an undirected graph. We are interested in undirected graphs in this paper.

We now introduce two classes of Cayley graphs. Because of their excellent features, they are the most popular, versatile and efficient topological structures of interconnection networks (see, for example, [34]).

Example 2.2. A *circulant graph* $G(n; \pm S)$, where $S = \{s_1, s_2, \dots, s_k\} \subseteq \{1, 2, \dots, \lfloor \frac{1}{2}n \rfloor\}$ with $s_1 < s_2 < \dots < s_k$ and $n \geq 3$, has vertex-set $V = \{0, 1, \dots, n-1\}$ and edge-set $E = \{ij : |j-i| \equiv s_i \pmod{n} \text{ for some } s_i \in S\}$.

Clearly, $G(n; \pm 1)$ is a cycle C_n and $G(n; \pm\{1, 2, \dots, \lfloor \frac{1}{2}n \rfloor\})$ is a complete graph K_n . The two graphs shown in Figure 1 are $G(8; \pm\{1, 3\})$ and $G(8; \pm\{1, 3, 4\})$.

Note that the identity element of the ring group $\mathbb{Z}_n$ ($n \geq 2$) is just the zero element, and the inverse of any $i \in \mathbb{Z}_n$ is $n-i$. If let $S \subseteq \{1, 2, \dots, n-1\}$ and $S^{-1} = S$, then Cayley graph $C_{\mathbb{Z}_n}(S)$ is a circulant graph $G(n; S)$ if $n \geq 3$, and $C_{\mathbb{Z}_2}(S) = K_2$. Thus, circulant graphs are vertex-transitive by Lemma 2.1.

Li and Li [19] showed that $G(n; \pm S)$ is λ' -optimal and $\lambda'(G(n; \pm S)) = 4k - 2$ if $k \geq 2$ and $s_k < \frac{n}{2}$.

Example 2.3. The *hypercube* Q_n has the vertex-set consisting of 2^n binary strings of length n , two vertices being linked by an edge if and only if they differ in exactly one position. Hypercubes Q_1, Q_2, Q_3 and Q_4 are shown in Figure 2.

It is easy to see that the hypercube Q_n is Cartesian products $K_2 \times K_2 \times \dots \times K_2$ of n complete graph K_2 . Let $(\mathbb{Z}_2)^n = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \dots \times \mathbb{Z}_2$ and

$$e_0 = \underbrace{0 \cdots 0}_n \quad \text{and} \quad e_i = \underbrace{0 \cdots 0}_{i-1} 1 \underbrace{0 \cdots 0}_{n-i} \quad \text{for each } i = 1, 2, \dots, n. \quad (2.1)$$

Then e_0 is the identity element of $(\mathbb{Z}_2)^n$ and, by Lemma 2.1, Q_n is a Cayley graph $C_{(\mathbb{Z}_2)^n}(S)$ and so is vertex-transitive, where $S = \{e_1, e_2, \dots, e_n\}$, each of which is self-inverse and, hence, $S = S^{-1}$.

Esfahanian [4] showed that the hypercube Q_n is λ' -optimal, i.e., $\lambda'(Q_n) = 2n - 2$ for $n \geq 2$.

Now, we introduce the *replacement product*. There are several equivalent definitions of the replacement product proposed by Hoory et al. [13] and Reingold et al. [28]. Here, we adopt the definition proposed by Hoory et al. [13]. Let G_1 be a δ_1 -regular graph on n vertices and G_2 be a δ_2 -regular graph on δ_1 vertices. For every vertex $x \in V(G_1)$, we label on all edges incident with x , say $e_x^1, e_x^2, \dots, e_x^{\delta_1}$.

Definition 2.4. Let G_1 be a δ_1 -regular graph on n vertices and G_2 be a δ_2 -regular graph on δ_1 vertices. The *replacement product* of G_1 and G_2 is a graph, denoted by $G_1 \mathbb{R} G_2$, where $V(G_1 \mathbb{R} G_2) = V(G_1) \times V(G_2)$, two distinct vertices (x, i) and (y, j) , where $x, y \in V(G_1)$ and $i, j \in V(G_2)$, are linked by an edge in $G_1 \mathbb{R} G_2$ if and only if either $x = y$ and $ij \in E(G_2)$, or $xy \in E(G_1)$ and $e_x^i = xy = e_y^j$.

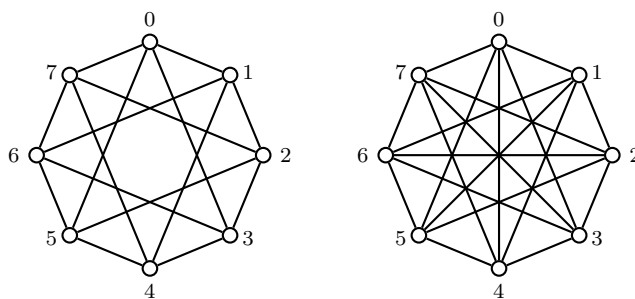


Figure 1 (a) $G(8; \pm\{1, 3\})$; (b) $G(8; \pm\{1, 3, 4\})$

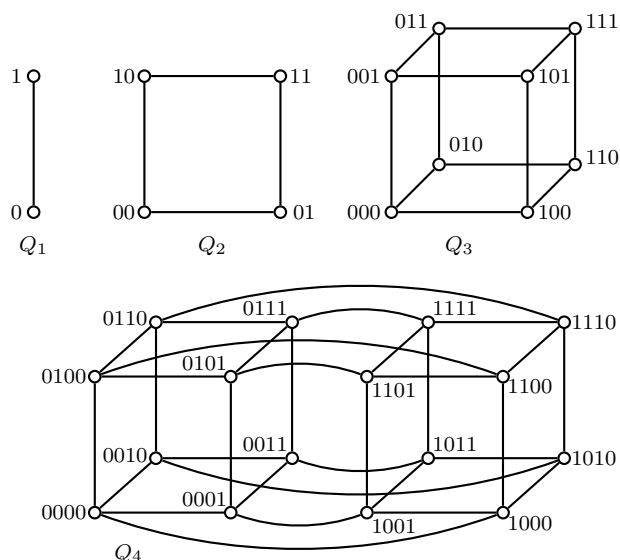


Figure 2 The n -cubes Q_1 , Q_2 , Q_3 and Q_4

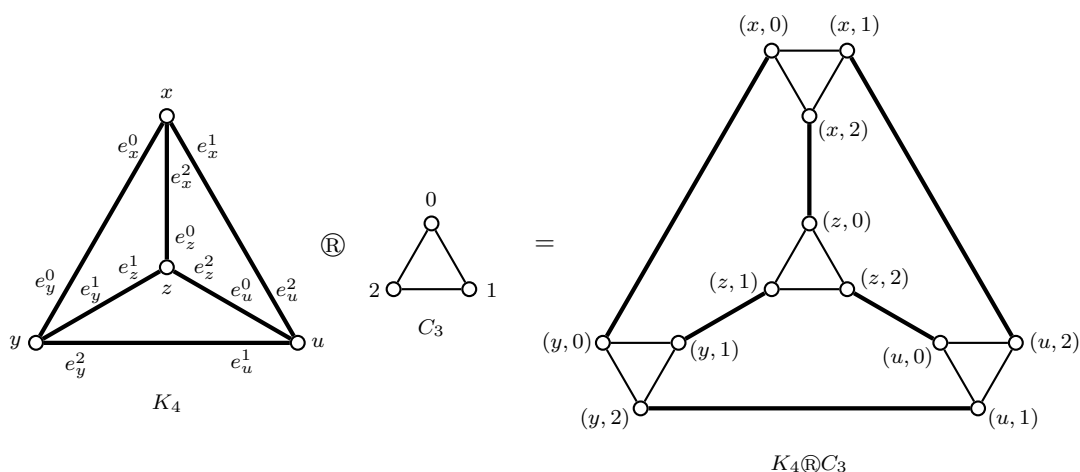


Figure 3 $K_4 @ C_3$

Figure 3 shows the replacement product of K_4 and C_3 with given labelling of edges around vertices of K_4 .

