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On Seymour's Second Neighborhood conjecture of m-free digraphs[☆]

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ABSTRACT

This paper gives an approximate result related to Seymour's Second Neighborhood conjecture, that is, for any m-free digraph G, there exists a vertex $v \in V(G)$ and a real number λ_m such that $d^{++}(v) \geq \lambda_m d^+(v)$, and $\lambda_m \to 1$ while $m \to +\infty$. This result generalizes and improves some known results in a sense.

1. Introduction

Throughout this article, all digraphs are finite, simple and digonless. As usual, for a vertex v of the digraph G, we denote by $N_G^+(v)$ the set of out-neighbors of v, $N_G^{++}(v)$ the set of vertices at distance 2 from v. Let $d_G^+(v) = |N_G^+(v)|$ (the out-degree of v) and $d_G^{++}(v) = |N_G^{++}(v)|$. We will omit the subscript if the digraph is clear from the context.

In 1990, Seymour [3] proposed the following conjecture.

Conjecture 1.1 (*Seymour's Second Neighborhood Conjecture*). *For any digraph G, there exists a vertex v in G such that $d^{++}(v) \geq d^+(v)$.*

We call the vertex v in Conjecture 1.1 a *Seymour vertex*. In 1996, Fisher [5] showed that any tournament has a Seymour vertex. In 2001, Kaneko and Locke [8] showed that any digraph with the minimum outdegree less than 7 has a Seymour vertex. In 2007, Fidler and Yuster [4] proved that any tournament minus a star or a sub-tournament, and any digraph G with minimum degree $|V(G)| - 2$ have Seymour vertices. In 2008, Hamidoune [7] proved that any vertex-transitive digraph has a Seymour vertex. In 2013, Lladó [10] proved that any digraph with large connectivity has a Seymour vertex. In 2016, Cohn et al. [2] gave a probabilistic statement about Seymour's conjecture and proved that almost surely there are a large number of Seymour vertices in random tournaments and even more in general random digraphs. For a general digraph, Conjecture 1.1 is still open.

Another approach to Conjecture 1.1 is to determinate the maximum value of λ such that there is a vertex v in G satisfying $d^{++}(v) \geq \lambda\, d^+(v)$ for any digraph G. In 2003, Chen, Shen and Yuster [1] gave $\lambda = 0.657298\cdots$, which is the unique real root of the polynomial $2x^3 + x^2 - 1$. Furthermore, they improved this bound to $0.67815\cdots$ mentioned in the end of the article [1].

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A digraph G is called to be *m-free* if G contains no directed cycles of G with length at most m. In 2010, Zhang and Zhou [11] showed that for any 3-free digraph G, there exists a vertex v in G such that $d^{++}(v) \geq \lambda\, d^{+}(v)$, where $\lambda = 0.6751\cdots$ is the only real root in the interval $(0, 1)$ of the polynomial $x^3 + 3x^2 - x - 1$. In this paper, we consider general m-free digraphs and obtain the following result.

Theorem 1.2. *Let m be an arbitrarily fixed integer with $m \geq 3$ and G be an m-free digraph, then there exists a vertex v in G such that $d^{++}(v) \geq \lambda_m d^{+}(v)$, where λ_m is the only real root in the interval $(0, 1)$ of the polynomial*

$$g_m(x) = 2x^3 - (m-3)x^2 + (2m-4)x - (m-1). \tag{1.1}$$

Furthermore, λ_m is increasing with m, and $\lambda_m \to 1$ while $m \to +\infty$.

Since G is simple and digonless, G is 2-free. When $m = 2$, the polynomial defined in (1.1) is exactly $2x^3 + x^2 - 1$, and our result can be considered to be a generalization of Chen et al.'s result. When $m = 3$, $\lambda_3 = 0.6823\cdots$, which improves Zhang et al.'s value on λ_3. When $m = 4$, $\lambda_4 = 0.7007\cdots$. From Theorem 1.2, we immediately get the following corollary.

