



# The vulnerability of the diameter of the enhanced hypercubes

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## ABSTRACT

Concern about fault tolerance in the design of interconnection networks has raised interest in the study of graphs such that deleting some vertices increases the diameter only moderately. For an interconnection network  $G$ , the  $(\omega - 1)$ -fault diameter  $D_\omega(G)$  is the maximum diameter of a subgraph obtained by deleting fewer than  $\omega$  vertices of  $G$ , and the  $\omega$ -wide diameter  $d_\omega(G)$  is the least  $\ell$  such that any two vertices are joined by  $\omega$  internally-disjoint paths of length at most  $\ell$ . The enhanced hypercube  $Q_{n,k}$  is a variant of the well-known  $n$ -dimensional hypercube  $Q_n$  in which an edge is added from each vertex  $x_n, \dots, x_1$  to the vertex obtained by complementing  $x_k, \dots, x_1$ . Yang, Chang, Pai, and Chan gave an upper bound for  $d_{n+1}(Q_{n,k})$  and  $D_{n+1}(Q_{n,k})$  and posed the problem of finding the wide diameter and fault diameter of  $Q_{n,k}$ . By constructing internally disjoint paths between any two vertices in the enhanced hypercube, for  $n \geq 3$  and  $2 \leq k \leq n$  we prove that  $D_\omega(Q_{n,k}) = d_\omega(Q_{n,k}) = d(Q_{n,k})$  for  $1 \leq \omega < n - \lfloor \frac{k}{2} \rfloor$ ;  $D_\omega(Q_{n,k}) = d_\omega(Q_{n,k}) = d(Q_{n,k}) + 1$  for  $n - \lfloor \frac{k}{2} \rfloor \leq \omega \leq n + 1$ , where  $d(Q_{n,k})$  is the diameter of  $Q_{n,k}$ . These results mean that interconnection networks modeled by enhanced hypercubes are extremely robust.

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## 1. Introduction

An interconnection network is conveniently represented by an undirected graph. The vertices (or edges) of the graph represent the nodes (or links) of the network. Throughout this paper, vertex and node, edge and link, graph and network are used interchangeably. Reliability and efficiency are important criteria in the design of interconnection networks. In the study of fault-tolerance and transmission delay of networks, wide diameter and fault diameter are important parameters that have been studied by many researchers. They combine connectivity with diameter to measure simultaneously the fault-tolerance and efficiency of parallel processing computer networks. These parameters were studied by several authors for some Cartesian product graphs [7,27,28] and for the hypercube and its variants [3,5,10,12,19–21].

Let  $u$  and  $v$  be two vertices in a network  $G$ . A  $u, v$ -path is a path with endpoints  $u$  and  $v$ . The distance between  $u$  and  $v$ , denoted by  $d(u, v)$ , is the minimum length (number of edges) of a  $u, v$ -path. The diameter of  $G$ , denoted by  $d(G)$ , is the maximum distance between vertices. The connectivity  $\kappa(G)$  is the minimum number of vertices whose removal results in a disconnected or 1-vertex network. We say that  $G$  is  $\omega$ -connected when  $1 \leq \omega \leq \kappa(G)$ .

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Failures are inevitable when a network is put in use. Therefore, it is important to consider faulty networks. By the definition of the connectivity, if  $G$  is a  $\omega$ -connected graph then removal of fewer than  $\omega$  vertices from  $G$  still keeps the resulting graph connected. The  $(\omega - 1)$ -fault diameter of a graph  $G$ , denoted by  $D_\omega(G)$ , is the maximum diameter among subgraphs obtained from  $G$  by deleting fewer than  $\omega$  vertices; it measures the worst-case effect on the diameter when vertex faults occur. Note that  $D_\omega(G)$  is well-defined if and only if  $G$  is  $\omega$ -connected. Moreover,

$$d(G) = D_1(G) \leq D_2(G) \leq \dots \leq D_{k-1}(G) \leq D_k(G).$$

The problem we are interested in is to determine the precise value of  $D_\omega(G)$  or derive the relations between  $D_\omega(G)$  and  $d(G)$  for a given  $\omega$  connected graph  $G$ . The answer to this problem is quite important for some applications in a real-time system. Generally speaking, this is a quite difficult problem. However, the precise values of  $D_\omega(G)$  have been determined for some well-known networks  $G$  (see, for example, Section 13.6 in the text-book [26]).