By Definition 2.4, we can obtain the following proposition.

Proposition 2.5. $G_1 @ G_2$ is $(\delta_2 + 1)$ -regular and has $n\delta_1$ vertices. Moreover, the vertex-set of $G_1 @ G_2$ can be partitioned into $\{X_1, X_2, \dots, X_n\}$ such that $G[X_i] \cong G_2$ for each $i \in I_n$.

The *inflation* or *inflated graph* of G is a graph obtained from G by replacing each vertex x by a complete graph $K_{d_G(x)}$ and joining each edge to a different vertex of $K_{d_G(x)}$. Inflation graphs have been studied in [3, 6, 7, 14, 21, 27]. Clearly, if G is n -regular then $G @ K_n$ is the inflation graph of G . In special, Liu and Zhang [21] showed that $Q_n @ K_n$ is a Cayley graph.

The *lexicographic product* $G_1[G_2]$ of two graphs G_1 and G_2 is a graph with vertex-set $V(G_1) \times V(G_2)$, and in which two vertices (x, i) and (y, j) are adjacent if and only if either $x = y$ and $ij \in E(G_2)$ or $xy \in E(G_1)$, without the condition “ $e_x^i = xy = e_y^j$ ”. Thus, the replacement product graph $G_1 @ G_2$ is a subgraph of the lexicographic product graph $G_1[G_2]$. In special, Li et al. [17] showed that $G_1[G_2]$ is a Cayley graph if G_1 and G_2 are Cayley graphs.

The replacement product of two graphs is an important constructing method, which can obtain a larger graph from two smaller graphs, and so it has been widely used to address many fundamental problems in such areas as graph theory, combinatorics, probability, group theory, in the study of expander graphs and

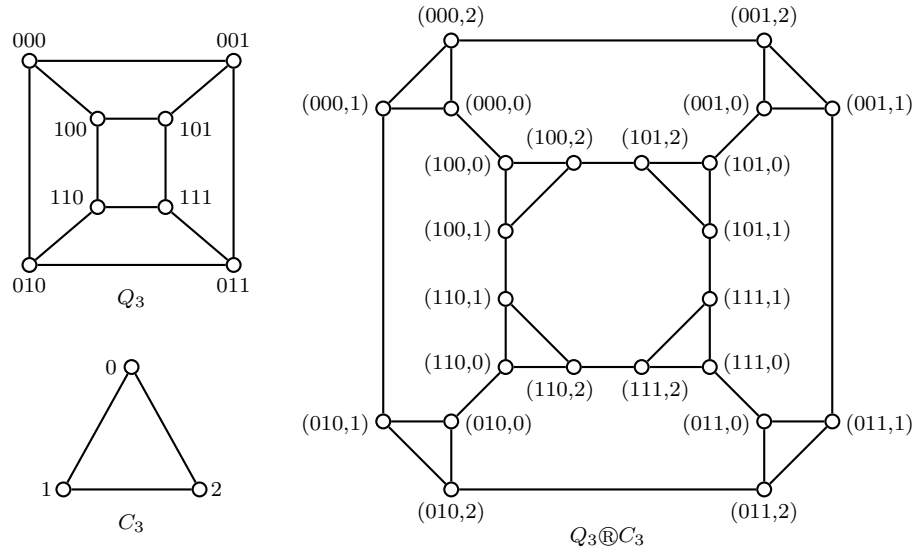


Figure 4 The cube-connected cycle $CCC(3) = Q_3 \otimes C_3$

graph-based coding schemes [1, 2, 10, 13, 15, 16, 28]. The replacement product has been also used in the designing of an interconnection networks. For example, the well-known n -dimensional cube-connected cycle CCC_n is a replacement product $Q_n \otimes C_n$, where Q_n is a hypercube and C_n is a cycle of length n (see [26]). The graph shown in Figure 4 is $Q_3 \otimes C_3 = CCC_3$. In addition, n -dimensional hierarchical hypercube is a replacement product $Q_{2^n} \otimes Q_n$ (see [22]).

For simplicity, when a replacement product graph $G_1 \otimes G_2$ is mentioned, if no otherwise specified, we always assume that G_1 is a δ_1 -regular graph with n vertices and G_2 is a δ_2 -regular graph with δ_1 vertices. Moreover, we simply write $\kappa_i = \kappa(G_i)$, $\lambda_i = \lambda(G_i)$ and $\delta_i = \delta(G_i)$ for each $i = 1, 2$, and write xG_2 for $\{x\} \times G_2$ for any $x \in V(G_1)$, and let $I_n = \{1, 2, \dots, n\}$.

In this paper, we also need some notation. For a subset $X \subset V(G)$, use $G[X]$ to denote the subgraph of G induced by X . For two disjoint subsets X and Y in $V(G)$, use $[X, Y]$ to denote the set of edges between X and Y in G . In particular, $E_G(X) = [X, \overline{X}]$ and let $d_G(X) = |E_G(X)|$, where $\overline{X} = V(G) \setminus X$.

For a λ' -connected graph G , there is certainly a subset $X \subset V(G)$ with $|X| \geq 2$ such that $E_G(X)$ is a λ' -cut and, both $G[X]$ and $G[\overline{X}]$ are connected. Such an X is called a λ' -fragment of G . A λ' -fragment X of G with minimum cardinality is called a λ' -atom of G . The λ' -atom has been successfully used in the study of restricted edge-connectivity of graphs (see, for example, [23, 25, 30, 37]).

3 Edge-connectivity of $G_1 \otimes G_2$

In this section, we investigate the edge-connectivity of replacement product graph $G_1 \otimes G_2$. By Definition 2.4, it is easy to see that if G_1 and G_2 are connected, then $G_1 \otimes G_2$ is also connected. We now establish the upper and lower bounds on the edge-connectivity for replacement product graphs.

Theorem 3.1. *If both G_1 and G_2 are connected, then*

$$\min\{\lambda_1, \lambda_2\} \leq \lambda(G_1 \otimes G_2) \leq \min\{\lambda_1, \delta_2 + 1\}. \quad (3.1)$$

Furthermore,

$$\min\{\lambda_1, \lambda_2 + 1\} \leq \lambda(G_1 \otimes G_2) \quad \text{if} \quad \kappa_1 \geq 2. \quad (3.2)$$

Proof. Let $G = G_1 \otimes G_2$. Clearly,

$$\lambda(G) \leq \delta(G) = \delta_2 + 1. \quad (3.3)$$

Let $S \subset V(G_1)$ and $E_{G_1}(S)$ be a λ_1 -cut of G_1 , and $T = \{(x, i) : x \in S, i \in V(G_2)\}$. Then $E_G(T)$ is an edge-cut of G . Since there is an edge xy in G_1 if and only if there is exactly one edge between $V(xG_2)$ and $V(yG_2)$ in G , $xy \in E_{G_1}(S)$ if and only if there are two vertices i and j of G_2 such that $((x, i), (y, j)) \in E_G(T)$. Therefore, $|E_G(T)| = |E_{G_1}(S)| = \lambda_1$ and

$$\lambda(G) \leq |E_G(T)| = \lambda_1. \quad (3.4)$$

Combining (3.3) with (3.4), we establish the upper bound on $\lambda(G_1 \mathbb{R} G_2)$ in (3.1). We now show the lower bound in (3.1).

