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Equal relation between the extra connectivity and pessimistic diagnosability for some regular graphs

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ABSTRACT

Extra connectivity and the pessimistic diagnosis are two crucial subjects for a multiprocessor system's ability to tolerate and diagnose faulty processor. The pessimistic diagnosis strategy is a classic strategy based on the PMC model in which isolates all faulty vertices within a set containing at most one fault-free vertex. In this paper, the result that the pessimistic diagnosability $t_p(G)$ equals the extra connectivity $\kappa_1(G)$ of a regular graph G under some conditions are shown. Furthermore, the following new results are gotten: the pessimistic diagnosability $t_p(S_n^2) = 4n - 9$ for split-star networks S_n^2; $t_p(\Gamma_n) = 2n - 4$ for Cayley graphs generated by transposition trees Γ_n; $t_p(\Gamma_n(\Delta)) = 4n - 11$ for Cayley graph generated by the 2-tree $\Gamma_n(\Delta)$; $t_p(BP_n) = 2n - 2$ for the burnt pancake networks BP_n. As corollaries, the known results about the extra connectivity and the pessimistic diagnosability of many famous networks including the alternating group graphs, the alternating group networks, BC networks, the k-ary n-cube networks etc. are obtained directly.

1. Introduction

It is well known that a topological structure of an interconnection network can be modeled by a loopless undirected graph $G = (V, E)$, where vertices in V represent the processors and the edges in E represent the communication links. In this paper, we use graphs and networks interchangeably. The *connectivity* $\kappa(G)$ of a connected graph G is the minimum number of vertices removed to get the graph disconnected or trivial. In a multiprocessor system, some processors may fail, connectivity is used to determine the reliability and fault tolerance of a network. However, a connectivity is not suitable for large-scale processing systems because it is almost impossible for all processors adjacent to, or all links incident to, the same processors to fail simultaneously. To compensate for this shortcoming, it seems reasonable to generalize the notion of classical connectivity by imposing some conditions or restrictions on the components of G when we delete the set of faulty processors. Fábrega and Fiol [17] introduced the *extra connectivity* of interconnection networks as follows.

Definition 1. A vertex set $S \subseteq V(G)$ is called to be an *h-extra vertex cut* if $G - S$ is disconnected and every component of $G - S$ has at least $h + 1$ vertices. The *h-extra connectivity* of G, denoted by $\kappa_h(G)$, is defined as the cardinality of a minimum h-extra vertex cut, if exists.

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It is obvious that $\kappa_0(G) = \kappa(G)$ for any graph G that is not a complete graph. The 1-extra connectivity is usually called extra connectivity. Regarding the computational complexity of the problem [17], there is no known polynomial-time algorithm for finding $\kappa_h(G)$ even for $h = 2$. The problem of determining the extra connectivity of numerous networks has received a great deal of attention in recent years. For a general integer h, Yang and Meng determined the h-extra connectivity of the hypercubes [49] and the folded hypercubes [50], respectively. Chang et al. studied the $\{2,3\}$-extra connectivity for the hypercube-like networks [3] and the 3-extra connectivity for the folded hypercubes [4]; Hsieh et al. [29] determined the 2-extra connectivity of k-ary n-cubes; Li et al. [35] derived the 3-extra connectivity of the Cayley graphs generated by transposition generating trees; Lin et al. obtained the $\{1,2,3\}$-extra connectivity of the split-star networks [37] and the alternating group networks [38], respectively; Guo and Lu [26] studied the h-extra connectivity $(1 \le h \le 3)$ of bubble-sort star graphs and Lü [39] obtained the $\{2,3\}$-extra connectivity of balanced hypercubes etc.

The diagnosis of a system is the process of appraising the faulty processors. A number of models have been proposed for diagnosing faulty processors in a network. Preparata et al. [40] first introduced a graph theoretical model, the so-called *PMC model* (i.e., Preparata, Metze and Chien's model), for system level diagnosis in multiprocessor systems. The pessimistic diagnosis strategy proposed by Kavianpour and Friedman [33] is a classic diagnostic model based on the PMC model. In this strategy, all faulty processors to be isolated within a set having at most one fault-free processor.

Definition 2. A system is *t/t-diagnosable* if, provided the number of faulty processors is bounded by t, all faulty processors can be isolated within a set of size at most t with at most one fault-free vertex mistaken as a faulty one. The *pessimistic diagnosability* of a system G, denoted by $t_p(G)$, is the maximal number of faulty processors so that the system G is t/t-diagnosable.

The pessimistic diagnosability of many interconnection networks has been explored. Using the pessimistic strategy, Chwa and Hakimi [12] characterized the diagnosable systems, and Sullivan [42] gave a polynomial time algorithm for determining the diagnosability of a system. Kavianpour and Kim [33] had shown that the hypercubes were $(2n-2)/(2n-2)$-diagnosable. Fan [18] derived the diagnosability of the Möbius cubes using the pessimistic strategy. Wang [47] had shown that the enhanced hypercubes were $2n/2n$-diagnosable. Wang et al. [48] gave the pessimistic diagnosability of the k-ary n-cubes. Tsai in [44] and [45] obtained the pessimistic diagnosability of the alternating group graphs AG_n and the hypercube-like networks (BC networks), respectively. Recently, the pessimistic diagnosability of the (n,k)-arrangement graphs, the (n,k)-star graphs and the balanced hypercubes, the bubble-sort star graphs and augmented k-ary n-cubes were determined in [24] and [25], respectively. For more results related with the diagnosability, you are referred to see [2,20,27,34,36], etc.

Based on the importance of the extra connectivity and the pessimistic diagnosability and motivated by the recent researches on the extra connectivity and pessimistic diagnosability of some graphs, including some famous networks, our object is to propose the relationship between extra connectivity and pessimistic diagnosability of regular graphs with some given conditions. In this paper, the result that the pessimistic diagnosability $t_p(G)$ equals the extra connectivity $\kappa_1(G)$ of a regular graph G under some conditions are shown. Furthermore, the following new results are gotten: the pessimistic diagnosability $t_p(S_n^2) = 4n - 9$ for split-star networks S_n^2; $t_p(\Gamma_n) = 2n - 4$ for Cayley graphs generated by transposition trees Γ_n; $t_p(\Gamma_n(\Delta)) = 4n - 11$ for Cayley graphs generated by the 2-tree $\Gamma_n(\Delta)$; $t_p(BP_n) = 2n - 2$ for the burnt pancake networks BP_n. As corollaries, the known results about the extra connectivity and the pessimistic diagnosability of many famous networks including the alternating group graphs, the alternating group networks, BC networks and the k-ary n-cube networks, etc. are obtained directly.

The remainder of this paper is organized as follows. Section 2 introduces necessary definitions and properties of some graphs. In Section 3, we determines the equal relationship between extra connectivity and pessimistic diagnosability of regular graphs with some given conditions. In Section 4, we concentrates on the applications to some famous networks. The pessimistic diagnosability and the extra connectivity of many famous networks, such as the alternating group graph AG_n, the alternating group network AN_n, the k-ary n-cube networks Q_n^k, the BC networks X_n, the split-star networks S_n^2, the Cayley graphs generated by transposition trees Γ_n, the Cayley graphs generated by 2-trees $\Gamma_n(\Delta)$ and the burnt pancake networks BP_n are obtained directly. Finally, our conclusions are given in Section 5.

2. Preliminaries

In this section, we give some terminologies and notations of combinatorial network theory. For notations not defined here, the reader is referred to [1].

We use a graph, denoted by $G = (V(G), E(G))$, to represent an interconnection network, where $V(G)$ is the vertex set of G; $E(G)$ is the edge set of G. For a vertex $u \in V(G)$, let $N_G(u)$ (or $N(u)$ if there is no ambiguity) denote a set of vertices in G adjacent to u. For a vertex set $U \subseteq V(G)$, let $N_G(U) = \bigcup_{v \in U} N_G(v) - U$ and $G[U]$ be the subgraph of G induced by U. If $|N_G(u)| = k$ for any vertex in G, then G is *k-regular*. For any two vertices u and v in G, let $cn(G; u, v)$ denote the number of vertices who are the neighbors of both u and v, that is, $cn(G; u, v) = |N_G(u) \cap N_G(v)|$. Let $cn(G) = \max\{cn(G; u, v) : u, v \in V(G)\}$, $l(G) = \max\{cn(G; u, v) : (u, v) \in E(G)\}$. Let $|V(G)|$ be the size of vertex set and $|E(G)|$ be the size of edge set. Throughout this paper, all graphs are finite, undirected without loops.

Let $[n]=\{1,2,\ldots,n\}$ and $\langle n\rangle=\{-1,-2,\ldots,-n,1,2,\ldots,n\}$. For a finite group A and a subset S of A such that $1\notin S$ and $S=S^{-1}$ (where 1 is the identity element of A), the *Cayley graph* $\mathrm{Cay}(A;S)$ on A with respect to S is defined to have vertex set A and edge set $\{(g,gs)|g\in A,s\in S\}$. A Cayley graph is $|S|$-regular, and is connected if and only if S generates Γ. Moreover, A Cayley graph is $|S|$-connected if S is a minimal generating set of Γ.

2.1. The alternating group graphs

Jwo et al. [32] introduced the alternating group graph as an interconnection network topology for computing systems.

Definition 3. Let A_n be the alternating group of degree n with $n\geq 3$. Set $S=\{(1\ 2\ i),(1\ i\ 2)\mid 3\leq i\leq n\}$. The *alternating group graph*, denoted by AG_n, is defined as the Cayley graph $AG_n=\mathrm{Cay}(A_n,S)$.

It is clear that AG_3 is a triangle, AG_n is a $(2n-4)$-connected and $(2n-4)$-regular graph with $n!/2$ vertices. Each AG_n contains n sub-alternating group graphs $AG_n^0, AG_n^1,\ldots,AG_n^{n-1}$. For each $i\in[n]$, AG_n^i is isomorphic to AG_{n-1}. For each vertex $v\in AG_n^i$, v has exactly two neighbors that are not contained in AG_n^i, which are called the extra neighbors of v.

Lemma 1. ([28]) *The extra neighbors of every vertex of AG_n are in different subgraphs AG_n^i for $n\geq 4$. For any two different vertices u,v, $cn(AG_n:u,v)=1$ if u and v are adjacent; otherwise, $cn(AG_n:u,v)\leq 2$.*

Lemma 2. ([44]) *Let AG_n be the n-dimensional alternating group graph for $n\geq 4$. If U is a subset of $V(AG_n)$ and $2\leq|U|\leq 8n-25$, then $|N_{AG_n}(U)|\geq 4n-11$.*

Lemma 3. ([28]) *Let F be a vertex-cut of AG_n for $n\geq 5$. If $|F|\leq 4n-11$, then AG_n-F satisfies one of the following conditions:*

(1) *AG_n-F has two components, one of which is a trivial component.*
(2) *AG_n-F has two components, one of which is an edge. Moreover, if $|F|=4n-11$, F is formed by the neighbor of the edge.*

2.2. The alternating group networks

The alternating group network AN_n was first proposed by Y. Ji [31] to improve upon the alternating group graph AG_n, studied by Jwo and others [32].