By Menger's Theorem [17], in a  $\omega$ -connected network there exist  $\omega$  internally disjoint paths joining any two vertices (internally disjoint means that the only shared vertices are the endpoints). Given a  $\omega$ -connected graph  $G$ , fix  $\omega$  with  $1 \leq \omega \leq k$ . The  $\omega$ -wide diameter of  $G$ , denoted by  $d_\omega(G)$ , is the least  $\ell$  such that for any  $u, v \in V(G)$  there exist  $\omega$  internally disjoint  $u, v$ -paths of length at most  $\ell$ . Throughout this paper, we abuse terminology by writing “disjoint paths” to mean “internally disjoint paths”. Note that  $d_1(G)$  is just the diameter  $d(G)$  of  $G$ . From the definition,

$$d(G) = d_1(G) \leq d_2(G) \leq \dots \leq d_{k-1}(G) \leq d_k(G).$$

An ideal interconnection network has connectivity as large as possible and diameter as small as possible. The wide diameter  $d_\omega(G)$  combines connectivity  $\kappa(G)$  with diameter  $d(G)$ , where  $1 \leq \omega \leq \kappa(G)$ . Hence  $d_\omega(G)$  is a more suitable parameter than  $\kappa(G)$  and  $d(G)$  to measure fault-tolerance and efficiency of parallel processing computer networks. Determining the precise value of  $d_\omega(G)$  is significant for a given graph and an integer  $\omega$  with  $1 \leq \omega \leq \kappa(G)$ . However, Hsu [9] proved that this problem is NP-complete.

From the definitions, it follows that  $D_\omega(G) \leq d_\omega(G)$  when  $G$  is  $\omega$ -connected. Equality holds for some well-known networks [6,14].

As a topology for an interconnection network of a multiprocessor system, the hypercube is a widely used and well-known model, since it possesses many attractive properties such as regularity, symmetry, logarithmic diameter, high connectivity, recursive construction, ease of bisection, and relatively low link complexity [13,18,25]. We study an important variant of the hypercube  $Q_n$ , the enhanced hypercube  $Q_{n,k}$  proposed by Tzeng and Wei [22]; it has a smaller diameter, improved mean node distance and cost efficiency when compared to the hypercube; its properties have been studied in [2,15,23,24,29]. For  $2 \leq k \leq n$ , the  $n$ -dimensional enhanced hypercube  $Q_{n,k}$  is obtained from the hypercube  $Q_n$  in which an edge is added from each vertex to the vertex obtained by complementing its last  $k$  bits. As a special case of the enhanced hypercube, the folded hypercube  $FQ_n$  was studied in [11,30,31]. We give the basic properties of  $Q_{n,k}$  in Section 2.

It was shown by Liu [15] that  $\kappa(Q_{n,k}) = n + 1$ . Thus, the wide diameter  $d_\omega(Q_{n,k})$  and the fault diameter  $D_\omega(Q_{n,k})$  are well-defined when  $\omega \leq n + 1$ . Yang, Chang, Pai, and Chan [29] gave an upper bound for  $d_{n+1}(Q_{n,k})$  and  $D_{n+1}(Q_{n,k})$ , and they posed the problem of finding the wide diameter and fault diameter of  $Q_{n,k}$ . In this paper, for  $n \geq 3$  and  $2 \leq k \leq n$ , we prove

$$D_\omega(Q_{n,k}) = d_\omega(Q_{n,k}) = \begin{cases} d(Q_{n,k}) & \text{for } 1 \leq \omega < n - \lfloor \frac{k}{2} \rfloor; \\ d(Q_{n,k}) + 1 & \text{for } n - \lfloor \frac{k}{2} \rfloor \leq \omega \leq n + 1. \end{cases}$$

The special case  $k = n$  (folded hypercube) was obtain earlier by Simó and Yebra [20], along with the same values for edge deletions. For enhanced hypercubes also, our arguments yield the same values for edge deletions as for vertex deletions.

To prove this result, in Section 3 we explicitly construct disjoint paths of bounded length joining any two vertices in  $Q_{n,k}$ . The proofs of our main results are in Section 4.

## 2. Properties of $Q_{n,k}$

Let  $x_n \dots x_1$  be an  $n$ -bit binary string. We call the rightmost bit the first bit and the leftmost bit the  $n$ th bit. For simplicity we use  $a^i$  to mean that the bit  $a$  is repeated  $i$  times; for example,  $01^30^2 = 011100$ . The Hamming distance between strings  $u$  and  $v$ , denoted by  $H(u, v)$ , is the number of positions where the two strings differ.