Let F be a λ -cut in G . Then there are two λ -fragments associated with F in G , say, X and $\overline{X}$. Let $\{V_1, V_2, \dots, V_n\}$ be a partition of $V(G)$ satisfying the property in Proposition 2.5.

Assume for each $i \in I_n$, either $V_i \subset X$ or $V_i \subset \overline{X}$. Let $Y = \{i : V_i \subset X, i \in I_n\}$. Then $Y \subset V(G_1)$, $E_{G_1}(Y)$ is an edge-cut of G_1 and $|E_{G_1}(Y)| = |F|$, and so

$$\lambda(G) = |F| = |E_{G_1}(Y)| \geq \lambda_1. \quad (3.5)$$

Assume now that there exists some $i \in I_n$ such that $V_i \cap X \neq \emptyset$ and $V_i \cap \overline{X} \neq \emptyset$. Then

$$\lambda(G) = |F| \geq |[V_i \cap X, V_i \cap \overline{X}]| \geq \lambda(G[V_i]) = \lambda(G_2) = \lambda_2. \quad (3.6)$$

Combining (3.5) with (3.6), we establish the lower bound on $\lambda(G_1 \mathbb{R} G_2)$ in (3.1).

To prove (3.2), let (x, i) be any vertex of $V_x \cap X$ and (x, j) be any vertex of $V_x \cap \overline{X}$. Since $G[V_x] \cong G_2$ and G_2 is λ_2 -connected, there exist λ_2 edge-disjoint paths $P_1, P_2, \dots, P_{\lambda_2}$ between (x, i) and (x, j) in $G[V_x] \subset G$. Let $(y, k) \in N_G((x, i))$ and $(z, \ell) \in N_G((x, j))$, where $\{y, z\} \subseteq N_{G_1}(x)$. Since $\kappa_1 \geq 2$, there exist at least two internally vertex-disjoint paths between y and z in G_1 , one of them avoids x . By the connectedness of G_2 , there exists a path Q between (y, k) and (z, ℓ) in G that avoids the vertices of V_x . Let $P_0 = \langle (x, i), Q, (x, j) \rangle$. Thus, $P_0, P_1, P_2, \dots, P_{\lambda_2}$ are $\lambda_2 + 1$ edge-disjoint paths between (x, i) and (x, j) . Since $(x, i) \in V_x \cap X$ and $(x, j) \in V_x \cap \overline{X}$, it is easy to find $|E(P_i) \cap F| \geq 1$ for each $i \in \{0, 1, 2, \dots, \lambda_2\}$ and so $\lambda(G) = |F| \geq \lambda_2 + 1$ as required. $\square$

Combining the Whitney's inequality $\kappa(G) \leq \lambda(G) \leq \delta(G)$ with Theorem 3.1, we obtain the following results immediately.

Corollary 3.2. Suppose that both G_1 and G_2 are connected. Then

- (a) $\lambda(G_1 \mathbb{R} G_2) = 1$ if $\lambda_1 = 1$;
- (b) $\lambda(G_1 \mathbb{R} G_2) = \lambda_1$ if $\lambda_2 \geq \lambda_1$;
- (c) $\lambda(G_1 \mathbb{R} G_2) = \lambda_1$ if $\kappa_1 \geq 2$ and $\lambda_2 \geq \lambda_1 - 1$;
- (d) $\lambda(G_1 \mathbb{R} G_2) = \min\{\lambda_1, \delta_2 + 1\}$ if $\kappa_1 \geq 2$ and $\lambda_2 = \delta_2$.

Lemma 3.3. $\lambda(G) \leq \frac{1}{2}\Delta(G)$ for any connected graph that contains cut-vertices.

Proof. Suppose that x is a cut-vertex of G and $G - x$ has k components, where $k \geq 2$. Then $\lambda(G) \leq \frac{1}{k}|N(x)| \leq \frac{1}{2}\Delta(G)$. $\square$

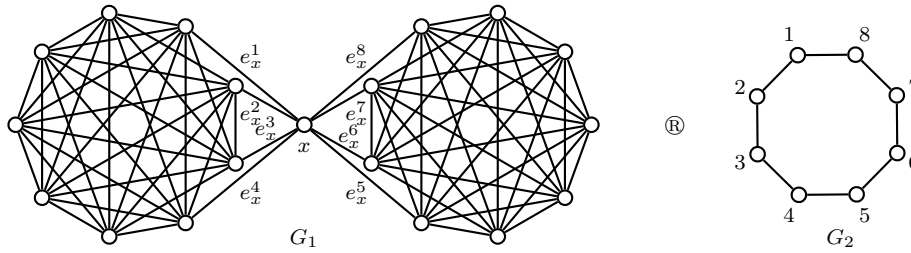
Corollary 3.4. $\lambda(G \mathbb{R} K_n) = \lambda(G)$ for any n -regular connected graph G .

Proof. Clearly, $\lambda(G) \leq \delta(G) = n$ and $\lambda(K_n) = \delta(K_n) = n - 1$. If $\kappa(G) \geq 2$, then $\lambda(G \mathbb{R} K_n) = \lambda(G)$ by Theorem 3.1. If $\kappa(G) = 1$, by Lemma 3.3, then $\lambda(G) \leq \frac{n}{2} < n$ and so $\lambda(K_n) \geq \lambda(G)$. By Corollary 3.2(b), the result follows. $\square$

Corollary 3.5. $\lambda(G \mathbb{R} C_n) = \min\{\lambda(G), 3\}$ for any 2-connected n -regular graph G .

Example 3.6. $\lambda(K_4 \mathbb{R} C_3) = \lambda(K_4) = 3$, and $\lambda(CCC_n) = \lambda(Q_n \mathbb{R} C_n) = \min\{\lambda(Q_n), 3\} = \min\{n, 3\} = 3$ if $n \geq 3$.

Remark 3.7. We conclude this section with a remark on Theorem 3.1. The condition " $\kappa_1 \geq 2$ " in (3.2) is necessary. For example, two graphs G_1 and G_2 are shown in Figure 5. It is easy to see that $\kappa_1 = 1$, $\lambda_1 = 4$, and $\lambda_2 = \delta_2 = 2$, $G_1 \mathbb{R} G_2$ is 3-regular, and $\lambda(G_1 \mathbb{R} G_2) = 2 < \min\{4, 3\} = \min\{\lambda_1, \lambda_2 + 1\}$, which contradicts the lower bound on $\lambda(G_1 \mathbb{R} G_2)$ given in (3.2).