Definition 4. ([31]) Let A_n be an alternating group of degree $n\geq 3$ and let $S=\{(1\ 2\ 3),(1\ 3\ 2),(1\ 2)(3\ i)|\ 4\leq i\leq n\}$. The *alternating group network*, denoted by AN_n, is defined as the Cayley graph $\mathrm{Cay}(A_n,S)$.

By the definition, we can get some properties about AN_n [31]. AN_n is a regular graph with $n!/2$ vertices and $n!(n-1)/4$ edges. AN_3 is a triangle. AN_4 contains four copies of AN_3. AN_n contains n copies of AN_{n-1}, say $AN_n^0, AN_n^1,\ldots,AN_n^{n-1}$. For each $i\in[n]$, AN_n^i is isomorphic to AN_{n-1}. By Theorem 1 in [52], AN_n is $(n-1)$-regular and $(n-1)$-connected.

Lemma 4. ([27]) *Let AN_n be the alternating group network for $n\geq 3$.*

(1) *Each vertex in AN_n has exactly one extra neighbor.*
(2) *AN_n has no 4-cycle and 5-cycle.*
(3) *Let u and v be any two distinct vertices of AN_n, then $cn(AN_n:u,v)\leq 1$.*

Lemma 5. ([53]) *Let F be a vertex-cut of AN_n for $n\geq 5$. If $|F|\leq 2n-5$, then AN_n-F satisfies one of the following conditions:*

(1) *AN_n-F has two components, one of which is a trivial component.*
(2) *AN_n-F has two components, one of which is an edge. Moreover, if $|F|=2n-5$, F is formed by the neighbor of the edge.*

2.3. BC networks

Definition 5. The 1-dimensional BC network X_1 is a complete graph with two vertices. The *n-dimensional BC network* X_n is defined as follows: $V(X_n)=V(G_1)\cup V(G_2)$ and $E(X_n)=E(G_1)\cup E(G_2)\cup M$, where $G_1,G_2\in L_{n-1}$, and M is a perfect matching between $V(G_1)$ and $V(G_2)$, where $L_k=\{X_k:X_k$ is an k-dimensional BC network$\}$.

Lemma 6. ([19,46,54]) *Let $G=X_n\in L_n$ for $n\geq 1$. Then G is n-regular n-connected and triangle-free. Any two vertices has at most two common neighbors in G.*

Lemma 7. ([54]) *For any $X_n \in L_n$, let $F \subseteq V(X_n)$ with $|F| \leq 2n-3$ be a vertex-cut of X_n. Then $X_n - F$ has two components, one of which is a trivial component.*

2.4. The k-ary n-cube networks

Definition 6. The *k-ary n-cube*, denoted by Q_n^k, where $k \geq 2$ and $n \geq 1$ are integers, is a graph consisting of k^n vertices, each of these vertices has the form $u = u_{n-1}u_{n-2}\cdots u_0$, where $u_i \in \{0, 1, \ldots, k-1\}$ for $0 \leq i \leq n-1$. Two vertices $u = u_{n-1}u_{n-2}\cdots u_0$ and $v = v_{n-1}v_{n-2}\cdots v_0$ in Q_n^k are adjacent if and only if there exists an integer j, where $0 \leq j \leq n-1$, such that $u_j = v_j \pm 1 (\mathrm{mod}\ k)$ and $u_i = v_i$ for every $i \in \{0, 1, \ldots, n-1\} \setminus \{j\}$. In this case, (u, v) is a *j-dimensional edge*.

For convenience, "(mod k)" does not appear in similar expressions in the remainder of the paper. Note that each vertex has degree $2n$ for $k \geq 3$ and has degree n for $k = 2$. Clearly, Q_1^k is a cycle of length k, Q_n^2 is an *n-dimensional hypercube*, Q_2^k is a $k \times k$ *wrap-around mesh*.

Q_n^k can be partitioned over the jth-dimension, for a $j \in [n-1]$, into k disjoint subcubes, denoted by $Q_{n-1}^k[0]$, $Q_{n-1}^k[1]$, $\ldots$, $Q_{n-1}^k[k-1]$, by deleting all the j-dimensional edges from Q_n^k. For convenience, abbreviate these as $Q[0], Q[1], \ldots,$ $Q[k-1]$ if there is no ambiguity. Moreover, $Q[i]$ for $0 \leq i \leq k-1$ is isomorphic to the k-ary $(n-1)$-cube. For each vertex $u \in V(Q[i])$, the neighbor which is not in $V(Q[i])$ is called the *extra neighbor*. For $i \in [k-1]$, $u \in V(Q[i])$, the two extra neighbors of u are in different subgraphs $Q[i+1]$ and $Q[i-1]$, respectively.

Lemma 8. *Let Q_n^k be a k-ary n-cube, where $k \geq 2$ and $n \geq 1$ are integers.*

(1) ([15]) *Q_n^k is $2n$-regular and $2n$-connected for $k \geq 3$ and n-regular and n-connected for $k = 2$.*
(2) ([14,22,29]) *For any $x, y \in V(Q_n^k)$, $k \geq 2$,*

$$cn(Q_n^k : x, y) = \begin{cases} 1 & \text{if } (x, y) \in E(Q_n^k) \text{ and } k = 3; \\ 2 & \text{if } (x, y) \notin E(Q_n^k) \text{ and } N_{Q_n^k}(x) \cap N_{Q_n^k}(y) \neq \emptyset; \\ 0 & \text{otherwise}. \end{cases}$$

Lemma 9.

(1) ([16]) *If $F \subseteq V(Q_n^2)$ with $|F| \leq 2n-3$ is a vertex cut of Q_n^2 for $n \geq 2$, then $Q_n^2 - F$ has two components, one of which is a trivial component.*
(2) ([14,22]) *If $F \subseteq V(Q_n^3)$ with $|F| \leq 4n-4$ is a vertex cut of Q_n^3 for $n \geq 2$, then $Q_n^3 - F$ has two components, one of which is a trivial component.*
(3) ([14,23]) *If $F \subseteq V(Q_n^k)$ is a vertex cut of Q_n^k with $|F| \leq 4n-3$ for $n \geq 2$ and $k \geq 4$, then $Q_n^k - F$ has two components, one of which is a trivial component.*

2.5. Split-star networks S_n^2

Cheng et al. [8] propose the Split-star networks as alternatives to the star graphs and companion graphs with the alternating group graphs.

Definition 7. Given two positive integers n and k with $n > k$, note that $[n] = \{1, 2, \ldots, n\}$, and let $\mathcal{P}_n$ be a set of $n!$ permutations on $[n]$. The *n-dimensional Split-star network*, denoted by S_n^2, such that $V(S_n^2) = \mathcal{P}_n$, $E(S_n^2) = \{(p, q) |$ p (resp. q) can be obtained from q (resp. p) by either a 2-exchange or a 3-rotation$\}$. Where

(1) A 2-*exchange interchanges* the symbols in 1st position and 2nd position.
(2) A 3-*rotation rotates* the symbols in three positions labeled by the vertices of a triangle in which three vertices of the triangle are 1, 2 and k for some $k \in \{3, 4, \ldots, n\}$.

Let $V_n^{n:i}$ be the set of all vertices in S_n^2 with the nth position having value i, i.e., $V_n^{n:i} = \{p | p = x_1x_2\cdots x_{n-1}i,\ x_j \in \{1, 2, \ldots, i-1, i+1, \ldots n\}\ (1 \leq j \leq n-1)$ are do not care symbols$\}$. The set $\{V_n^{n:i} | 1 \leq i \leq n\}$ forms a partition $V(S_n^2)$. Let $S_n^{2:i}$ denote the subgraph of S_n^2 induced by $V_n^{n:i}$, i.e., $S_n^{2:i} = S_n^2[V_n^{n:i}]$. It is easy to know that $S_n^{2:i}$ is isomorphic to S_{n-1}^2. Every vertex $v \in S_n^{2:i}$ has exactly two neighbors, called extra neighbors, outside of $S_n^{2:i}$; moreover these two neighbors belong to different $S_n^{2:j}$s where $j \neq i$. We call these neighbors as the extra neighbors of v. We call these edges, whose end-vertices belong to different subgraphs, as *cross edges*. Let $S_{n,E}^2$ be a subgraph of S_n^2 induced by the set of even permutations, in which the adjacency rule is precisely the 3-rotation. We know that $S_{n,E}^2$ is the alternating group graph AG_n [32]. Let $S_{n,O}^2$ be a subgraph of S_n^2 induced by the set of odd permutations, in which the adjacency rule is precisely the 3-rotation. We have that $S_{n,O}^2$ is also isomorphic to AG_n and $S_{n,O}^2$ is isomorphic $S_{n,E}^2$ via the 2-exchange $\phi(a_1a_2a_3\cdots a_n) = a_2a_1a_3\cdots a_n$. Hence, there are $\frac{n!}{2}$ matching edges between $S^2_{n,O}$ and $S^2_{n,E}$. Indeed, the Split-star network S^2_n is introduced in [9] which is the companion graph of AG_n.

Lemma 10. ([7–9]) *Let S^2_n be the n-dimensional split-star network.*

(1) *S^2_n is $(2n-3)$-regular and $\kappa(S^2_n) = 2n-3$ for $n \geq 2$.*
(2) *Two extra neighbors of every vertex in $S^{2:i}_n$ are in distinct induced subgraphs and these two extra neighbors are adjacent. For any two vertices in the same subgraph $S^{2:i}_n$, their extra neighbors in other subgraphs are different. There is one to one correspondence between the subgraph $S^2_{n,O}$ and the subgraph $S^2_{n,E}$.*
(3) *Let x, y be any two vertices of S^2_n, then*

$$cn(S^2_n : x, y) \leq \begin{cases} 1 & \text{if } d(x,y)=1; \\ 2 & \text{if } d(x,y)=2; \\ 0 & \text{if } d(x,y)\geq 3. \end{cases}$$

Lemma 11. ([37]) *If $F \subseteq V(S^2_n)$ with $|F| \leq 4n-10$ is a vertex cut of S^2_n for $n \geq 4$, then $S^2_n - F$ has two components, one of which is a trivial component.*

2.6. Cayley graphs generated by transposition trees Γ_n

Note that $\mathcal{P}_n$ is a group of all permutations on $[n]$. For convenience, (ij), which is called a *transposition*, denotes the permutation that swaps the elements at position i and j, that is $(ij)p_1p_2\dots p_i\dots p_j\dots p_n = p_1p_2\dots p_j\dots p_i\dots p_n$.