Corollary 1.3. *For every $\varepsilon > 0$, there is a positive integer m such that every m-free digraph contains a vertex v with $d^{++}(v) \geq (1-\varepsilon)\, d^{+}(v)$.*

The first conclusion in Theorem 1.2 is our main result. The proof proceeds by induction on the number of vertices. In the induction step, we assume to the contrary that $d^{++}(v) < \lambda_m d^{+}(v)$ for any vertex v in G, where λ_m is the unique real root of $g_m(x)$ in the interval $(0, 1)$. Then we show that the assumption leads to a contradiction. To this end, we need the following lemmas.

Lemma 1.4. *For $m \geq 3$, the polynomial $g_m(x)$ defined in (1.1) is strictly increasing and has a unique real root in the interval $(0, 1)$.*

Proof. Since $g_m(x) = 2x^3 - (m-3)x^2 + (2m-4)x - (m-1)$, we have

$$g'_m(x) = 6x^2 - 2(m-3)x + (2m-4) = 6x^2 + 2x + (2m-4)(1-x).$$

Clearly, $g'_m(x) > 0$ when $m \geq 3$ and $x \in (0, 1)$, which implies $g_m(x)$ is strictly increasing in $[0, 1]$. Since $g_m(0) = -m + 1 < 0$ and $g_m(1) = 2 > 0$, it follows that there is a unique real root in the interval $(0, 1)$ of the polynomial. ∎

Lemma 1.5 (*Hamburger et al. [6]*)**.** *If one can delete t edges from a digraph G to make it acyclic, then there exists a vertex v in G such that $d^{+}(v) \leq \sqrt{2t}$.*

Lemma 1.6 (*Liang and Xu [9]*)**.** *If an m-free digraph G is obtained from a tournament by deleting t edges, then one can delete from G additional $t/(m-2)$ edges so that the resulting digraph is acyclic.*

Combining Lemma 1.5 with Lemma 1.6, we can easily get the following lemma.

Lemma 1.7. *If an m-free digraph G is obtained from a tournament by deleting t edges, then there exists a vertex v in G such that $d^{+}(v) \leq \sqrt{2t/(m-2)}$.*

Proof. From Lemma 1.6, an m-free digraph G is obtained from a tournament by deleting t edges, then we can delete $t/(m-2)$ edges from G to make it acyclic. From Lemma 1.5, there exists a vertex v in G such that $d^{+}(v) \leq \sqrt{2t/(m-2)}$. ∎

2. Proof of Theorem 1.2

We first prove the first conclusion by induction on the number of vertices. Theorem 1.2 is trivial for any digraph with 1 or 2 vertices. Assume that Theorem 1.2 holds for all digraphs with less than n vertices. Let G be an m-free digraph with n vertices, $n \geq 3$ and $m \geq 3$. Assume to the contrary that $d^{++}(v) < \lambda_m d^{+}(v)$ for any vertex v in G, where λ_m is the unique real root of $g_m(x)$ in the interval $(0, 1)$. Our purpose is to show that the assumption leads to a contradiction.

Let u be a vertex in G with minimum out-degree. Let $A = N^{+}(u)$, $B = N^{++}(u)$, $a = |A|$ and $b = |B|$. By our assumption, we have

$$b = d^{++}(u) < \lambda_m d^{+}(u) = \lambda_m a. \tag{2.1}$$

For any two disjoint subsets $X, Y \subseteq V(G)$, let $E(X, Y)$ denote the edges from X to Y and $e(X, Y) = |E(X, Y)|$. Since G is simple and digonless, we have that

$$e(X, Y) + e(Y, X) \leq |X| \cdot |Y|.$$

For simplicity, for any subset $S \subseteq V(G)$, use S to denote the subgraph of G induced by S. By the definitions of A and B, we have