The  $n$ -dimensional hypercube  $Q_n$  is the graph whose vertices are the  $n$ -bit binary strings and whose edges are the pairs of vertices differing in exactly one position. An edge of  $Q_n$  is a  $j$ -dimensional edge if the two endpoints differ in the  $j$ th position. For  $1 \leq j \leq n$ , let  $E_j$  denote the set of  $j$ -dimensional edges in  $Q_n$ .

As a variant of the hypercube, the  $n$ -dimensional folded hypercube  $FQ_n$ , proposed first by El-Amawy and Latifi [1], is obtained from the hypercube  $Q_n$  by making each vertex  $u$  adjacent to its complementary vertex, denoted  $\bar{u}$  and obtained from  $u$  by subtracting each bit from 1. Such an edge is often called a complementary edge.

As we have defined, the enhanced hypercube  $Q_{n,k}$  for  $2 \leq k \leq n$  is obtained from the hypercube  $Q_n$  by adding the edge  $uv$  whenever  $u$  and  $v$  are related by  $u = x_n \dots x_1$  and  $v = x_n \dots x_{k+1} \bar{x}_k \bar{x}_{k-1} \dots \bar{x}_1$ , that is, the  $k$  bits  $x_k, \dots, x_1$  are complemented. Such an edge is called a  $k$ -complementary edge. For convenience, we use  $E_0$  to denote the set of  $k$ -complementary edges. Thus  $E(Q_{n,k}) = E(Q_n) \cup E_0$ . When  $k = n$ , we have  $Q_{n,n} = FQ_n$ ; hence the enhanced hypercube is a generalization

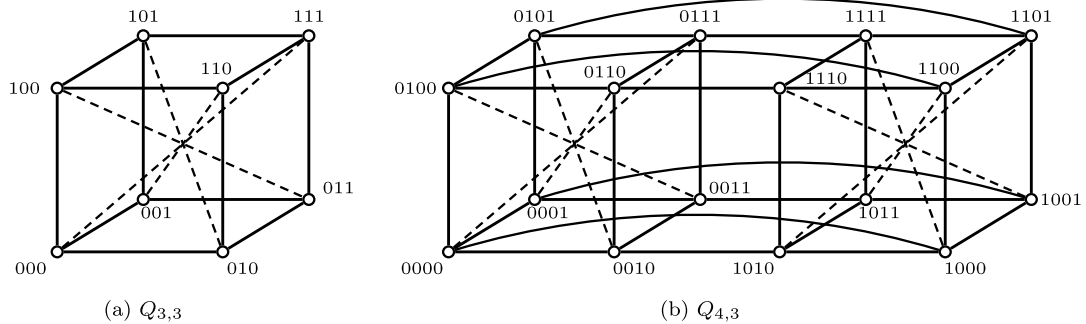


Fig. 1. Enhanced hypercubes  $Q_{3,3}$  and  $Q_{4,3}$ .

of the folded hypercube. The graphs shown in Fig. 1 are  $Q_{3,3}$  and  $Q_{4,3}$ , where the hypercube edges and 3-complementary edges are represented by solid lines and dashed lines, respectively.

A graph  $G$  is *vertex-transitive* if for any  $u, v \in V(G)$  there is some  $\sigma \in \text{Aut}(G)$ , the automorphism group of  $G$ , such that  $\sigma(u) = v$ ; it is *edge-transitive* if for any  $xy, uv \in E(G)$  there is some  $\sigma \in \text{Aut}(G)$  such that  $\{\sigma(x), \sigma(y)\} = \{u, v\}$ . The hypercube  $Q_n$  and folded hypercube  $FQ_n$  are vertex-transitive and edge-transitive, and the enhanced hypercube  $Q_{n,k}$  is vertex-transitive but not edge-transitive when  $k < n$  [16,25,29].

The Cartesian product  $G \square H$  of graphs  $G$  and  $H$  is the graph with vertex set  $V(G) \times V(H)$ , in which vertices  $(u, v)$  and  $(u', v')$  are adjacent whenever  $uu' \in E(G)$  and  $v = v'$ , or  $u = u'$  and  $vv' \in E(H)$ . By the definition of  $Q_{n,k}$ , we have  $Q_{n,k} = Q_{n-k} \square FQ_k$ . Although  $Q_{n,k}$  is not edge-transitive when  $2 \leq k < n$ , we have the following properties.