Definition 8. Let $\mathcal{P}_n$ be symmetric group on $[n]$, and the generating set S to be a set of transpositions. A graph $G(S)$ with vertex set $[n]$, where there is an edge between i and j if and only if the transposition (ij) belongs to S, is called the *transposition generating graph*. When $G(S)$ is a tree, we call $G(S)$ a *transposition tree*. The Cayley graphs $Cay(\mathcal{P}_n, S)$ obtained by transposition trees are called *Cayley graphs generated by transposition trees*, denoted by Γ_n.

If $G(S) \cong K_{1,n-1}$, $Cay(\mathcal{P}_n, S)$ is called the *star graph*, denoted by S_n. If $G(S) \cong P_n$, that is the transposition tree is a path P_n with n vertices, then $Cay(\mathcal{P}_n, S)$ is called the *bubble-sort graph*, denoted by B_n.

Let Γ^i_n be the subgraph of Γ_n spanned by vertices corresponding to permutations with i in the last position. Then Γ_n can be divided into n subgraphs $\Gamma^1_{n-1}, \Gamma^2_{n-1}, \cdots, \Gamma^n_{n-1}$ and each Γ^i_{n-1} is isomorphic to Γ_{n-1} for $i \in [n]$. For $u \in V(\Gamma^i_{n-1})$, denoted by $u' = u(1n)$ the unique neighbor of u outside Γ^i_{n-1}, called the extra neighbor of u.

Lemma 12. *Let Γ_n be the Cayley graphs generated by transposition trees for $n \geq 3$.*

(1) ([6]) $\kappa(\Gamma_n) = n-1$.
(2) ([6]) *Γ_n has the girth 4 unless Γ_n is the star graph which has girth 6. Γ_n does not have $K_{2,3}$ as a subgraph.*
(3) ([51]) *For any two distinct vertices $u, v \in \Gamma_n$, $|N_{\Gamma_n}(u) \cap N_{\Gamma_n}(v)| = 1$ if $\Gamma_n = S_n$; otherwise $|N_{\Gamma_n}(u) \cap N_{\Gamma_n}(v)| \leq 2$.*

Lemma 13. ([6,51]) *If $F \subseteq V(\Gamma_n)$ with $|F| \leq 2n-5$ is a vertex cut of Γ_n for $n \geq 4$, then $\Gamma_n - F$ has two components, one of which is a trivial component.*

2.7. Cayley graphs generated by 2-trees

Definition 9. Let Γ be the alternating group, the set of even permutations on $\{1, 2, \dots, n\}$, and the generating set Δ to be a set of 3-cycles. To get an undirected Cayley graph, we will assume that whenever a 3-cycle (abc) is in Δ, so is its inverse, (acb). Since (abc), (bca) and (cab) represent the same permutation, the set $\{a, b, c\}$ uniquely represents this 3-cycle and its inverse. So we can depict Δ via a hypergraph with vertex set $[n]$, where a hyperedge of size 3 corresponds to each pair of a 3-cycle and its inverse in Δ.

It is easy to see that the Cayley graph generated by the 3-cycles in Δ is connected if its corresponding hypergraph H is connected. Since an interconnection network needs to be connected, we require H graph to be connected.

In general, this graph may have extra K_3's formed by vertices that do not correspond to a 3-cycle in Δ. We will avoid this possibility by considering a simpler case when H has a tree-like structure. Such a graph is built by the following procedure. We start from K_3, then repeatedly add a new vertex, joining it to exactly two adjacent vertices of the previous graph. Any graph obtained by this procedure is called a 2-tree. If v is a vertex of a 2-tree H with the property that H can be generated in such a way that v is the last vertex added, then v is called a leaf of the 2-tree.

The *alternating group graph* AG_n [31], can be viewed as the Cayley graph generated by the graph having a tree-like (in fact, star-like) structure of triangles.

It is easy to prove that if two 2-trees are isomorphic, then the corresponding Cayley graphs will also be isomorphic; hence without loss of generality we may assume that vertex n is the tail of the 2-tree. For $n \geq 4$, the vertices corresponding to even permutations ending with i induce a subgraph $\Gamma^i_{n-1}(\Delta)$ that is also a Cayley graph generated by a 2-tree Δ', which is obtained by deleting the edges corresponding to the two 3-cycles in Δ containing n. Thus we obtain the following result of the recursive structure of $\Gamma_n(\Delta)$:

Lemma 14. ([10]) *Let $\Gamma_n(\Delta)$ be a Cayley graph generated by the 2-tree Δ, $\Delta' = \Delta - \{n\}$, $n \geq 4$. Then*

(1) *$\Gamma_n(\Delta)$ consists of n vertex-disjoint subgraphs, $\Gamma^1_{n-1}(\Delta), \Gamma^2_{n-1}(\Delta), \dots, \Gamma^n_{n-1}(\Delta)$, each is isomorphic to $\Gamma_{n-1}(\Delta')$.*
(2) *$\Gamma^i_{n-1}(\Delta)$ has $(n-1)!/2$ vertices, and it is $(2n-6)$-regular for all i.*
(3) *There are exactly $(n-2)!$ independent edges between $\Gamma^i_{n-1}(\Delta)$ and $\Gamma^j_{n-1}(\Delta)$ for all $i \neq j$.*
(4) *Each vertex in $\Gamma^i_{n-1}(\Delta)$ has exactly two neighbors outside $\Gamma^i_{n-1}(\Delta)$; these two outside neighbors are in different $\Gamma^k_{n-1}(\Delta)$'s, and there is an edge between them. Thus every vertex forms a triangle with its two outside neighbors.*
(5) *$\Gamma_n(\Delta)$ does not contain $K_4 - e$, that is, K_4 with an edge deleted, and $K_{2,3}$ as a subgraph. For any two vertices u and v, $|N(u) \cap N(v)| = 1$ if $d(u, v) = 1$, $|N(u) \cap N(v)| \leq 2$ otherwise.*

Lemma 15. ([5]) *Let $G = \Gamma_n(\Delta)$ be a Cayley graph generated by the 2-tree Δ for $n \geq 4$. Then G is maximally connected, i.e., G is $(2n-4)$-regular and $(2n-4)$-connected.*

Lemma 16. ([5]) *Let $G = \Gamma_n(\Delta)$ be a Cayley graph generated by the 2-tree Δ for $n \geq 4$, and let T be a set of vertices in G such that $|T| \leq 4n - 11$. If $n \geq 5$, then $G - T$ satisfies one of the following conditions:*

(1) *$G - T$ is connected.*
(2) *$G - T$ has two components, one of which is a singleton.*
(3) *$G - T$ has two components, one of which is a K_2. Moreover, $|T| = 4n - 11$, and the set T is formed by the neighbors of the two vertices in the K_2.*

When $n = 4$, there are two additional possibilities. In both cases, $G - T$ has two components, one of which is a 4-cycle. The other component is either a 4-cycle if $|T| = 4$ or a path with 3 vertices if $|T| = 5$.

2.8. Burnt pancake networks BP_n

Gates and Papadimitriou [21] introduced the burnt pancake problem in 1979. Burnt pancake problem relates to the construction of networks of parallel processors.

Let n be a positive integer. We use $[n]$ to denote the set $\{1, 2, \dots, n\}$. To save space, the negative sign may be placed on the top of an expression. Thus, $\bar{i} = -i$. We use $\langle n \rangle$ to denote the set $[n] \cup \{\bar{i} | i \in [n]\}$. A *signed permutation* of $[n]$ is an n-permutation $u_1 u_2 \cdots u_n$ of $\langle n \rangle$ such that $|u_1||u_2|\cdots|u_n|$ taking the absolute value of each element, forms a permutation of $[n]$. For a signed permutation $u = x_1 x_2 \cdots x_i \cdots x_n$ of $\langle n \rangle$, the i-th prefix reversal of u, denoted by u^i is $u^i = \bar{x}_i \bar{x}_{i-1} \cdots \bar{x}_1 x_{i+1} \cdots x_n$, $1 \leq i \leq n$. For example, let $u = 1\bar{2}4\bar{3}5$; then u is a signed permutation of [5], $u^2 = 2\bar{1}4\bar{3}5$, $u^5 = \bar{5}3\bar{4}2\bar{1}$.

Definition 10. An *n-dimensional burnt pancake network* BP_n is defined to be an n-regular graph G with $n!2^n$ vertices, each of which has a unique label from the signed permutation of $\langle n \rangle$. Two vertices u and v are adjacent in BP_n if and only if $u^i = v$ for some unique i $(1 \leq i \leq n)$. Such an edge uv is called an *i-dimensional edge* and v is called the *i-neighbor* of u. It is seen that every vertex has a unique i-neighbor for $1 \leq i \leq n$.

Lemma 17. ([11,13,30]) *An n-dimensional burnt pancake network BP_n has the following combinatorial properties.*

(1) *BP_n is n-regular with $n! \times 2^n$ vertices and $n! \times 2^{n-1}$ edges.*
(2) *$\kappa(BP_n) = n$, the girth of BP_n $(n \geq 3)$ is $g(BP_n) = 8$.*
(3) *BP_n can be decomposed into 2n vertex-disjoint subgraphs, denoted BP^i_n, by fixing the symbol in the last position n, in which the symbol in the nth position is i, where $i \in [n]$. Obviously, BP^i_n is isomorphic to BP_{n-1}. The number of cross edges between any two subgraphs, BP^i_n and BP^j_n $(i \neq j, i, j \in [n])$, is $|E(i, j)| = (n-2)! \times 2^{n-2}$ if $i \neq \bar{j}$; otherwise, $|E(i, j)|$=0. For a vertex $v \in V(BP^i_n)$, v has exactly one neighbor outside BP^i_n, called the* extra neighbor *of v.*

Lemma 18. ([41]) *For any subset $F \subseteq V(BP_n)$ with $|F| \leq 2n - 2$ is a vertex-cut of BP_n for $n \geq 4$, then $BP_n - F$ satisfies one of the following conditions.*

(1) $BP_n - F$ *has two connected components, one of which is a trivial component;*
(2) $BP_n - F$ *has two connected components, one of which is an edge. Furthermore, F is the neighborhood of this edge with* $|F| = 2n - 2$.

3. Main result

In this section, the relationship between the pessimistic diagnosability under the PMC model and the extra connectivity with some restricted conditions will be proposed.