Figure 5 Two graphs G_1 and G_2 in Remark 3.7

4 Restricted edge-connectivity of $G_1 \otimes G_2$

In this section, we investigate the restricted edge-connectivity of the replacement product of two regular graphs.

Theorem 4.1. *If both G_1 and G_2 are connected, then $\lambda'(G_1 \otimes G_2) \leq \min\{\lambda_1, 2\delta_2\}$.*

Proof. Let $G = G_1 \otimes G_2$. Since G is $(\delta_2 + 1)$ -regular and $\delta_2 + 1 \geq 2$, it is easy to see that G is λ' -connected. By Theorem 1.1,

$$\lambda'(G) \leq \xi(G) = 2\delta_2. \quad (4.1)$$

Let $X \subset V(G_1)$ such that $[X, \overline{X}]_{G_1}$ is a λ_1 -cut of G_1 . Then $G_1[X]$ and $G_1[\overline{X}]$ are both connected. Let $Y = \{(x, i) : x \in X, i \in V(G_2)\}$. Then $G[Y]$ and $G[\overline{Y}]$ are connected, and $|Y| \geq |V(G_2)| = \delta_1 \geq 2$, $|\overline{Y}| \geq |V(G_2)| = \delta_1 \geq 2$. Hence, $[Y, \overline{Y}]_G$ is a restricted edge-cut of G . There is an edge xy in G_1 if and only if there is exactly one edge between $V(xG_2)$ and $V(yG_2)$ in G , so $xy \in [X, \overline{X}]_{G_1}$ if and only if there are two vertices i and j of G_2 such that $((x, i), (y, j)) \in [Y, \overline{Y}]_G$. Therefore, $|[Y, \overline{Y}]_G| = |[X, \overline{X}]_{G_1}| = \lambda_1$ and

$$\lambda'(G) \leq |[Y, \overline{Y}]_G| = \lambda_1. \quad (4.2)$$

Combining (4.1) with (4.2), the result follows. $\square$

Theorem 4.2. $\lambda'(G \otimes K_n) = \lambda(G)$ for any n -regular connected graph G .

Proof. By Corollary 3.4, $\lambda(G \otimes K_n) = \lambda(G)$. By Theorem 4.1, $\lambda'(G \otimes K_n) \leq \lambda(G)$, and so

$$\lambda(G) = \lambda(G \otimes K_n) \leq \lambda'(G \otimes K_n) \leq \lambda(G).$$

The result follows. $\square$

For $\delta_1 \leq 3$, Theorem 4.2 shows that $\lambda'(G_1 \otimes G_2) = \lambda(G_1)$. In the following discussion, we always assume $\delta_1 \geq 4$.

Lemma 4.3. *Suppose that both G_1 and G_2 are connected and $\delta_1 \geq 4$, F is a λ' -cut of $G_1 \otimes G_2$ and $\{X_1, X_2, \dots, X_n\}$ is a partition of $V(G_1 \otimes G_2)$ satisfying the property in Proposition 2.5. If there is some $i \in I_n$ such that $G[X_i]$ is disconnected in $G - F$, then*

$$\lambda'(G_1 \otimes G_2) \geq \min\{\kappa_1 + \lambda_2 - 1, 2\lambda_2, \lambda'_2 + 2\}. \quad (4.3)$$

Proof. Let $G = G_1 \otimes G_2$. Since F is a λ' -cut of G , there is some $X \subset V(G)$ with $|X| \geq 2$ such that $F = E_G(X)$. Without loss of generality we assume $|\overline{X}| \geq |X|$.

If there exist two distinct $j, k \in I_n$ such that $G[X_j]$ and $G[X_k]$ are disconnected in $G - F$, then

$$|F| \geq \lambda(G[X_j]) + \lambda(G[X_k]) = 2\lambda_2. \quad (4.4)$$

Now assume that there exists exactly one integer, say $j \in I_n$, such that $G[X_j]$ is disconnected in $G - F$. Then $X_j \cap X \neq \emptyset$ and $X_j \cap \overline{X} \neq \emptyset$. Consider the following two cases.

Case 1. $X \subset X_j$. In this case, $\overline{X} = (V(G) \setminus X_j) \cup (X_j \setminus X)$. Thus

$$|F| = |[X, \overline{X}]| = |[X, (V(G) \setminus X_j)]| + |[X, X_j \setminus X]| = |X| + |[X, X_j \setminus X]|.$$

If $|X| = \delta_1 - 1$, then $|[X, X_j \setminus X]| = \delta_2$, and so $|F| = \delta_1 - 1 + \delta_2 \geq 2\delta_2 \geq 2\lambda_2$. If $2 \leq |X| \leq \delta_1 - 2$, then $[X, X_j \setminus X]$ is a restricted edge-cut of $G[X_j]$, and so $|F| \geq |X| + \lambda'(G[X_j]) \geq \lambda'_2 + 2$. Hence, in this case,

$$|F| \geq \min\{2\lambda_2, \lambda'_2 + 2\}. \quad (4.5)$$

Case 2. $X \not\subseteq X_j$. Since $|\overline{X}| \geq |X|$, $\overline{X} \not\subseteq X_j$. Equivalently, there exist at least two sets X_k and X_ℓ other than X_j such that $X_k \subset X$ and $X_\ell \subset \overline{X}$. Since $\kappa(G_1 - u) \geq \kappa_1 - 1 \geq 0$ for any vertex $u \in V(G_1)$, there are at least $\kappa_1 - 1$ internally vertex-disjoint paths between any two distinct vertices x and y in $G_1 - u$. By the definition of G , it is easy to see that there are at least $\kappa_1 - 1$ internally vertex-disjoint paths $P_1, P_2, \dots, P_{\kappa_1-1}$ between X_k and X_ℓ in $G - X_j$. Let $F' = F \setminus [X_j \cap X, X_j \cap \overline{X}]$. Since $X_k \subset X$ and $X_\ell \subset \overline{X}$, $|E(P_i) \cap F'| \geq 1$ for each $i \in \{1, 2, \dots, \kappa_1 - 1\}$ and $|F'| \geq \kappa_1 - 1$. Thus, we have

$$|F| = |[X, \overline{X}]| = |F'| + |[X_j \cap X, X_j \cap \overline{X}]| \geq \kappa_1 - 1 + \lambda(G[X_j]) = \kappa_1 + \lambda_2 - 1,$$

i.e.,

$$|F| \geq \kappa_1 + \lambda_2 - 1. \quad (4.6)$$

Note that if $\kappa_1 = 1$, then $|F| \geq |[X_j \cap X, X_j \cap \overline{X}]| \geq \lambda_2$ and so (4.6) also holds.

By (4.4)–(4.6), the inequality (4.3) is established. $\square$

Theorem 4.4. Suppose that both G_1 and G_2 are connected and $\delta_1 \geq 4$. Then

$$\min\{\lambda_1, \kappa_1 + \lambda_2 - 1, 2\lambda_2, \lambda'_2 + 2\} \leq \lambda'(G_1 \mathbin{\text{\textcircled{R}}} G_2) \leq \min\{\lambda_1, 2\delta_2\}. \quad (4.7)$$

Furthermore, if $\kappa_1 \geq \lambda_1 - \lambda_2 + 1$ (or $\kappa_1 \geq \lambda_2 + 1$) and G_2 is λ' -optimal, then

$$\lambda'(G_1 \mathbin{\text{\textcircled{R}}} G_2) = \min\{\lambda_1, 2\delta_2\}. \quad (4.8)$$

Proof. Let $G = G_1 \mathbin{\text{\textcircled{R}}} G_2$. By Theorem 4.1, we only need to show the lower bound on $\lambda'(G_1 \mathbin{\text{\textcircled{R}}} G_2)$ in (4.7). To the end, let $\{X_1, X_2, \dots, X_n\}$ be a partition of $V(G)$ satisfying the property in Proposition 2.5 and F be a λ' -cut of G . There is some $X \subset V(G)$ with $|X| \geq 2$ such that $F = E_G(X)$. Without loss of generality we assume $|\overline{X}| \geq |X|$.