**Proposition 1.** *Permuting the first  $k$  positions and/or permuting the last  $n - k$  positions in the names of the vertices of  $Q_{n,k}$  does not change the graph.*

**Proof.** Exchanging the  $i$ th bit and  $j$ th bit among the first  $k$  or among the last  $n - k$  preserves the adjacency relation, since the number of coordinates in which two vertices differ is not changed by exchanging such coordinates. An arbitrary permutation is obtained by a succession of such exchanges.  $\square$

**Proposition 2.** *Given  $u, v \in V(Q_{n,k})$ , let  $r$  and  $s$  be the numbers of positions in which  $u$  and  $v$  differ among the first  $k$  and last  $n - k$  positions, respectively. The distance between  $u$  and  $v$  is computed by  $d(u, v) = s + \min\{r, k - r + 1\}$ .*

**Proof.** The distance is the minimum number of steps to change  $u$  into  $v$ . All steps change one bit, except that a  $k$ -complementary edge changes the first  $k$  bits. If no  $k$ -complementary edge is used, then the number of steps is at least the Hamming distance, and this suffices. For this reason, a shortest path uses at most one  $k$ -complementary edge. If a  $k$ -complementary edge is used, then the  $k - r$  positions in which  $u$  and  $v$  agree among the first  $k$  must be changed individually.  $\square$

Proposition 2 immediately yields  $d(Q_{n,k}) = (n - k) + \lceil \frac{k}{2} \rceil$ , which equals  $n - \lfloor \frac{k}{2} \rfloor$ . This was observed by Tzeng and Wei [22], along with an algorithm for finding shortest paths joining vertices. Note that if  $u$  and  $v$  differ in more than  $\lceil \frac{k}{2} \rceil$  positions among the first  $k$ , then every shortest  $u, v$ -path contains exactly one  $k$ -complementary edge.

### 3. Construction of paths

Many properties of interconnection networks were investigated by different construction methods of paths [4,8]. In this section we will prove our main results by constructing disjoint paths of bounded length joining any two vertices in  $Q_{n,k}$ .

Let  $P$  be a path  $u_0 \rightarrow u_1 \rightarrow \dots \rightarrow u_{\ell-1} \rightarrow u_\ell$  from the vertex  $u_0$  to a vertex  $u_\ell$  in  $Q_{n,k}$ . The path  $P$  traverses the edges  $u_0u_1, u_1u_2, \dots, u_{\ell-1}u_\ell$ . Since the edge  $u_{j-1}u_j$  is in  $E_{d_j}$  for some  $d_j \in \{0, 1, \dots, n\}$ , and every vertex is incident with exactly one edge in  $E_{d_j}$ , we can represent the path  $P$  from  $u_0$  by the list  $(d_1, \dots, d_\ell)$ , where  $d_j$  indicates the type of the edge joining  $u_{j-1}$  and  $u_j$ . For example, in  $Q_{5,3}$  the path originating from 00000 determined by  $(2, 0, 5)$  is  $00000 \xrightarrow{2} 00010 \xrightarrow{0} 00101 \xrightarrow{5} 10101$ . Note that the length of a path determined by a list  $I$  is the number of elements in  $I$ .

We use the following two lemmas to construct disjoint paths. A *proper segment* of a list is a string of consecutive elements that is not the full list. A *proper initial segment* is a proper segment containing the first element of a list. For example, the  $\ell - 1$  proper initial segments of  $(d_1, d_2, \dots, d_\ell)$  are listed as follows:  $(d_1)$ ,  $(d_1, d_2)$ ,  $\dots$ ,  $(d_1, d_2, \dots, d_{\ell-1})$ .

**Lemma 1.** *No two cyclic permutations of a list of distinct elements have a common proper initial segment.*

**Table 1**  
Two cases for  $r < d(Q_{n,k})$ .

Case	Condition	$I$
A	$i \leq k - i$	$(1, \dots, i, k + 1, \dots, k + j)$
B	$i > k - i$	$(0, i + 1, \dots, k, k + 1, \dots, k + j)$

**Proof.** Let  $I_1 = (d_1, d_2, \dots, d_\ell)$  and  $I_j = (d_j, \dots, d_\ell, d_1, \dots, d_{j-1})$  for  $2 \leq j \leq \ell$ .