Lemma 19. *Let G be a k-regular graph. Let u and v be two distinct vertices in G, if* $cn(G; u, v) \le 2$, *then* $|N_G(\{u, v\})| \ge 2k - 2 - l$, *where* $l = l(G) = \max\{cn(G; u, v) : (u, v) \in E(G)\}$, *i.e.,* $l = l(G)$ *be the maximum number of common neighbors between any two adjacent vertices.*

Proof. Since $cn(G; u, v) \le 2$, if u is non-adjacent to v, then $|N_G(\{u, v\})| = |N_G(u)| + |N_G(v)| - cn(G; u, v) \ge 2k - 2 \ge 2k - 2 - l$. Otherwise, u is adjacent to v, $|N_G(\{u, v\})| = |N_G(u)| - 1 + |N_G(v)| - 1 - cn(G; u, v) \ge 2(k - 1) - l$. As a result, $|N_G(\{u, v\})| \ge 2k - 2 - l$. □

Tsai and Chen [43] derived the following result which characterizes a graph for t/t-diagnosability.

Lemma 20. ([43]) *A graph G is* t/t*-diagnosable if and only if for each vertex set* $S \subseteq V(G)$ *with* $|S| = p$, $0 \le p \le t - 1$, $G - S$ *has at most one trivial component and each nontrivial component C of* $G - S$ *satisfies* $|V(C)| \ge 2(t - p) + 1$.

The following result is useful.

Lemma 21. ([18]) *Let G be a connected graph and* $U \subseteq V(G)$. *Then,* $|N_{V(G)-U}(U)| \ge \kappa(G)$ *if* $|V(G) - U| \ge \kappa(G)$, *otherwise,* $|N_{V(G)-U}(U)| = |V(G) - U|$.

Theorem 1. *Let G be a k-regular k-connected* ($k \ge 5$) *graph with order N. Let U be a subset of* $V(G)$ *and* $l = l(G)$ *be the maximum number of common neighbors between any two adjacent vertices. Suppose further that all of the following conditions hold:*

(1) $N \ge 4k - 2$.
(2) $cn(G) \le 2$.
(3) *If* $2 \le |U| \le 2(2k - 4 - l)$, *then* $|N_G(U)| \ge 2k - 2 - l$.
(4) *Let* $F \subseteq V(G)$ *be a vertex-cut of G. If* $|F| \le 2k - 3 - l$, *then* $G - F$ *has a large component and a small component which is a trivial component.*

Then, $t_p(G) = 2k - 2 - l = \kappa_1(G)$.

Proof. We first prove $t_p(G) \le 2k - 2 - l$. Suppose $t_p(G) \ge 2k - 2 - l + 1$, then G is $(2k - 2 - l + 1)/(2k - 2 - l + 1)$-diagnosable. Let (u, v) be an edge of G such that $|N_G(u) \cap N_G(v)| = l$. Let $S = N_G(\{u, v\})$. Then $|S| = 2k - 2 - l \le t_p(G) - 1$. An edge $\{u, v\}$ is a connected component of $G - S$, say C. By Lemma 20, $|V(C)| \ge 2(t_p(G) - |S|) + 1 \ge 2[(2k - 2 - l + 1) - (2k - 2 - l)] + 1 = 3$, which is a contradiction. Thus, $t_p(G) \le 2k - 2 - l$.

Secondly, we show $t_p(G) \ge 2k - 2 - l$, i.e., G is $(2k - 2 - l)/(2k - 2 - l)$-diagnosable. Suppose G is not $(2k - 2 - l)/(2k - 2 - l)$-diagnosable, by Lemma 20, there exists a vertex set $S \subseteq V(G)$ with $|S| = p$, $0 \le p \le 2k - 3 - l$ such that $G - S$ contains more than one trivial components or contains a nontrivial component C with $|V(C)| \le 2(2k - 2 - l - p)$. The following cases should be considered.

Case 1. $G - S$ contains more than one trivial components.

Suppose $C_1 = \{u\}$ and $C_2 = \{v\}$ are two distinct trivial components of $G - S$. By Condition (2) and Lemma 19, $|N_G(\{u, v\})| \ge 2k - 2 - l$. Note that $N_G(\{u, v\}) \subseteq S$, this implies that $|S| \ge 2k - 2 - l$, which is a contradiction.

Case 2. $G - S$ contains a nontrivial component C with $2 \le |V(C)| \le 2(2k - 2 - l - p)$.

Suppose $p \le 1$. Since the connectivity of G is $k \ge 5 > p$, $G - S$ is connected. It implies $C = G - S$. By $|V(C)| = |V(G)| - |S| = N - p \ge N - 1$, Condition (1) and $l \le cn(G) \le 2$, one has $|V(C)| \ge 4k - 3 \ge 2(2k - 2 - l - p) + 1$ which is a contradiction.

Now consider $2 \le p \le 2k - 3 - l$. Since $2 \le |V(C)| \le 2(2k - 2 - l - p)$, so $2 \le |V(C)| \le 2(2k - 4 - l)$. By condition (3), $|N_G(V(C))| \ge 2k - 2 - l$. Since C is a connected component of $G - S$, $N_G(V(C)) \subseteq S$. This implies $p = |S| \ge 2k - 2 - l$, which is a contradiction for the fact that $p = |S| \le 2k - 3 - l$. Thus, $t_p(G) \le 2k - 2 - l$.

Next we prove $2k - 2 - l = \kappa_1(G)$. Let (u, v) be an edge of G such that $|N_G(u) \cap N_G(v)| = l$. Let $S = N_G(\{u, v\})$. Then $|S| = 2k - 2 - l$. If $G - S = \{(u, v)\}$, then $|V(G)| = |S| + 2 = 2k - l < 4k - 2$ for $k \ge 5$ which contradicts with Condition (1). If $G - S$ has a trivial component which contains only one vertex, say $\{x\}$, then $G - S$ has at least two components: $\{x\}$ and the edge (u, v). By $cn(G) \le 2$, then $|S| \ge 2k - 2 - l + (k - 4) = 3k - 6 - l$. Note $3k - 6 - l > 2k - 2 - l$ for $k \ge 5$, it is a contradiction. Thus, $G-S$ has no trivial component, i.e., S is an extra vertex cut of G, which implies $\kappa_1(G)\le 2k-2-l$. On the other hand, by condition (4), $\kappa_1(G)\ge 2k-2-l$. Thus, $\kappa_1(G)=2k-2-l$.

By above discussion, $t_p(G)=2k-2-l=\kappa_1(G)$. □

4. Application to some interconnection networks

As applications of Theorem 1, in this section, we determine the pessimistic diagnosability and extra connectivity for some well-known interconnection networks, including the alternating group graph AG_n, the alternating group network AN_n, the k-ary n-cube networks Q_n^k, BC networks X_n, split-star networks S_n^2, Cayley graphs generated by transposition trees Γ_n, Cayley graphs generated by 2-trees, burnt pancake networks BP_n.

4.1. Application to the alternating group graphs AG_n

Remark 1. It is known that $\kappa_1(AG_n)=4n-11$ for $n\ge 5$ determined by Lin et al. [38] and $t_p(AG_n)=4n-11$ obtained by Tsai [44]. As a corollary of Theorem 1, we immediately obtain the following result which contains the above result.

Corollary 1. *Let AG_n be the n-dimensional alternating group graph for $n\ge 5$. Then $t_p(AG_n)=4n-11=\kappa_1(AG_n)$.*

Proof. Obviously, $N=|V(AG_n)|=\frac{n!}{2}$, $k=2n-4\ge 6$ for $n\ge 5$, $l=l(AG_n)=1$.

Note that $N=\frac{n!}{2}\ge 4(2n-4)-2$ for $n\ge 5$, Conditions (1) in Theorem 1 holds. Conditions $(2)-(4)$ in Theorem 1 hold by Lemmas 1, 2 and 3, respectively. Thus, AG_n satisfies all conditions in Theorem 1, $t_p(AG_n)=4n-11=\kappa_1(AG_n)$ for $n\ge 5$. □

4.2. Application to the alternating group networks

Zhou [53] derived $\kappa_1(AN_n)=2n-5$ for $n\ge 4$. However, $t_p(AN_n)$ has not been determined so far. We can deduce the result as a corollary of Theorem 1 as following. Notice that for AN_n, $k=n-1$ and $l=1$ in Theorem 1.

Lemma 22. *Let AN_n be the n-dimensional alternating group network for $n\ge 4$. If U is a subset of $V(AN_n)$ and $2\le |U|\le 2(2k-4-l)=4n-14$, then $|N_{AN_n}(U)|\ge 2n-5$.*

Proof. The Lemma can be proved by using the induction on n. It is easy to verify that $|N_{AN_4}(U)|\ge 3$ for $|U|=2$ by Lemma 19. We assume that the lemma is true for AN_m, where m is an integer with $5\le m\le n-1$, we will prove the result for AN_n.

Recall that AN_n is constructed by n disjoint AN_{n-1}'s, denoted by AN_n^i for $i\in[n]$. Let $U_i=U\cap V(AN_n^i)$ and $\overline{AN_n^i}=AN_n-AN_n^i$ for $i\in[n]$. Without loss of generality, we may assume that $|U_1|\ge|U_2|\ge\ldots\ge|U_n|$. The following cases should be considered.

Case 1. $|U_1|\le 1$.

In this case, $|U_i|\le 1$ for all $i\in[n]$. Clearly, $2\le|U|\le n$ because of $i\le n$. The Lemma follows if $|U|=2$ by Lemma 19. Now assume that $3\le|U|\le n$. Since AN_n is $(n-1)$-regular and AN_n^i is isomorphic to AN_{n-1}, $|N_{AN_n}(U)|\ge 3\kappa(AN_n^i)=3(n-2)\ge 2n-5$ for $n\ge 7$.

Case 2. $2\le|U_1|\le 4n-19$.

By the inductive hypothesis in AN_n^1, $|N_{AN_n^1}(U_1)|\ge 2(n-1)-5=2n-7$. If $U=U_1$, $|N_{AN_n}(U)|=|N_{AN_n^1}(U_1)|+|N_{\overline{AN_n^1}}(U_1)|\ge 2n-7+|U_1|\ge 2n-5$. Assume $U\ne U_1$ in the following. If $|U_2|=1$, $|N_{AN_n^2}(U_2)|=\kappa(AN_n^2)=n-2$. Note that AN_n^1 and AN_n^2 are vertex disjoint, $|N_{AN_n}(U)|\ge|N_{AN_n^1}(U_1)|+|N_{AN_n^2}(U_2)|\ge 3n-9\ge 2n-5$ for $n\ge 5$. Now consider $2\le|U_2|\le|U_1|\le 4n-19$, by the inductive hypothesis in AN_n^2, $|N_{AN_n^2}(U_2)|\ge 2(n-1)-5=2n-7$. Thus, $|N_{AN_n}(U)|\ge|N_{AN_n^1}(U_1)|+|N_{AN_n^2}(U_2)|\ge 4n-14\ge 2n-5$ for $n\ge 5$.