By Lemma 4.3, we only need to show that $\lambda'(G) \geq \lambda_1$ if $G[X_i]$ is connected in $G - F$ for each $i \in I_n$.

In this case, either $X_i \subset X$ or $X_i \subset \overline{X}$ for each $i \in I_n$. Thus, we can assume $F = E_G(Y \times V(G_2))$, where $Y \subset V(G_1)$. By the definition of G , $|F| = |E_G(Y \times V(G_2))| = |E_{G_1}(Y)|$. Since $|E_{G_1}(Y)| \geq \lambda_1$, we have $|F| \geq \lambda_1$, and so the lower bound on $\lambda'(G_1 \mathbin{\text{\textcircled{R}}} G_2)$ in (4.7) is established.

We now show the equality (4.8). If $\kappa_1 \geq \lambda_1 - \lambda_2 + 1$ (or $\kappa_1 \geq \lambda_2 + 1$) and G_2 is λ' -optimal, then $\lambda'_2 = \xi(G_2) = 2\delta_2 - 2$, and so $\lambda_2 = \delta_2$. Thus, we have $\kappa_1 + \lambda_2 - 1 \geq \lambda_1$ (or $\kappa_1 + \lambda_2 - 1 \geq 2\lambda_2$), and so

$$\min\{\lambda_1, \kappa_1 + \lambda_2 - 1, 2\lambda_2, \lambda'_2 + 2\} = \min\{\lambda_1, 2\delta_2\}. \quad (4.9)$$

Comparing (4.7) with (4.9), the equality (4.8) is established. $\square$

Note that if G_2 is λ' -optimal then $\delta_1 = |V(G_2)| \geq 4$, and so $\lambda'(G_1 \mathbin{\text{\textcircled{R}}} G_2)$ is well-defined. By Theorem 4.4, we obtain the following corollary immediately.

Corollary 4.5. Assume G_1 and G_2 are connected. If $\kappa_1 = \lambda_1$ and G_2 is λ' -optimal, then

$$\lambda'(G_1 \mathbin{\text{\textcircled{R}}} G_2) = \min\{\lambda_1, 2\delta_2\}.$$

A connected graph G is *super- λ* if every λ -cut isolates a vertex in G . It is clear that G is super- λ if and only if $\lambda'(G) > \lambda(G)$. By Theorem 4.4, we obtain the following results immediately.

Theorem 4.6. Suppose that G_1 and G_2 are two connected graphs. If $\kappa_1 \geq \lambda_1 - \lambda_2 + 1 \geq 2$ (or $\kappa_1 \geq \lambda_2 + 1$) and G_2 is λ' -optimal, then

- (a) $G_1 \mathbin{\text{\textcircled{R}}} G_2$ is λ' -optimal if and only if $\lambda_1 \geq 2\delta_2$;
- (b) $G_1 \mathbin{\text{\textcircled{R}}} G_2$ is super- λ if and only if $\lambda_1 > \delta_2 + 1$.

Proof. Let $G = G_1 \mathbb{R} G_2$. Clearly, $\xi(G) = 2\delta_2$, and $\lambda_2 = \delta_2 \geq 2$ since G_2 is λ' -optimal.

Since $\kappa_1 \geq \lambda_1 - \lambda_2 + 1$ (or $\kappa_1 \geq \lambda_2 + 1$) and G_2 is λ' -optimal, by Theorem 4.4 we have that

$$\lambda'(G) = \min\{\lambda_1, 2\delta_2\}. \quad (4.10)$$

Thus, G is λ' -optimal if and only if $\lambda'(G) = \xi(G) = 2\delta_2$, i.e., $\lambda_1 \geq 2\delta_2$ from (4.10).

Since $\kappa_1 \geq 2$ and $\lambda_2 = \delta_2$, by Corollary 3.2(d) we have that

$$\lambda(G) = \min\{\lambda_1, \delta_2 + 1\}. \quad (4.11)$$

Note $2\delta_2 > \delta_2 + 1$ for $\delta_2 \geq 2$. It follows that G is super- λ if and only if $\lambda'(G) > \lambda(G)$, that is $\lambda_1 > \delta_2 + 1$ from (4.11). $\square$

Corollary 4.7. Assume G_1 and G_2 are two connected graphs with $\delta_1 \geq 4$. If $\kappa_1 = \lambda_1 \geq 2$ and G_2 is λ' -optimal, then

- (a) $G_1 \mathbb{R} G_2$ is λ' -optimal if and only if $\lambda_1 \geq 2\delta_2$;
- (b) $G_1 \mathbb{R} G_2$ is super- λ if and only if $\lambda_1 > \delta_2 + 1$.

Corollary 4.8. $\lambda'(G \mathbb{R} C_n) = \min\{\lambda(G), 4\}$ if G is an n -regular graph and $\kappa(G) \geq 3$.

Example 4.9. By Corollary 4.8, it is easy to see that $\lambda'(K_4 \mathbb{R} C_3) = \min\{\lambda(K_4), 4\} = \min\{3, 4\} = 3$, and

$$\lambda'(CCC_n) = \lambda'(Q_n \mathbb{R} C_n) = \min\{\lambda(Q_n), 4\} = \min\{n, 4\} = \begin{cases} 3, & \text{if } n = 3, \\ 4, & \text{if } n \geq 4. \end{cases}$$

5 Replacement product of Cayley graphs

In this section, we investigate the restricted edge-connectivity of the replacement product of two Cayley graphs by a semidirect product of two groups. We will further confirm that under certain conditions on the underlying groups and generating sets, the replacement product of two Cayley graphs is indeed a Cayley graph. Using this result, we will give a necessary and sufficient condition for such Cayley graphs to be λ' -optimal. Based on this condition, we will construct an example to answer Problem 1.4.

We first recall the notion of semidirect product of two groups. Let $A = (A, \circ)$ and $B = (B, *)$ be two finite groups. A *group homomorphism* from A to B is a mapping $\phi : A \rightarrow B$ satisfying $\phi(a \circ b) = \phi(a) * \phi(b)$. Let e_A and e_B be identities in A and B , respectively, throughout this section. Group homomorphisms have two important and useful properties.

Proposition 5.1. Let A and B be two finite groups, and ϕ be a group homomorphism from A to B . Then

- (a) $\phi(e_A) = e_B$;
- (b) $\phi(a^{-1}) = (\phi(a))^{-1}$ for any $a \in A$.

An *action* of B on A is a group homomorphism $\phi : B \rightarrow \text{Aut}(A)$ defined by $\phi(b) = \phi_b$ and $\phi(b_1 b_2) = \phi(b_1) \phi(b_2) = \phi_{b_1} \phi_{b_2}$.