$$\sum_{v\in A} d_G^+(v) = |E(A)| + e(A,B). \tag{2.2}$$

By the choice of u, $d^+(v) \ge d^+(u) = a$ for any $v \in V(G)$, and so

$$\sum_{v\in A} d_G^+(v) \ge |A| \cdot d^+(u) = a^2. \tag{2.3}$$

Since $|E(A)| \le a(a-1)/2$, we have

$$e(A,B) = \sum_{v\in A} d_G^+(v) - |E(A)| \ge a^2 - a(a-1)/2 = a(a+1)/2.$$

It follows that there exists $v \in A$ such that $e(v,B) \ge e(A,B)/a \ge (a+1)/2$. Since $b = |B| \ge e(v,B)$ for any $v \in A$, it follows that $\lambda_m a > b \ge e(v,B) \ge (a+1)/2 > a/2$, which implies

$$\lambda_m > 1/2. \tag{2.4}$$

The subgraph A can be obtained from a tournament of order a by deleting t edges. Let $\theta = t/a^2$. Since $0 \le t \le a(a-1)/2$, we have $0 \le \theta \le (a-1)/2a < 1/2$ and

$$|E(A)| = a(a-1)/2 - t = (1/2-\theta)a^2 - a/2 < (1/2-\theta)a^2. \tag{2.5}$$

Combining (2.2), (2.3) with (2.5), we have that

$$e(A,B) = \sum_{v\in A} d_G^+(v) - |E(A)| > a^2 - (1/2-\theta)a^2 = (1/2+\theta)a^2. \tag{2.6}$$

Since G is m-free, it follows that the subgraph A is m-free. From Lemma 1.7, there is a vertex $w_0 \in A$ such that

$$d_A^+(w_0) \le \sqrt{2t/(m-2)} = a\sqrt{2\theta/(m-2)}. \tag{2.7}$$

Let $d_B^+(w_0) = |N_B^+(w_0)|$, then $d_B^+(w_0) \le |B| = b$. Since $d_A^+(w_0) + d_B^+(w_0) = d_G^+(w_0)$, it follows from (2.1) that $d_A^+(w_0) = d_G^+(w_0) - d_B^+(w_0) \ge d_G^+(w_0) - b \ge a - \lambda_m a = (1-\lambda_m)a$, that is,

$$d_A^+(w_0) \ge (1-\lambda_m)a. \tag{2.8}$$

Combining (2.7) with (2.8), we have $\sqrt{2\theta/(m-2)}a > (1-\lambda_m)a$, that is,

$$\theta > (m-2)(1-\lambda_m)^2/2. \tag{2.9}$$

Since A is m-free and $|A| = a < n$, by induction hypothesis there is a vertex $w_1 \in A$ such that $|N_A^{++}(w_1)| \ge \lambda_m |N_A^+(w_1)|$, where λ_m is the unique real root of $g_m(x)$ in the interval $(0,1)$.

Let $X = N_A^+(w_1)$, $Y = N_B^+(w_1)$ and $|Y| = d$. It follows from (2.1) that

$$d = |Y| \le |B| = b < \lambda_m a. \tag{2.10}$$

By the induction hypothesis,$|A-X| \ge |N_A^{++}(w_1)| \ge \lambda_m|X|$, that is, $(1+\lambda_m)|X| \le |A| = a$. By (2.4) $\lambda_m > \frac{1}{2}$, we have

$$|X| \le \frac{a}{1+\lambda_m} < \frac{2a}{3}.$$

By the choice of u, we have $d_G^+(w_1) \ge d_G^+(u) = a$, and so

$$d = |Y| = |N_G^+(w_1)| - |X| > a - \frac{2a}{3} = \frac{a}{3}. \tag{2.11}$$

Combining (2.10) with (2.11), we have

$$a/3 < d < \lambda_m a. \tag{2.12}$$

For any $y \in Y$, use $d_{V-A-Y}^+(y)$ to denote the number of out-neighbors of y in G not in $A \cup Y$. Since $d_G^{++}(w_1) < \lambda_m d_G^+(w_1)$ and $d_A^{++}(w_1) \ge \lambda_m d_A^+(w_1)$, we have

$$d_{V-A-Y}^+(y) \le d_G^{++}(w_1) - d_A^{++}(w_1) < \lambda_m d_G^+(w_1) - \lambda_m d_A^+(w_1) = \lambda_m d.$$