Case 3. $4n-18\le|U_1|\le 4n-14$.

Since the connectivity of AN_n^1 is $n-2$, and $\frac{(n-1)!}{2}-|U_1|\ge n-2=\kappa(AN_n^1)$ for $n\ge 5$, by Lemma 21, $|N_{AN_n^1}(U_1)|\ge n-2$. By Lemma 4, $|N_{\overline{AN_n^1}}(U_1)|=|U_1|$. If $U=U_1$, $|N_{AN_n}(U)|\ge|N_{AN_n^1}(U_1)|+|N_{\overline{AN_n^1}}(U_1)|\ge(n-2)+4n-18=5n-20\ge 2n-5$ for $n\ge 5$. In the following, we assume the case of $U\ne U_1$. Note that $U\ne U_1$ and $|U-U_1|\le 3$, so $1\le|U_2|\le 3$.

If $|U_2|=1$, recall that AN_n is $(n-1)$-regular and AN_n^i is isomorphic to AN_{n-1}, $|N_{AN_n^2}(U_2)|=\kappa(AN_n^2)=n-2$. Hence, $|N_{AN_n}(U)|\ge|N_{AN_n^1}(U_1)|+|N_{AN_n^2}(U_2)|\ge 2n-4\ge 2n-5$ for $n\ge 5$. Now suppose that $2\le|U_2|\le 3$. Since $\frac{(n-1)!}{2}-|U_2|\ge n-2=\kappa(AN_n^2)$ for $n\ge 5$, by Lemma 21, $|N_{AN_n^2}(U_2)|\ge n-2$. Thus, $|N_{AN_n}(U)|\ge|N_{AN_n^1}(U_1)|+|N_{AN_n^2}(U_2)|\ge 2(n-2)\ge 2n-5$ for $n\ge 5$.

By the above cases, the Lemma holds. □

Corollary 2. *Let AN_n be the n-dimensional alternating group network for $n\ge 6$. Then $t_p(AN_n)=2n-5=\kappa_1(AN_n)$.*

Proof. Note that $N = |V(AN_n)| = \frac{n!}{2} \geq 4(n-1) - 2$ for $n \geq 6$, Condition (1) in Theorem 1 holds. Conditions (2)–(4) in Theorem 1 hold by Lemmas 4, 5 and 22, respectively. So AN_n satisfies all conditions in Theorem 1, and $t_p(AN_n) = 2n - 5 = \kappa_1(AN_n)$ for $n \geq 6$. □

4.3. Application to BC networks

Note that $L_n = \{X_n : X_n$ is an n-dimensional BC network$\}$. For a BC network $X_n \in L_n$, the connectivity is $k = n \geq 5$, $l = 0$ and $N = |V| = 2^n \geq 4n - 2$ for $n \geq 5$ in Theorem 1. As a directive corollary of Theorem 1, we can get the result $\kappa_1(X_n) = t_p(X_n) = 2n - 2$ in which Zhu [54] determined $\kappa_1(X_n) = 2n - 2$ for $n \geq 4$. Fan and Lin [20] obtained $t_p(X_n) = 2n - 2$ for $n \geq 4$.

Lemma 23. *For any $X_n \in L_n$, if $U \subseteq V(X_n)$ with $2 \leq |U| \leq 4n - 8$ for $n \geq 3$, then $|N_{X_n}(U)| \geq 2n - 2$.*

Proof. We prove the lemma by using introduction on n. If $n = 3$, $2 \leq |U| \leq 4n - 8 = 4$, it is not difficult to see that $|N_{X_3}(U)| \geq 4$. Assume that the lemma is true for X_{m-1}, where m is an integer with $4 \leq m \leq n - 1$. We consider X_n for $n \geq 4$ as follows.

Since X_n is n-regular n-connected triangle-free and $C(X_n) = 2$, if $|U| = 2$, then $|N_{X_n}(U)| \geq 2n - 2$. Now consider $3 \leq |U| \leq 4n - 8$. Note that X_n contains two copies of X_{n-1}, say X^1_{n-1} and X^2_{n-1}, respectively. Let $U_i = U \cap V(X^i_{n-1})$ for $i \in \{1, 2\}$. Without loss of generality, we may assume that $|U_1| \geq |U_2|$. It implies that $2 \leq |U_1|$.

Case 1. $2 \leq |U_1| \leq 4n - 12$. By the inductive hypothesis in X^1_{n-1}, $|N_{X^1_{n-1}}(U_1)| \geq 2n - 4$. If $|U_2| = 0$, then $U = U_1$. $|N_{X_n}(U)| \geq |N_{X^1_{n-1}}(U_1)| + |N_{\overline{X^1_{n-1}}}(U_1)| \geq (2n - 4) + 2 \geq 2n - 2$. If $|U_2| = 1$, $|N_{X^2_{n-1}}(U_2)| = \kappa(X^2_{n-1}) = n - 1$. Thus $|N_{X_n}(U)| \geq |N_{X^1_{n-1}}(U_1)| + |N_{X^2_{n-1}}(U_2)| \geq (2n - 4) + (n - 1) = 3n - 5 \geq 2n - 2$ for $n \geq 4$. Now consider $2 \leq |U_2| \leq |U_1| \leq 4n - 12$ for $n \geq 4$, so $|N_{X^2_{n-1}}(U_2)| \geq 2n - 4$. Thus, $|N_{X_n}(U)| \geq |N_{X^1_{n-1}}(U_1)| + |N_{X^2_{n-1}}(U_2)| \geq 2(2n - 4) = 4n - 8 \geq 2n - 2$ for $n \geq 4$.

Case 2. $4n - 11 \leq |U_1| \leq 4n - 8$.

If $U = U_1$, by definition, $|N_{\overline{X^1_{n-1}}}(U_1)| = |U_1| \geq 4n - 11$. Thus, $|N_{X_n}(U)| \geq |N_{\overline{X^1_{n-1}}}(U_1)| \geq 4n - 11 \geq 2n - 4$ for $n \geq 4$. Now assume that $U \neq U_1$. Since the connectivity of X^1_{n-1} is $n - 1$ and $|V(X^1_{n-1})| - (4n - 8) \geq \kappa(X^1_{n-1}) = n - 1$ for $n \geq 4$, by Lemma 21, $|N_{X^1_{n-1}}(U_1)| \geq n - 1$. Note that $U \neq U_1$ and $|U - U_1| \leq 3$, so $1 \leq |U_2| \leq 3$. If $|U_2| = 1$, $|N_{X^2_{n-1}}(U_2)| = \kappa(X^2_{n-1}) = n - 1$. Hence, $|N_{X_n}(U)| \geq |N_{X^1_{n-1}}(U_1)| + |N_{X^2_{n-1}}(U_2)| \geq 2n - 2$ for $n \geq 4$. Now suppose that $2 \leq |U_2| \leq 3$. Since $|V(X^2_{n-1})| - 3 \geq \kappa(B_2) = n - 1$ for $n \geq 4$, by Lemma 21, $|N_{X^2_{n-1}}(U_2)| \geq \kappa(X^2_{n-1}) = n - 1$. So $|N_{X_n}(U)| \geq |N_{X^1_{n-1}}(U_1)| + |N_{X^2_{n-1}}(U_2)| \geq 2n - 2$ for $n \geq 4$.

By the above cases, the proof is completed. □

By Lemmas 6, 7 and 23 and Theorem 1, we obtain the following Corollary 3.

Corollary 3. *For any $X_n \in L_n$, $t_p(X_n) = 2n - 2 = \kappa_1(X_n)$ for $n \geq 5$.*

It is not difficult to check that the hypercube Q_n, the crossed cube CQ_n, the Möbius cubes MQ_n, the twisted cubes TQ_n are all n-regular n-connected triangle-free BCs, then the following known result is derived directly.

Corollary 4. ([20]) *Every pessimistic diagnosability of the hypercube Q_n, the crossed cube CQ_n, the Möbius cubes MQ_n and the twisted cubes TQ_n is $2n - 2$ for $n \geq 6$.*

4.4. Application to the k-ary n-cube networks Q^k_n

Lemma 24. *Let Q^k_n be a k-ary n-cube, where $k \geq 2$ and $n \geq 1$ are integers.*

(1) *For $n \geq 3$, let U be a subset of $V(Q^2_n)$ with $2 \leq |U| \leq 4n - 8$. Then $|N_{Q^2_n}(U)| \geq 2n - 2$.*
(2) *For $n \geq 3$, let U be a subset of $V(Q^3_n)$ and $2 \leq |U| \leq 8n - 10$, then $|N_{Q^3_n}(U)| \geq 4n - 3$.*
(3) *For $n \geq 3$ and $k \geq 4$, let U be a subset of $V(Q^k_n)$ and $2 \leq |U| \leq 8n - 8$, then $|N_{Q^k_n}(U)| \geq 4n - 2$.*

Proof. Since the proof for the three cases are similar, we take (2) as an example, the details for (1) and (3) are omitted.

Let $Q[0], Q[1], Q[2]$ represent the three disjoint subcubes obtained from Q^3_n by partition over one dimension. Let $U_i = U \cap V(Q[i])$ and $\overline{Q[i]} = Q^3_n - Q[i]$ for $i \in \{0, 1, 2\}$. Without loss of generality, we may assume that $|U_0| \geq |U_1| \geq |U_2|$.

The lemma is proved by the induction on n. When $n = 3$, it is easy to check $|N_{Q^3_3}(U)| \geq 9$ for $2 \leq |U| \leq 8n - 10 = 14$. We assume that the lemma is true for Q^3_{m-1}, where m is an integer with $4 \leq m \leq n - 1$. We consider Q^3_n for $n \geq 4$ as follows.

Case 1. $|U_0| \le 1$.

In this case, $|U_i| \le 1$ for all $0 \le i \le 2$. Clearly, $2 \le |U| \le 3$ because of $i \le 2$. The Lemma follows if $|U| = 2$ by Lemma 19. Now assume that $|U| = 3$. Since Q_n^3 is $2n$-regular and $Q[i]$ is isomorphic to Q_{n-1}^3, $|N_{Q_n^3}(U)| \ge 3\kappa(Q_{n-1}^3) = 3(2n-2) \ge 4n-3$ for $n \ge 3$.

Case 2. $2 \le |U_0| \le 8n-18$.

By the inductive hypothesis in $Q[0]$, $|N_{Q[0]}(U_0)| \ge 4(n-1)-3 = 4n-7$. If $U = U_0$, then $|N_{Q_n^3}(U)| = |N_{Q[0]}(U_0)| + |N_{\overline{Q[0]}}(U_0)| \ge 4n-7+2|U_0| \ge 4n-7+4 = 4n-3$. Assume $U \ne U_0$ in the following. Note that $|U| \le 8n-10$ and $|U_0| \ge |U_1| \ge |U_2|$, $|U_1| \le 4n-5$.