The *orbit* of $a \in A$ under the action ϕ of B is expressed as $a^B = \{\phi_b(a) \in A : b \in B\}$.

Example 5.2. Let $A = (\mathbb{Z}_2)^n$, $B = \mathbb{Z}_n$, and let e_i be an element in A defined in (2.1) for each $i = 0, 1, \dots, n$.

The action ϕ of B on A is defined as follows. For each $a = a_1 a_2 \cdots a_n \in A$,

$$\phi_i(a) = a_{1-i} a_{2-i} \cdots a_{n-i \pmod n} \quad \text{for each } i = 0, \dots, n-1 \in B.$$

For example, if $a = e_1$, then $\phi_i(e_1) = e_{i+1}$ for each $i = 0, 1, \dots, n-1$. Under ϕ the orbit $e_1^B = \{e_1, e_2, \dots, e_n\}$.

We now introduce the concept of the semidirect product of two finite groups following Robison [29].

The (*external*) *semidirect product* $A \rtimes_{\phi} B$ of groups A and B with respect to ϕ is the group with set $A \times B = \{(a, b) : a \in A, b \in B\}$ and the binary operation “ $*$ ” subject to

$$(a_1, b_1) * (a_2, b_2) = (a_1 \phi_{b_1}(a_2), b_1 b_2) \quad \text{for any } a_1, a_2 \in A \text{ and } b_1, b_2 \in B.$$

The identity is (e_A, e_B) . Since $\phi_b \in \text{Aut}(A)$ is an automorphism on A , by Proposition 5.1(a) we have

$$\phi_b(a) = e_A \Leftrightarrow a = e_A \quad \text{for any } a \in A \text{ and } b \in B. \quad (5.1)$$

By (5.1), it is easy to verify that the inverse $(a, b)^{-1}$ of (a, b) is $(\phi_{b^{-1}}(a^{-1}), b^{-1})$, i.e.,

$$(a, b)^{-1} = (\phi_{b^{-1}}(a^{-1}), b^{-1}).$$

It is also easy to check that the set $\{(a, e_B) : a \in A\}$ forms a normal subgroup of $A \rtimes_{\phi} B$ isomorphic to A , and the set $\{(e_A, b) : b \in B\}$ forms a subgroup of $A \rtimes_{\phi} B$ isomorphic to B . Thus, $A \rtimes_{\phi} B \cong A \rtimes B$, a *semidirect product* of two subgroups A and B of a group Γ , where A is normal.

The *direct product* $A \times B$ is a special case of $A \rtimes_{\phi} B$, in which the action $\phi(b)$ is the identity automorphism of A for any $b \in B$, and so $(a_1, b_1) * (a_2, b_2) = (a_1 a_2, b_1 b_2)$. Thus the semidirect product is a generalization of the direct product of two groups.

Many groups can be expressed as a semidirect product of two groups. For example, using the semidirect product, Feng [8] and Ganesan [9] determined the automorphism groups of some Cayley graphs generated by transposition sets; Zhou [38] determined the full automorphism group of the alternating group graph. The semidirect product of groups is also used to prove that some networks are Cayley graphs. For example, using the semidirect product, Zhou et al. [39] showed that the dual-cube DC_n is a Cayley graph $C_{(\Gamma \times \Gamma) \rtimes_{\phi} \mathbb{Z}_2}(S)$, where $\Gamma = (\mathbb{Z}_2)^n$, the action $\phi : \mathbb{Z}_2 \rightarrow \text{Aut}(\Gamma \times \Gamma)$ is defined by

$$\phi_i(\alpha, \beta) = \begin{cases} (\alpha, \beta), & \text{if } i = 0, \\ (\beta, \alpha), & \text{if } i = 1, \end{cases}$$

and $S = \{(e_0, e_1, 0), \dots, (e_0, e_n, 0), (e_0, e_0, 1)\}$.

Assumption 5.3. Let A and B be two groups with generating sets S_A and S_B , respectively, $|S_A| = |B| \geq 2$, ϕ be such an action of B on A that $S_A = x^B$ for some $x \in S_A$, and $S = \{(e_A, b) : b \in S_B\} \cup \{(x, e_B)\}$.

Theorem 5.4. Under Assumption 5.3, S generates $A \rtimes_{\phi} B$. Moreover, if $S_B = S_B^{-1}$ and $x = x^{-1}$, then $S = S^{-1}$ and $C_{A \rtimes_{\phi} B}(S)$ is a replacement product of $C_A(S_A)$ and $C_B(S_B)$.

Remark 5.5. Before proving this result, we make some remarks on the theorem.

(a) Since Cayley graphs under our discussion are undirected, by the definition of Cayley graphs, it is clear that the conditions “ $S_A = S_A^{-1}$, $S_B = S_B^{-1}$ and $S = S^{-1}$ ” are necessary to guarantee that Cayley graphs $C_A(S_A)$, $C_B(S_B)$ and $C_{A \rtimes_{\phi} B}(S)$ are undirected. By Proposition 5.1(b) for any action ϕ of B on A , $(x, e_B)^{-1} = (\phi_{e_B}(x^{-1}), e_B) = (x^{-1}, e_B)$. Thus, the condition “ $S = S^{-1}$ ” means that

$$\begin{aligned} \{(e_A, b) : b \in S_B\} \cup \{(x, e_B)\} &= (\{(e_A, b) : b \in S_B\} \cup \{(x, e_B)\})^{-1} \\ &= \{(e_A, b^{-1}) : b \in S_B\} \cup \{(x^{-1}, e_B)\}, \end{aligned}$$

which implies that the condition “ $S = S^{-1}$ ” is equivalent to the condition “ $S_B = S_B^{-1}$ and $x = x^{-1}$ ”.

Furthermore, since $S_A = x^B$ under the action ϕ , for any $a \in S_A$, there is some $b \in B$ such that $a = \phi_b(x)$. By Proposition 5.1(b), we have

$$x = x^{-1} \Leftrightarrow a = \phi_b(x) = \phi_b(x^{-1}) = (\phi_b(x))^{-1} = a^{-1} \quad \text{for any } a \in S_A.$$

(b) The original and simple statement of Theorem 5.4 is due to Alon et al. [1, Theorem 2.3] (as a special case of zig-zag products without proof), and a comparatively complete statement is given by Hoory et al. [13, Theorem 11.22] without the conditions “ $x = x^{-1}$ and $S_B = S_B^{-1}$ ”, and with an unperfect proof. We give a complete proof here.

Proof of Theorem 5.4. By the explanation in Remark 5.5(a), we only need to prove that S generates $A \rtimes_{\phi} B$ and $C_{A \rtimes_{\phi} B}(S)$ is a replacement product of $C_A(S_A)$ and $C_B(S_B)$.

We first show that S generates $A \rtimes_{\phi} B$. To the end, we only need to show that any $(a, b) \in A \rtimes_{\phi} B$ can be expressed as products of a sequence of elements of S .