Noting that $d_G^+(y) \geq d_G^+(u) = a$ and $\sum_{y\in Y} d_Y^+(y) = |E(Y)| \leq d(d-1)/2$, we obtain

$$\begin{aligned}
e(Y,A) &= \sum_{y\in Y} |N_A^+(y)| \\
&\geq \sum_{y\in Y} (a - d_{V-A-Y}^+(y) - d_Y^+(y)) \\
&> (a-\lambda_m d)\, d - \sum_{y\in Y} d_Y^+(y) \\
&\geq (a-\lambda_m d)\, d - d(d-1)/2 \\
&> (a-\lambda_m d - d/2)\, d,
\end{aligned}$$

that is

$$e(Y,A) > (a-\lambda_m d - d/2)d. \tag{2.13}$$

Combining (2.1), (2.6), (2.9) with (2.13), we have

$$\begin{aligned}
\lambda_m a^2 &\geq ab \\
&\geq e(A,B) + e(B,A) \\
&\geq e(A,B) + e(Y,A) \\
&> (1/2+\theta)\, a^2 + (a-\lambda_m d - d/2)\, d \\
&> [1/2 + (m-2)(1-\lambda_m)^2/2]\, a^2 + (a-\lambda_m d - d/2)\, d \\
&= -(\lambda_m + 1/2)d^2 + ad + [1/2 + (m-2)(1-\lambda_m)^2/2]\, a^2,
\end{aligned}$$

that is,

$$\lambda_m a^2 > -(\lambda_m + 1/2)\, d^2 + ad + [1/2 + (m-2)(1-\lambda_m)^2/2]\, a^2, \tag{2.14}$$

where $a/3 < d < \lambda_m a$ (see (2.12)). For $a/3 \leq z \leq \lambda_m\, a$, let the function

$$f(z) = -(\lambda_m + 1/2)z^2 + az + [1/2 + (m-2)(1-\lambda_m)^2/2]\, a^2.$$

Since $f(z)$ is a quadratic function with a negative leading coefficient, the following inequality holds.

$$f(z) \geq \min\{f(a/3), f(\lambda_m a)\} \text{ for any } z \in [a/3, \lambda_m a]. \tag{2.15}$$

Combining (2.14) with (2.15), we have

$$\lambda_m a^2 > f(d) \geq \min\{f(a/3), f(\lambda_m a)\}. \tag{2.16}$$

We first note that, since

$$f(\lambda_m a) = \frac{a^2[-2\lambda_m^3 + (m-3)\lambda_m^2 - (2m-6)\lambda_m + (m-1)]}{2},$$

if $\lambda_m a^2 > f(\lambda_m a)$, then

$$\lambda_m a^2 > \frac{a^2[-2\lambda_m^3 + (m-3)\lambda_m^2 - (2m-6)\lambda_m + (m-1)]}{2},$$

that is

$$g_m(\lambda_m) = 2\lambda_m^3 - (m-3)\lambda_m^2 + (2m-4)\lambda_m - (m-1) > 0.$$

This fact shows that λ_m is not a root of the polynomial $g_m(x)$, which contradicts our assumption on λ_m.

It follows that $\lambda_m a^2 \leq f(\lambda_m a)$, and so $\lambda_m a^2 > f(a/3)$ by (2.16). Since

$$f(a/3) = \frac{a^2[9(m-2)\lambda_m^2 - (18m-34)\lambda_m + (9m-4)]}{18}$$

we have

$$\lambda_m a^2 > \frac{a^2[9(m-2)\lambda_m^2 - (18m-34)\lambda_m + (9m-4)]}{18}.$$