If $|U_1| = 1$, $|N_{Q[1]}(U_1)| = \kappa(Q[1]) = 2n-2$. Note that $Q[0]$ and $Q[1]$ are vertex disjoint, $|N_{Q_n^3}(U)| \ge |N_{Q[0]}(U_0)| + |N_{Q[1]}(U_1)| \ge (4n-7)+(2n-2) = 6n-9 \ge 4n-3$ for $n \ge 4$. Now consider $2 \le |U_1| \le 4n-5 \le 8n-18$ for $n \ge 4$, by the inductive hypothesis in $Q[1]$, $|N_{Q[1]}(U_1)| \ge 4(n-1)-3 = 4n-7$. Thus, $|N_{Q_n^3}(U)| \ge |N_{Q[0]}(U_0)| + |N_{Q[1]}(U_1)| \ge 2(4n-7) = 8n-14 \ge 4n-3$ for $n \ge 4$.

Case 3. $8n-17 \le |U_0| \le 8n-10$.

If $U = U_0$, $|N_{Q_n^3}(U)| \ge |N_{\overline{Q[0]}}(U_0)| = 2|U_0| \ge 2(8n-17) \ge 4n-3$ for $n \ge 4$. In the following, we assume the case of $U \ne U_0$. Since the connectivity of $Q[0]$ is $2n-2$, note that $U \ne U_0$, so $2 \le |U_0| \le 8n-11$. Since $|V(Q_{[0]}) - U_0| = 3^{n-1} - |U_0| \ge 3^{n-1} - (8n-11) \ge 2n-2 = \kappa(Q[0])$ for $n \ge 4$, and by Lemma 21, $|N_{Q[0]}(U_0)| \ge 2n-2$.

Note that $U \ne U_0$ and $|U - U_0| \le 7$, so $1 \le |U_1| \le 7$.

If $|U_1| = 1$ and $|U_2| = 0$, recall that the connectivity of Q_n^3 is $2n$ and $Q[i]$ is isomorphic to Q_{n-1}^k, $|N_{Q[1]}(U_1)| = \kappa(Q[1]) = 2n-2$. Note that each vertex in $Q[0]$ (resp. $Q[1]$) has an extra neighbor in $Q[2]$. Hence, $|N_{Q_n^3}(U)| \ge |N_{Q[0]}(U_0)| + |N_{Q[1]}(U_1)| + |N_{Q[2]}(U_0)| \ge 4n-4+(8n-17) = 12n-21 \ge 4n-3$ for $n \ge 4$. If $|U_i| = 1$ for $i = 1, 2$, $|N_{Q[i]}(U_i)| = \kappa(Q[i]) = 2n-2$. Hence, $|N_{Q_n^3}(U)| \ge |N_{Q[0]}(U_0)| + |N_{Q[1]}(U_1)| + |N_{Q[2]}(U_2)| \ge 3(2n-2) = 6n-6 \ge 4n-3$ for $n \ge 4$. Now suppose that $2 \le |U_1| \le 7$. Since $7 < 8n-17$ for $n \ge 4$, by the inductive hypothesis in $Q[1]$, $|N_{Q[1]}(U_1)| \ge 4(n-1)-3 = 4n-7$. Thus, $|N_{Q_n^3}(U)| \ge |N_{Q[0]}(U_0)| + |N_{Q[1]}(U_1)| \ge (2n-2)+(4n-7) = 6n-9 \ge 4n-3$ for $n \ge 4$.

The proof is complete. □

Remark 2. Esfahanian [16] obtained $\kappa_1(Q_n^2) = 2n-2$ for $n \ge 3$ and Day [14] got $\kappa_1(Q_n^3) = 4n-3$, $\kappa_1(Q_n^k) = 4n-2$ for $k \ge 4$. Kavianpour and Kim [33] proved that $t_p(Q_n^2) = 2n-2$ for $n \ge 3$ and Wang et al. [48] derived $t_p(Q_n^3) \ge 4n-3$ for $n \ge 4$ and $t_p(Q_n^k) \ge 4n-2$ for $k \ge 4$ and $n \ge 4$. These results can be gotten directly as corollary of Theorem 1 as following.

Since $k^n \ge 4\kappa(Q_n^k) - 2$ for $k \ge 3$ and $n \ge 3$ ($k = 2$ and $n \ge 5$), Condition (1) in Theorem 1 holds. By Lemmas 8, 9 and 24, Conditions (2)–(4) in Theorem 1 holds.

Corollary 5. *Let Q_n^k be a k-ary n-cube, where $k \ge 2$ and $n \ge 1$ are integers. Then*

(1) $t_p(Q_n^2) = 2n-2 = \kappa_1(Q_n^2)$ *for* $n \ge 5$;
(2) $t_p(Q_n^3) = 4n-3 = \kappa_1(Q_n^3)$ *for* $n \ge 3$;
(3) $t_p(Q_n^k) = 4n-2 = \kappa_1(Q_n^k)$ *for* $n \ge 3$ *and* $k \ge 4$.

4.5. Application to the split-star networks S_n^2

Lin et al. [37] proved $\kappa_1(S_n^2) = 4n-9$ for $n \ge 4$. However, $t_p(S_n^2)$ has not been determined so far. We can deduce the result by Theorem 1 in which for S_n^2, $k = 2n-3$ and $l = 1$.

Lemma 25. *Let S_n^2 be the n-dimensional split-star network for $n \ge 4$. If U is a subset of $V(S_n^2)$ and $2 \le |U| \le 8n-22$, then $|N_{S_n^2}(U)| \ge 4n-9$.*

Proof. We prove the lemma by using the induction on n. Since S_4^2 is constructed by four disjoint triangles S_3^2, it is easy to verify that $|N_{S_4^2}(U)| \ge 7$ for $2 \le |U| \le 10$. By the inductive hypothesis, we assume that the lemma is true for S_m^2, where m is an integer with $5 \le m \le n-1$. Now we consider S_n^2.

Recall that S_n^2 is constructed by n disjoint S_{n-1}^2s, denoted by $S_n^{2:i}$ for $i \in [n]$. Let $U_i = U \cap V(S_n^{2:i})$ and $\overline{S_n^{2:i}} = S_n^2 - S_n^{2:i}$ for $i \in [n]$. Without loss of generality, we may assume that $|U_1| \ge |U_2| \ge \ldots \ge |U_n|$. The following cases should be considered.

Case 1. $|U_1| \le 1$.

In this case, $|U_i| \le 1$ for all $i \in [n]$. Clearly, $2 \le |U| \le n$ because of $U = \bigcup_{i=1}^{n} U_i$. If $|U| = 2$, by Lemma 19, $|N_{S_n^2}(U)| \ge 2(2n-3)-2-1 = 4n-9$, the lemma follows. Now assume that $3 \le |U| \le n$. Since S_n^2 is $(2n-3)$-regular and $S_n^{2:i}$ is isomorphic to S_{n-1}^2, $|N_{S_n^2}(U)| \ge 3\kappa(S_n^{2:i}) = 3(2n-5) \ge 4n-9$ for $n \ge 5$.

Case 2. $2 \le |U_1| \le 8n-30$.

By the inductive hypothesis in $S_n^{2:1}$, $|N_{S_n^1}(U_1)| \ge 4(n-1)-9 = 4n-13$. Since $|U| \le 8n-22$ and $|U_1| \ge |U_2| \ge \ldots \ge |U_n|$, $|U_2| \le 4n-11$. If $U = U_1$, by Lemma 10(2), $|N_{S_n^2}(U)| = |N_{S_n^{2:1}}(U_1)| + |N_{\overline{S_n^{2:1}}}(U_1)| \ge 4n-13+2|U_1| \ge 4n-9$. Assume $U \ne U_1$ in the following. If $|U_2| = 1$, $|N_{S_n^{2:1}}(U_1)| = \kappa(S_n^{2:1}) = 2n-5$. Note that $S_n^{2:1}$ and $S_n^{2:2}$ are vertex disjoint, $|N_{S_n^2}(U)| \ge |N_{S_n^{2:1}}(U_1)| + |N_{S_n^{2:2}}(U_2)| \ge 4n-13+2n-5 = 6n-18 \ge 4n-9$ for $n \ge 5$. Now consider $2 \le |U_2| \le 4n-11$. Note that $4n-11 \le 8n-30$ for $n \ge 5$, by the inductive hypothesis in $S_n^{2:2}$, $|N_{S_n^{2:2}}(U_2)| \ge 4(n-1)-9 = 4n-13$. Thus, $|N_{S_n^2}(U)| \ge |N_{S_n^{2:1}}(U_1)| + |N_{S_n^{2:2}}(U_2)| \ge 8n-26 \ge 4n-9$ for $n \ge 5$.

Case 3. $8n-29 \le |U_1| \le 8n-22$.

By Lemma 10(2), $|N_{\overline{S_n^{2:1}}}(U_1)| = 2|U_1|$. If $U = U_1$, $|N_{S_n^2}(U)| \ge |N_{\overline{S_n^{2:1}}}(U_1)| = 2|U_1| \ge 16n-58 \ge 4n-9$ for $n \ge 5$. In the following, we assume the case of $U \ne U_1$. Since the connectivity of $S_n^{2:1}$ is $2n-5$, and $(n-1)! - |U_1| \ge 2n-5 = \kappa(S_n^{2:1})$ for $n \ge 5$, by Lemma 21, $|N_{S_n^{2:1}}(U_1)| \ge 2n-5$. Note that $U \ne U_1$ and $|U - U_1| \le 7$, so $1 \le |U_2| \le 7$.

If $|U_2| = 1$, recall that S_n^2 is $(2n-3)$-regular and $S_n^{2:i}$ is isomorphic to S_{n-1}^2, $|N_{S_n^{2:2}}(U_2)| = \kappa(S_n^{2:2}) = 2n-5$. Hence, $|N_{S_n^2}(U)| \ge |N_{\overline{S_n^{2:1}}}(U_1)| - |U - U_1| \ge 16n-65 \ge 4n-9$ for $n \ge 5$. Now suppose that $2 \le |U_2| \le 7$. Since $7 \le 8n-30$ for $n \ge 5$, by the inductive hypothesis in $S_n^{2:1}$, $|N_{S_n^{2:1}}(U_1)| \ge 4(n-1)-9 = 4n-13$. Thus, $|N_{S_n^2}(U)| \ge |N_{S_n^{2:1}}(U_1)| + |N_{S_n^{2:2}}(U_2)| \ge (2n-5)+(4n-13) = 6n-18 \ge 4n-9$ for $n \ge 5$.