By the hypothesis, S_A is a generating set of A and is the orbit x^B of some $x \in S_A$ under the action ϕ of B on A . Since $(a, b) = (a, e_B) * (e_A, b)$, it can be written as a product of elements from the set $\{(s_a, e_B) : s_a \in S_A\} \cup \{(e_A, s_b) : s_b \in S_B\}$. Since $S_A = x^B$, for $s_a \in S_A$ there is some $b \in B$ such that $s_a = \phi_b(x)$, where b can be expressed as products of a sequence of elements of S_B since S_B is a generating set of B by the hypothesis. Also since for any $b \in B$ and $\phi_b(x) \in S_A$,

$$(s_a, e_B) = (\phi_b(x), e_B) = (e_A, b) * (x, e_B) * (e_A, b^{-1}),$$

the element (s_a, e_B) can be expressed as products of a sequence of elements of S . This implies that S generates the group $A \rtimes_{\phi} B$.

We now show that $C_{A \rtimes_{\phi} B}(S)$ is a replacement product of $C_A(S_A)$ and $C_B(S_B)$. By Remark 5.5, under Assumption 5.3, Cayley graphs $C_A(S_A)$, $C_B(S_B)$ and $C_{A \rtimes_{\phi} B}(S)$ are well-defined and undirected, and so satisfy the requirements in Definition 2.4.

Let (y, i) and (z, j) be two distinct vertices in $C_{A \rtimes_{\phi} B}(S)$, where $y, z \in A = V(C_A(S_A))$ and $i, j \in B = V(C_B(S_B))$. Since $C_{A \rtimes_{\phi} B}(S)$ is a Cayley graph, we have that

$$\begin{aligned} (y, i)(z, j) \in E(C_{A \rtimes_{\phi} B}(S)) &\Leftrightarrow (y, i)^{-1} * (z, j) \\ &= (\phi_{i^{-1}}(y^{-1}), i^{-1}) * (z, j) \\ &= (\phi_{i^{-1}}(y^{-1})\phi_{i^{-1}}(z), i^{-1}j) \\ &= (\phi_{i^{-1}}(y^{-1}z), i^{-1}j) \\ &\in S = \{(e_A, b) : b \in S_B\} \cup \{(x, e_B)\}. \end{aligned} \quad (5.2)$$

If $(\phi_{i^{-1}}(y^{-1}z), i^{-1}j) \in \{(e_A, b) : b \in S_B\}$, then $y = z$ by (5.1), and $ij \in E(C_B(S_B))$, which means that the edge $(y, i)(y, j)$ of $C_{A \rtimes_{\phi} B}(S)$ is an edge in $C_A(S_A) \mathbb{R} C_B(S_B)$.

If $(\phi_{i^{-1}}(y^{-1}z), i^{-1}j) = (x, e_B)$, then $i = j$ and $\phi_{i^{-1}}(y^{-1}z) = x$. Since $\phi_{i^{-1}}\phi_i = \phi(i^{-1})\phi(i) = \phi(i^{-1}i) = \phi(e_B)$ is the identity automorphism of A , we have $\phi_{i^{-1}}^{-1} = \phi_i$. Thus, $y^{-1}z = \phi_{i^{-1}}^{-1}(x) = \phi_i(x) \in x^B = S_A$, i.e., $z = y\phi_i(x)$ and $yz \in E(C_A(S_A))$. Therefore, if we use e_y^i and e_z^i to label the edge $yz \in C_A(S_A)$ for each $(y, i)(z, i) \in E(C_{A \rtimes_{\phi} B}(S))$, i.e., $yz = e_y^i = e_z^i$, then the edge $(y, i)(z, i)$ of $C_{A \rtimes_{\phi} B}(S)$ is an edge in $C_A(S_A) \mathbb{R} C_B(S_B)$.

It follows that the structure of $C_{A \rtimes_{\phi} B}(S)$ satisfies the requirements of Definition 2.4, and so $C_{A \rtimes_{\phi} B}(S)$ is a replacement product of $C_A(S_A)$ and $C_B(S_B)$. $\square$

Example 5.6. Let $A = (\mathbb{Z}_2)^n$ and $B = \mathbb{Z}_n$. Then $e_A = e_0$ and $e_B = 0$. Let $S_A = \{e_1, e_2, \dots, e_n\}$, where e_i is defined in (2.1), and $e_i^{-1} = e_i$ for each $i \in \{1, 2, \dots, n\}$, and let $S_B = \pm\{s_1, s_2, \dots, s_k\}$. The Cayley graph $C_A(S_A)$ is a hypercube Q_n by Example 2.3 and the Cayley graph $C_B(S_B)$ is a circulant graph $G(n, \pm S)$ by Example 2.2. Let ϕ be the action of B on A defined in Example 5.2. Then S_A is the orbit e_1^B of $e_1 \in S_A$ under ϕ . Let $S = \{(e_A, s) : s \in S_B\} \cup \{(e_1, e_B)\}$. Then $S = S^{-1}$. By Theorem 5.4, S generates $A \rtimes_{\phi} B$, and $C_{A \rtimes_{\phi} B}(S)$ is a replacement product of $C_A(S_A)$ and $C_B(S_B)$.

In special cases, if $S_B = \{1, n-1\}$, then $S = \{(e_0, 1), (e_0, n-1), (e_1, 0)\}$. The Cayley graph $C_{(\mathbb{Z}_2)^n \rtimes_{\phi} \mathbb{Z}_n}(S) = Q_n \mathbb{R} C_n = CCC_n$. The cube-connected cycle $CCC(3)$, shown on the right-hand side in Figure 4, is a replacement product of Q_3 and C_3 , and is the Cayley graph $C_{\mathbb{Z}_2^3 \rtimes_{\phi} \mathbb{Z}_3}(\{(000, 1), (000, 2), (100, 0)\})$.

A graph G is κ -optimal if $\kappa(G) = \delta(G)$. The following theorem presents a necessary and sufficient condition for a Cayley graph $C_{A \rtimes_{\phi} B}(S)$ to be λ' -optimal if $C_A(S_A)$ is κ -optimal and $C_B(S_B)$ is λ' -optimal.

Theorem 5.7. Under Assumption 5.3, let $S = \{(e_A, s) : s \in S_B\} \cup \{(x, e_B)\}$ and $S = S^{-1}$. If Cayley graph $C_A(S_A)$ is κ -optimal and Cayley graph $C_B(S_B)$ is λ' -optimal, then Cayley graph $C_{A \rtimes_{\phi} B}(S)$ is λ' -optimal $\Leftrightarrow |S_A| \geq 2|S_B|$.

Proof. By Theorem 5.4, $C_{A \rtimes_{\phi} B}(S)$ is a replacement product of $C_A(S_A)$ and $C_B(S_B)$. Since $C_A(S_A)$ is κ -optimal, $\kappa(C_A(S_A)) = \lambda(C_A(S_A)) = \delta(C_A(S_A)) = |S_A| \geq 2$. Also since $C_B(S_B)$ is λ' -optimal, by Corollary 4.7(a), $C_{A \rtimes_{\phi} B}(S)$ is λ' -optimal if and only if $|S_A| \geq 2|S_B|$. $\square$

Example 5.8. By Example 5.6, the cube-connected cycle $CCC_n = Q_n \textcircled{R} C_n$ is 3-regular, $\xi(CCC_n) = 4$, $|S_{Q_n}| = n \geq 2$ and $|S_{C_n}| = 2$. Then

$$|S_A| = \begin{cases} 3 < 4 = 2|S_B|, & \text{if } n = 3, \\ n \geq 4 = 2|S_B|, & \text{if } n \geq 4. \end{cases}$$