Simplifying this inequality, we obtain

$$9(m-2)\lambda_m^2 - (18m-16)\lambda_m + (9m-4) < 0.$$

This implies

$$\lambda_m > \frac{9m-8-\sqrt{54m-8}}{9(m-2)}. \tag{2.17}$$

Now we show (2.17) is a contradiction to that λ_m is the only root in the interval $(0, 1)$ of the polynomial $g_m(x)$. We rewrite the polynomial $g_m(x)$ as

$$g_m(x) = \frac{1}{9}(p(x) - q(x)), \tag{2.18}$$

where

$$p(x) = 18x^3 + 9x^2 - 20x + 5,$$
$$q(x) = 9(m-2)x^2 - (18m-16)x + (9m-4).$$

The polynomial $q(x)$ has a real root

$$\varphi_m = \frac{9m - 8 - \sqrt{54m-8}}{9(m-2)}, \tag{2.19}$$

that is

$$q(\varphi_m) = 0. \tag{2.20}$$

Comparing (2.17) with (2.19), we have

$$\lambda_m \ge \varphi_m \quad \text{for } m \ge 3. \tag{2.21}$$

Since

$$\begin{aligned}\varphi_m &= 1 + \frac{10 - \sqrt{54m-8}}{9(m-2)}\\ &= 1 + \frac{108 - 54m}{9(m-2)(10 + \sqrt{54m-8})}\\ &= 1 - \frac{6}{10 + \sqrt{54m-8}},\end{aligned}$$

it is easy to see that φ_m is strictly increasing with m for $m \ge 3$. Thus we have

$$\varphi_m \ge \varphi_3 = 1 - \frac{6}{10 + \sqrt{154}} > 1 - \frac{3}{10} = \frac{7}{10}. \tag{2.22}$$

A simple calculation gives us that $p(x)$ is a strictly increasing function for $x > \frac{7}{10}$ and $p(\frac{7}{10}) = 1.584 > 0$. Noting that $g_m(x)$ is a strictly increasing function over the interval $[0, 1]$, and by (2.18), (2.20)– (2.22), we have

$$g_m(\lambda_m) > g_m(\varphi_m) = \frac{1}{9}[p(\varphi_m) - q(\varphi_m)] = \frac{1}{9}p(\varphi_m) > \frac{1}{9}p\left(\frac{7}{10}\right) > 0.$$

This fact shows that λ_m is not a root of the polynomial $g_m(x)$, a contradiction to our assumption, and so the first conclusion follows.

We now prove the second conclusion. Since $g_m(x) = 2x^3 - (m-3)x^2 + (2m-4)x - (m-1)$, $g_m(\lambda_m) = 0$ and

$$\begin{aligned}g_{m+1}(x) &= 2x^3 - (m-2)x^2 + (2m-2)x - m\\ &= 2x^3 - (m-3)x^2 + (2m-4)x - (m-1) - x^2 + 2x - 1\\ &= g_m(x) - (1-x)^2,\end{aligned}$$

for any $m \ge 3$ we have

$$g_{m+1}(\lambda_m) = g_m(\lambda_m) - (1-\lambda_m)^2 = -(1-\lambda_m)^2 < 0 = g_{m+1}(\lambda_{m+1}).$$

Since $g_m(x)$ is strictly increasing in the interval $(0, 1)$ for any $m \ge 3$ by Lemma 1.4, it follows that $\lambda_m < \lambda_{m+1}$, which implies that λ_m is increasing with m.

We rewrite $g_m(x)$ as

$$g_m(x) = 2x(x^2 - 1) + 2x^2 - (m-1)(1-x)^2.$$

It is easy to check that $\mu_m = \frac{\sqrt{m-1}}{\sqrt{m-1}+\sqrt{2}} \in (0, 1)$ is a real root of the polynomial $2x^2 - (m-1)(1-x)^2$. It follows that $g_m(\mu_m) = 2\mu_m(\mu_m^2 - 1) < 0 = g_m(\lambda_m)$. Since $g_m(x)$ is strictly increasing in the interval $(0, 1)$ by Lemma 1.4, we have

$$0 < \mu_m < \lambda_m < 1.$$

Since $\lim_{m\to+\infty}\mu_m = \lim_{m\to+\infty}\frac{\sqrt{m-1}}{\sqrt{m-1}+\sqrt{2}} = 1$, it follows that $\lim_{m\to+\infty}\lambda_m = 1$.

The proof of Theorem 1.2 is complete.

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