By the above cases, the lemma holds. □

Corollary 6. *Let S_n^2 be the n-dimensional split-star network for $n \ge 4$. Then $t_p(S_n^2) = 4n-9 = \kappa_1(S_n^2)$.*

Proof. To prove the theorem, we only need to verify that S_n^2 satisfies conditions in Theorem 1. Note that $k = 2n-3 \ge 5$ for $n \ge 4$, $l = 1$, $N = |V(S_n^2)| = n! \ge 4(2n-3)-2$ for $n \ge 4$, Condition (1) in Theorem 1 holds. By Lemmas 10 and 25, Conditions (2)–(3) in Theorem 1 holds. Condition (4) holds by Lemma 11. S_n^2 satisfies all conditions in Theorem 1, and thus $t_p(S_n^2) = 4n-9 = \kappa_1(S_n^2)$. □

4.6. Application to the Cayley graphs generated by transposition trees Γ_n

Let Γ_n be Cayley graphs generated by transposition trees. Yang et al. [51] determined $\kappa_1(\Gamma_n) = 2n-4$ for $n \ge 3$. However, $t_p(\Gamma_n)$ has not been known so far. By Theorem 1, we immediately the following result which contains the above result. Note that for Γ_n, $k = n-1$ and $l = 0$ in Theorem 1.

Lemma 26. *Let Γ_n be Cayley graphs generated by transposition trees for $n \ge 4$. If U is a subset of $V(\Gamma_n)$ and $2 \le |U| \le 4n-12$, then $|N_{\Gamma_n}(U)| \ge 2n-4$.*

Proof. The lemma is proved by induction on n. When $n = 4$, it is easy to check $|N_{\Gamma_n}(U)| \ge 4$ for $2 \le |U| \le 4n-12 = 4$. We assume that the lemma is true for Γ_m, where m is an integer with $4 \le m \le n-1$. We consider Γ_n for $n \ge 5$ as follows.

Recall that Γ_n can be decomposed into n copies of Γ'_{n-1}s, namely $\Gamma_{n-1}^1, \Gamma_{n-1}^2, \ldots, \Gamma_{n-1}^n$. Let $U_i = U \cap V(\Gamma_{n-1}^i)$ and $\overline{\Gamma_{n-1}^i} = \Gamma_n - \Gamma_{n-1}^i$ for $i \in [n]$. Without loss of generality, we may assume that $|U_1| \ge |U_2| \ge |U_3| \ge \ldots \ge |U_n|$.

Case 1. $|U_1| \le 1$.

In this case, $|U_i| \le 1$ for all $1 \le i \le n$. Since $|U| \ge 2$, it implies $|U_1| = |U_2| = 1$. Since Γ_n is $(n-1)$-regular and Γ_{n-1}^i is isomorphic to Γ_{n-1}, $|N_{\Gamma_n}(U)| \ge 2\kappa(\Gamma_{n-1}^i) = 2(n-2) = 2n-4$ for $n \ge 5$.

Case 2. $2 \le |U_1| \le 4n-16$.

By the inductive hypothesis in Γ_{n-1}^1, $|N_{\Gamma_{n-1}^1}(U_1)| \ge 2(n-1)-4 = 2n-6$. Note that $|U_i| \le |U_1| \le 4n-16$ for $i \in \{2, 3, \ldots, n\}$. If $|U_2| = 1$, $|N_{\Gamma_{n-1}^2}(U_2)| \ge \kappa(\Gamma_{n-1}^2) = n-2$, so $|N_{\Gamma_n}(U)| \ge |N_{\Gamma_{n-1}^1}(U_1)| + |N_{\Gamma_{n-1}^2}(U_2)| \ge (2n-6)+(n-2) = 3n-8 \ge 2n-4$ for $n \ge 5$. If $2 \le |U_2| \le 4n-16$, by the inductive hypothesis in Γ_{n-1}^2, $|N_{\Gamma_{n-1}^2}(U_2)| \ge 2(n-1)-4 = 2n-6$. Thus, $|N_{\Gamma_n}(U)| \ge |N_{\Gamma_{n-1}^1}(U_1)| + |N_{\Gamma_{n-1}^2}(U_2)| \ge 2(2n-6) = 4n-12 \ge 2n-4$ for $n \ge 5$. Now consider $|U_2| = 0$, then $|U_i| = 0$ for $i \in \{3, 4, \ldots, n\}$, it implies that $U = U_1$. So $|N_{\Gamma_n}(U)| \ge |N_{\Gamma_{n-1}^1}(U_1)| + |N_{\overline{\Gamma_{n-1}^1}}(U_1)| \ge 2n-6+|U_1| \ge 2n-6+2 = 2n-4$ for $n \ge 5$.

Case 3. $4n-15 \le |U_1| \le 4n-12$.

If $U = U_1$, by Lemma 12, $|N_{\overline{\Gamma_{n-1}^1}}(U_1)| = |U_1| \ge 4n-15$. Since $(n-1)! - (4n-12) \ge n-2$ for $n \ge 5$, by Lemma 21, $|N_{\Gamma_{n-1}^1}(U_1)| \ge \kappa(\Gamma_{n-1}^1) = n-2$. Thus, $|N_{\Gamma_n}(U)| = |N_{\overline{\Gamma_{n-1}^1}}(U_1)| + |N_{\Gamma_{n-1}^1}(U_1)| \ge 4n-15+(n-2) = 5n-17 \ge 2n-4$ for $n \ge 5$. In the following, we assume that $U \ne U_1$. It implies that $|U - U_1| \le 3$, so $1 \le |U_2| \le |U| - |U_1| \le 3$.

If $|U_2| = 1$, $|N_{\Gamma_{n-1}^2}(U_2)| = \kappa(\Gamma_{n-1}^2) = n-2$. Recall that $|N_{\Gamma_{n-1}^1}(U_1)| \ge n-2$. Hence, $|N_{\Gamma_n}(U)| \ge |N_{\Gamma_{n-1}^0}(U_0)| + |N_{\Gamma_{n-1}^1}(U_1)| \ge 2n-4$ for $n \ge 5$. Now suppose that $2 \le |U_2| \le 3$. Since $|U_2| \le 3 \le 4n-16$ for $n \ge 5$, by the inductive hypothesis in Γ_{n-1}^2, $|N_{\Gamma_{n-1}^2}(U_2)| \ge 2(n-1)-4 = 2n-6$. Thus, $|N_{\Gamma_n}(U)| \ge |N_{\Gamma_{n-1}^1}(U_1)| + |N_{\Gamma_{n-1}^2}(U_2)| \ge (n-2)+(2n-6) = 3n-8 \ge 2n-4$ for $n \ge 5$.

By the above cases, the proof is completed. □

Corollary 7. *Let Γ_n be Cayley graphs generated by transposition trees for $n \geq 6$. Then $t_p(\Gamma_n) = 2n-4 = \kappa_1(\Gamma_n)$ for $n \geq 6$.*

Proof. Note that $k = n-1 \geq 5$ and $N = |V(\Gamma_n)| = n! \geq 4(n-1)-2$ for $n \geq 6$, Condition (1) in Theorem 1 holds. By Lemmas 12 and 26, Conditions (2)–(3) in Theorem 1 holds. Condition (4) holds by Lemma 13. Thus, Γ_n satisfies all conditions in Theorem 1, $t_p(\Gamma_n) = 2n-4 = \kappa_1(\Gamma_n)$ for $n \geq 6$. □

Since the star graph and the bubble-sort graph are Cayley graph generated by transposition trees, The following corollary is gotten directly from Corollary 7.

Corollary 8. *Let S_n and B_n are the star graph and the bubble sort graph, then $t_p(S_n) = 2n-4 = \kappa_1(S_n)$ for $n \geq 6$, and $t_p(B_n) = 2n-4 = \kappa_1(B_n)$ for $n \geq 6$.*

4.7. Application to the Cayley graphs generated by 2-trees $\Gamma_n(\Delta)$

Lemma 27. *Let $\Gamma_n(\Delta)$ be a Cayley graph generated by the 2-tree Δ. For $n \geq 4$, let U be a subset of $V(\Gamma_n(\Delta))$ and $2 \leq |U| \leq 8n-26$. Then, $|N_{\Gamma_n(\Delta)}(U)| \geq 4n-11$.*

Proof. The lemma is proved by the induction on n. Since $\Gamma_4(\Delta)$ is constructed by 4 disjoint triangles, it is easy to verify that $|N_{\Gamma_4(\Delta)}(U)| \geq 5$ for $2 \leq |U| \leq 7$. By the inductive hypothesis, we assume that the lemma is true for $\Gamma_m(\Delta)$, where m is an integer with $5 \leq m \leq n-1$.

Note that $\Gamma_n(\Delta)$ is constructed by n disjoint $\Gamma_{n-1}(\Delta)$, denoted by $\Gamma_n^i(\Delta)$ for $i \in [n]$. Let $U_i = U \cap V(\Gamma_{n-1}^i(\Delta))$ and $\overline{\Gamma_{n-1}^i(\Delta)} = \Gamma_n(\Delta) - \Gamma_{n-1}^i(\Delta)$ for $i \in [n]$. Without loss of generality, we may assume that $|U_1| \geq |U_2| \geq \ldots \geq |U_n|$. The following three cases should be considered.

Case 1. $|U_1| \leq 1$.

In this case, $|U_i| \leq 1$ for all $i \in [n]$. Clearly, $2 \leq |U| \leq n$ because of $i \leq n$. The Lemma follows if $|U| = 2$ by Lemma 19. Now assume that $3 \leq |U| \leq n$. Since $\Gamma_n(\Delta)$ is $(2n-4)$-regular and $\Gamma_{n-1}^i(\Delta)$ is isomorphic to $\Gamma_{n-1}(\Delta)$, $|N_{\Gamma_n(\Delta)}(U)| \geq 3\kappa(\Gamma_{n-1}^i(\Delta)) = 3(2n-6) \geq 4n-11$ for $n \geq 5$.

Case 2. $2 \leq |U_1| \leq 8n-34$.