By Example 4.9 and Theorem 5.7,

$$CCC_n \text{ is } \begin{cases} \text{not } \lambda'\text{-optimal } (\lambda' = \lambda = 3 < 4 = \xi) \text{ if } n = 3; \\ \lambda'\text{-optimal (i.e., } \lambda' = 4 = \xi) \text{ if } n \geq 4. \end{cases}$$

Theorem 5.9. Let $A = (\mathbb{Z}_2)^n$ and $B = \mathbb{Z}_n$, $S_A = \{e_1, e_2, \dots, e_n\}$, where e_i is defined in (2.1), $S_B = \pm\{s_1, s_2, \dots, s_k\}$ with $k \geq 2$ and $s_k < \frac{n}{2}$, ϕ be the action of B on A defined in Example 5.2. Let $G = C_{A \rtimes_{\phi} B}(S)$ with order $v(G)$, where $S = \{(e_0, s) : s \in S_B\} \cup \{(e_1, 0)\}$. If $\frac{n}{2} < |S_B| < n - 1$, then G is not λ' -optimal, $\lambda(G) < \lambda'(G) = n < \frac{v(G)}{2}$ for $n \geq 3$, and $G[X] \cong C_B(S_B)$ for any λ' -atom X of G .

Proof. By Example 2.3, $C_A(S_A) \cong Q_n$, by Example 2.2 $C_B(S_B) \cong G(n; S_B)$, and by Theorem 5.4 the Cayley graph $G = C_{A \rtimes_{\phi} B}(S)$ is a replacement product of Q_n and $G(n; S_B)$. Since $k \geq 2$ and $s_k < \frac{n}{2}$, $G(n; S_B)$ is λ' -optimal by Example 2.2. Since Q_n is κ -optimal and $|S_A| = n < 2|S_B|$, G is not λ' -optimal by Theorem 5.7. By Corollary 4.5, $\lambda'(G) = \min\{n, 2|S_B|\} = n$. Since G is vertex-transitive and $|S_B| < n - 1$, we have $\lambda(G) = \delta(G) = |S| = |S_B| + 1 < n = \lambda'(G)$. Note that $v(G) = n \cdot 2^n$ and that $k \geq 2$ implies $n \geq 5$. It follows that

$$\lambda(G) < \lambda'(G) = n = \frac{n \cdot 2^n}{2^n} = \frac{v(G)}{2^n} < \frac{v(G)}{2} \quad \text{for } n \geq 3.$$

We now show the second conclusion. Let X be a λ' -atom of G and $F = E_G(X)$. Then $|X| \leq \frac{v(G)}{2}$ and F is a λ' -cut of G . We need to prove $G[X] \cong C_B(S_B)$. We first note that

$$|F| = \lambda'(G) = n < 2|S_B| = 4k. \quad (5.3)$$

Let $\{X_1, X_2, \dots, X_n\}$ be a partition of $V(G)$ satisfying the property in Proposition 2.5. Then $G[X_i] \cong C_B(S_B)$ for each $i \in I_{2^n}$. If there exists some $j \in I_{2^n}$ such that $G[X_j]$ is disconnected in $G - F$ then, by Lemma 4.3 and Example 2.2,

$$\begin{aligned} |F| &\geq \min\{\kappa(C_A(S_A)) + \lambda(C_B(S_B)) - 1, 2\lambda(C_B(S_B)), \lambda'(C_B(S_B)) + 2\} \\ &= \min\{n + 2k - 1, 4k\} = 4k, \end{aligned}$$

which contradicts (5.3). It follows that $G[X_i]$ is connected in $G - F$, i.e., either $X_i \subset X$ or $X_i \subset \overline{X}$ for each $i \in I_{2^n}$.

If both X and $\overline{X}$ contain at least two sets of $X_1, X_2, \dots, X_{2^n}$, then, by comparing the structure of G with that of Q_n , it is easy to see that the subset of edges in Q_n corresponding to F is a restricted edge-cut of Q_n . Hence, by Example 2.3, $|F| \geq \lambda'(Q_n) = 2n - 2 > n = \lambda'(G) = |F|$, a contradiction. Namely, $X = X_i$ or $\overline{X} = X_i$ for some $i \in I_{2^n}$.

Since $|X| \leq \frac{v(G)}{2}$, we have $X = X_i$ and $\overline{X} = V(G) \setminus X_i$ for some $i \in I_{2^n}$. Thus every λ' -cut of G isolates a subgraph which is isomorphic to $C_B(S_B)$. In other words, $G[X] \cong G[X_i] \cong C_B(S_B)$ for each $i \in I_{2^n}$. $\square$

Remark 5.10. We make some remarks on the conditions in Theorem 5.9.

The condition “ $k \geq 2$ ” is necessary. In fact, if $k = 1$, then $C_B(S_B)$ is a cycle C_n . By Example 5.8,

$$CCC_n \text{ is } \begin{cases} \text{not } \lambda'\text{-optimal and } \lambda' = \lambda = 3 \text{ if } n = 3; \\ \lambda'\text{-optimal if } n \geq 4. \end{cases}$$

The condition “ $|S_B| > \frac{n}{2}$ ” is necessary. Theorem 5.7 means that $C_{A \rtimes_{\phi} B}(S)$ is

$$\text{not } \lambda'\text{-optimal} \Leftrightarrow |S_A| < 2|S_B|, \quad \text{i.e.,} \quad |S_B| > \frac{1}{2}|S_A| = \frac{1}{2}n.$$

The condition “ $|S_B| < n - 1$ ” is also necessary. In fact, if $|S_B| = n - 1$ then $G(n; S_B)$ is a complete graph K_n by Example 2.2. Thus, $\lambda(G) = n = \lambda'(G)$, which contradicts our conclusion.

The following theorem gives a straight answer to Problem 1.4.

Theorem 5.11. *For a given odd integer $d (\geq 5)$ and any integer s with $1 \leq s \leq d - 3$, there is a Cayley graph G with degree d such that $\lambda'(G) = d + s < \frac{1}{2}v(G)$.*

Proof. In Theorem 5.9, let $n = d + s$ and $k = \frac{d-1}{2}$, then $|S_B| = d - 1$ and $G = C_{\mathbb{Z}_2^{d+s} \rtimes_{\phi} \mathbb{Z}_{d+s}}(S)$ is a Cayley graph. Since $1 \leq s \leq d - 3$, we have $\frac{d+s}{2} < |S_B| < d + s - 1$. By Theorem 5.9, G is not λ' -optimal, and

$$\lambda(G) = d < \lambda'(G) = d + s < \frac{(d+s) \cdot 2^{d+s}}{2} = \frac{v(G)}{2}.$$

The theorem follows. $\square$

6 Conclusion

In this paper, we investigate the restricted edge-connectivity of replacement product of two graphs. By means of the semidirect product of two groups, we further confirm that under certain conditions, the replacement product of two Cayley graphs is also a Cayley graph, and give a necessary and sufficient condition for such Cayley graphs to have maximum restricted edge-connectivity. Based on these results, for given odd integer d and integer s with $d \geq 5$ and $1 \leq s \leq d - 3$, we construct a Cayley graph with degree d whose restricted edge-connectivity is equal to $d + s$, which answers a problem proposed ten years ago.

In the proof of this result, the replacement product of graphs plays a key role. Thus, further properties of replacement products deserve further research.

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