By the inductive hypothesis in $\Gamma_{n-1}^1(\Delta)$, $|N_{\Gamma_{n-1}^1(\Delta)}(U_1)| \geq 4(n-1)-11 = 4n-15$. If $U = U_1$, $|N_{\Gamma_n(\Delta)}(U)| = |N_{\Gamma_{n-1}^1(\Delta)}(U_1)| + |N_{\overline{\Gamma_{n-1}^1(\Delta)}}(U_1)| \geq 4n-15+2|U_1| \geq 4n-11$. Assume $U \neq U_1$ in the following. If $|U_2| = 1$, $|N_{\Gamma_{n-1}^2(\Delta)}(U_2)| = \kappa(\Gamma_{n-1}^2(\Delta)) = 2n-6$. Note that $\Gamma_{n-1}^1(\Delta)$ and $\Gamma_{n-1}^2(\Delta)$ are vertex disjoint, $|N_{\Gamma_n(\Delta)}(U)| \geq |N_{\Gamma_{n-1}^1(\Delta)}(U_1)| + |N_{\Gamma_{n-1}^2(\Delta)}(U_2)| \geq 4n-15+(2n-6) \geq 6n-21$ for $n \geq 5$. Now consider $2 \leq |U_2| \leq |U_1| \leq 8n-34$, by the inductive hypothesis in $\Gamma_{n-1}^2(\Delta)$, $|N_{\Gamma_{n-1}^2(\Delta)}(U_2)| \geq 4(n-1)-11 = 4n-15$. Thus, $|N_{\Gamma_n(\Delta)}(U)| \geq |N_{\Gamma_{n-1}^1(\Delta)}(U_1)| + |N_{\Gamma_{n-1}^2(\Delta)}(U_2)| \geq 8n-30 \geq 4n-11$ for $n \geq 5$.

Case 3. $8n-33 \leq |U_1| \leq 8n-26$.

By Lemma 14, $|N_{\overline{\Gamma_{n-1}^1(\Delta)}}(U_1)| = 2|U_1|$. It is clear that the lemma holds if $U = U_1$. In the following, we assume the case of $U \neq U_1$. Since the connectivity of $\Gamma_{n-1}^1(\Delta)$ is $2n-6$, and by Lemma 21, $|N_{\Gamma_{n-1}^1(\Delta)}(U_1)| \geq 2n-6$. Note that $U \neq U_1$ and $|U - U_1| \leq 7$, so $1 \leq |U_2| \leq 7$.

If $|U_2| = 1$, $|N_{\Gamma_n(\Delta)}(U)| \geq |N_{\Gamma_{n-1}^1(\Delta)}(U_1)| + |N_{\overline{\Gamma_{n-1}^1(\Delta)}}(U_1)| - |U - U_1| \geq (2n-6) + 2|U_1| - 7 \geq 18n-79 \geq 4n-11$ for $n \geq 5$. Now suppose that $2 \leq |U_2| \leq 7$. Since $7 \leq 8n-32$ for $n \geq 5$, by the inductive hypothesis in $\Gamma_{n-1}^2(\Delta)$, $|N_{\Gamma_{n-1}^2(\Delta)}(U_2)| \geq 4(n-1)-11 = 4n-15$. Thus, $|N_{\Gamma_n(\Delta)}(U)| \geq |N_{\Gamma_{n-1}^1(\Delta)}(U_1)| + |N_{\Gamma_{n-1}^2(\Delta)}(U_2)| \geq (2n-6)+(4n-15) = 6n-21 \geq 2n-5$ for $n \geq 5$.

By the above cases, the lemma holds. □

Corollary 9. *Let $G = \Gamma_n(\Delta)$ be a Cayley graph generated by the 2-tree Δ for $n \geq 5$. Then $\kappa_1(G) = 4n-11 = t_p(G)$.*

Proof. Note that $k = 2n-4 \geq 5$ and $\frac{n!}{2} \geq 4(2n-4)-2$ for $n \geq 5$, Condition (1) in Theorem 1 holds. By Lemmas 14 and 27, Conditions (2) and (3) in Theorem 1 holds. Condition (4) holds by $|F| \leq 2k-3-l = 2(2n-4)-3-1 = 4n-12 < 4n-11$ and Lemma 16. Thus, $\Gamma_n(\Delta)$ satisfies all conditions in Theorem 1, and so $t_p(\Gamma_n(\Delta)) = 4n-11 = \kappa_1(\Gamma_n(\Delta))$ for $n \geq 5$. □

4.8. Application to the burnt pancake networks BP_n

Lemma 28. *Let BP_n be the n-dimensional burnt pancake network. For $n \geq 3$, let U be a subset of $V(BP_n)$ and $2 \leq |U| \leq 4n-8$, then $|N_{BP_n}(U)| \geq 2n-2$.*

Proof. If $|U| = 2$, by Lemma 17 and Lemma 19, for any two distinct vertices u and v, so $|N_{BP_n}(U)| \geq 2n-2$.

Recall that BP_n can be decomposed into $2n$ copies of BP_{n-1}'s, namely BP_{n-1}^i, for $i \in \langle n \rangle$. Let $U_i = U \cap V(BP_{n-1}^i)$ and $\overline{BP_{n-1}^i} = BP_n - BP_{n-1}^i$ for $i \in \langle n \rangle$. Without loss of generality, we may assume that $|U_1| \geq |U_2| \geq |U_3| \geq \ldots \geq |U_n| \geq |U_{\bar{n}}| \geq |U_{\overline{n-1}}| \geq |U_{\bar{1}}|$.

The lemma is proved by using the induction on n. If $n = 3$, it is easy to check $|N_{BP_n}(U)| \geq 4$ for $2 \leq |U| \leq 4n - 8 = 4$. We assume that the lemma is true for BP_m, where m is an integer with $4 \leq m \leq n - 1$. We consider BP_n for $n \geq 4$ as follows.

Case 1. $|U_1| \leq 1$.

In this case, $|U_i| \leq 1$ for all $1 \leq i \leq n$. Since $|U| \geq 2$, it implies that $|U_1| = |U_2| = 1$. Since BP_n is n-regular and BP_{n-1}^i is isomorphic to BP_{n-1}, $|N_{BP_n}(U)| \geq 2\kappa(BP_{n-1}^i) = 2(n-1) = 2n - 2$ for $n \geq 4$.

Case 2. $2 \leq |U_1| \leq 4n - 12$.

By the inductive hypothesis in BP_{n-1}^1, $|N_{BP_{n-1}^1}(U_1)| \geq 2(n-1) - 2 = 2n - 4$. Note that $|U_i| \leq |U_1| \leq 4n - 12$ for $i \in [n] \setminus \{1\}$. If $U = U_1$, $|N_{BP_n}(U)| = |N_{BP_{n-1}^1}(U_1)| + |N_{\overline{BP_{n-1}^1}}(U_1)| \geq 4n - 12 + |U_1| \geq 4n - 11$. Assume $U \neq U_1$ in the following. If $|U_2| = 1$, $|N_{BP_{n-1}^2}(U_2)| \geq \kappa(BP_{n-1}^2) = n - 1$, so $|N_{BP_n}(U)| \geq |N_{BP_{n-1}^1}(U_1)| + |N_{BP_{n-1}^2}(U_2)| \geq (2n-4) + (n-1) = 3n - 5 \geq 2n - 2$ for $n \geq 4$. If $2 \leq |U_2| \leq 4n - 12$, by the inductive hypothesis in BP_{n-1}^2, $|N_{BP_{n-1}^2}(U_2)| \geq 2(n-1) - 2 = 2n - 4$. Thus, $|N_{BP_n}(U)| \geq |N_{BP_{n-1}^1}(U_1)| + |N_{BP_{n-1}^2}(U_2)| \geq 2(2n-4) = 4n - 8 \geq 2n - 2$ for $n \geq 4$.

Case 3. $4n - 11 \leq |U_1| \leq 4n - 8$.

Since $(n-1)! - (4n-8) \geq n - 1$ for $n \geq 5$, by Lemma 21, $|N_{BP_{n-1}^1}(U_1)| \geq \kappa(BP_{n-1}^1) = n - 1$. If $U = U_1$, by Lemma 17, $|N_{\overline{BP_{n-1}^1}}(U_1)| = |U_1| \geq 4n - 11$. Thus, $|N_{BP_n}(U)| = |N_{\overline{BP_{n-1}^1}}(U_1)| + |N_{BP_{n-1}^1}(U_1)| \geq 4n - 11 + (n-1) = 5n - 2 \geq 2n - 2$ for $n \geq 4$. In the following, we assume that $U \neq U_1$. It implies that $|U - U_1| \leq 3$, so $1 \leq |U_2| \leq |U| - |U_1| \leq 3$.

If $|U_2| = 1$, $|N_{BP_{n-1}^2}(U_2)| = \kappa(BP_{n-1}^2) = n - 1$. Recall that $|N_{BP_{n-1}^1}(U_1)| \geq n - 1$. Hence, $|N_{BP_n}(U)| \geq |N_{BP_{n-1}^1}(U_1)| + |N_{BP_{n-1}^2}(U_2)| \geq 2n - 2$ for $n \geq 4$. Now suppose that $2 \leq |U_2| \leq 3$. Since $|U_2| \leq 3 \leq 4n - 12$ for $n \geq 4$, by the inductive hypothesis in BP_{n-1}^2, $|N_{BP_{n-1}^2}(U_2)| \geq 2(n-1) - 2 = 2n - 4$. Thus, $|N_{BP_n}(U)| \geq |N_{BP_{n-1}^1}(U_1)| + |N_{BP_{n-1}^2}(U_2)| \geq (n-1) + (2n-4) = 3n - 5 \geq 2n - 2$ for $n \geq 4$.

By the above cases, the proof is completed. □

Remark 3. The extra connectivity of BP_n was obtained by Song et al. [41], $\kappa_1(BP_n) = 2n - 2$ for $n \geq 4$. But $t_p(BP_n)$ is not known so far. By Theorem 1, we immediately the following result which contains the above result.

Corollary 10. *Let BP_n be the n-dimensional burnt pancake network for $n \geq 5$. Then $t_p(BP_n) = 2n - 2 = \kappa_1(BP_n)$.*

Proof. Note that $k = n \geq 5$ and $N = |V(BP_n)| = n! \geq 4n - 2$ for $n \geq 5$, Condition (1) in Theorem 1 holds. By Lemmas 17 and 28, Conditions (2) and (3) in Theorem 1 hold. Condition (4) holds by Lemma 18. BP_n satisfies all conditions in Theorem 1, and so $t_p(BP_n) = 2n - 2 = \kappa_1(BP_n)$ for $n \geq 5$. □

5. Concluding remarks

This paper establishes the close relationship between these two parameter: the extra connectivity and pessimistic diagnosability under the PMC model, by proving $t_p(G) = \kappa_1(G)$ for some regular graphs G with some conditions. As applications, the pessimistic diagnosability for each of split-star networks S_n^2, Cayley graphs generated by transposition trees Γ_n, Cayley graph generated by the 2-tree $\Gamma_n(\Delta)$ and the burnt pancake networks BP_n is gotten. As corollaries, the known results about the extra connectivity and the pessimistic diagnosability of many famous networks including the alternating group graphs [38,44], the alternating group networks [53], BC networks [54,20] and the k-ary n-cube networks [16,14,33,48] are obtained directly. The relationship between $t_p(G)$ and $\kappa_h(G)$ for some h without the condition $cn(G) \leq 2$ needs to be studied in the future.

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