Consider  $j$  and  $j'$  with  $1 \leq j' < j \leq \ell$ . A proper initial segment of  $I_j$  for  $j \geq 2$  contains  $d_j$  and not  $d_{j-1}$ , but this is not true for any initial segment of  $I_{j'}$ , since when  $d_j$  is not the first element,  $d_{j-1}$  also occurs in any initial segment containing  $d_j$ . Therefore,  $I_j$  and  $I_{j'}$  have no common proper initial segment.  $\square$

**Lemma 2.** Let  $S$  be a set of  $\ell$  distinct elements of  $\{0, 1, \dots, n\}$ . Let  $I_1, \dots, I_m$  be orderings of  $S$  such that no two have a common proper initial segment. If some element of  $\{0, 1, \dots, k\}$  is not in  $S$ , then for any vertex  $u$  in  $Q_{n,k}$ , the lists  $I_1, \dots, I_m$  determine disjoint paths to a single vertex.

**Proof.** Let  $S = \{d_1, \dots, d_\ell\}$  and  $[k] = \{1, \dots, k\}$ . If  $0 \notin S$ , then each path reaches the vertex  $v$  that differs from  $u$  in the positions of  $S$ . If  $0 \in S$ , then each path reaches the vertex  $v$  that differs from  $u$  in the positions of  $([k] - S) \cup (S - \{0\} - [k])$ .

If two subsets of  $S$  both contain 0 or both omit 0, then they produce paths to the same vertex only if they are the same set. If  $0 \in T \subseteq S$  and  $0 \notin T' \subseteq S$ , then  $T$  and  $T'$  produce paths to the same vertex only if they agree outside  $[k]$  and intersect  $[k]$  in complementary subsets. By the hypothesis that  $0, 1, \dots, k$  are not all present in  $S$ , this cannot occur.

Therefore, the paths from  $u$  determined by  $I_j$  and  $I_{j'}$  have a common internal vertex if and only if they have a common proper initial segment.  $\square$

**Lemma 3.** Let  $I = (d_1, \dots, d_\ell)$  with all  $d_i$  distinct and in  $\{0, 1, \dots, n\}$ . If some element of  $\{0, 1, \dots, k\}$  is not in  $I$ , then for any vertex  $u$  in  $Q_{n,k}$ , the  $\ell$  cyclic permutations of  $I$  determine disjoint paths to a single vertex.

**Proof.** The conclusion follows immediately from Lemmas 1 and 2.  $\square$

When  $G$  is the complete graph  $K_n$  with  $n \geq 3$ , we have  $D_\omega(G) = 1$  but  $d_\omega(G) = 2$  for  $2 \leq \omega \leq n - 1$ . Since  $Q_{2,2} = K_4$ , we consider  $Q_{n,k}$  with  $n \geq 3$  and  $2 \leq k \leq n$ .

**Theorem 1.** For any two distinct vertices  $u$  and  $v$  in  $Q_{n,k}$  with  $n \geq 3$  and  $2 \leq k \leq n$ , there exist  $n + 1$  disjoint  $u, v$ -paths of length at most  $d(Q_{n,k}) + 1$ , such that at least  $n - \lfloor \frac{k}{2} \rfloor - 1$  of the paths have length at most  $d(Q_{n,k})$ .

**Proof.** The vertex transitivity of  $Q_{n,k}$  and Proposition 1 allow us to assume  $u = 0^n$  and  $v = 0^{n-k-j}1^j0^{k-i}1^i$ , where  $0 \leq i \leq k$  and  $0 \leq j \leq n - k$ . We will construct the desired paths. Let  $r = \min\{i + j, k - i + j + 1\}$ . Recall that  $d(Q_{n,k}) = n - \lfloor \frac{k}{2} \rfloor$ . We consider two cases according to the relationship between  $r$  and  $d(Q_{n,k})$ .

*Case 1:  $r < d(Q_{n,k})$ .* We first specify a list  $I$  of length  $r$ , in two cases (Table 1). Note that  $r = i + j$  when  $i \leq k - i$  and  $r = k - i + j + 1$  when  $i > k - i$ .

Not all of  $0, 1, \dots, k$  appear in  $I$ , since having 0 requires  $i > k/2$  (Case B), and then also having 1 requires  $i = 0$ , a contradiction. Hence Lemma 3 applies, so the  $r$  cyclic permutations of  $I$  determine  $r$  disjoint  $u, v$ -paths of length  $r$ .

The remaining  $n - r + 1$  paths, with length  $r + 2$ , are specified by adding one of  $\{0, 1, \dots, n\} - I$  at both the beginning and the end of  $I$ . Let  $I'$  be the list obtained by adding  $h$ , and let  $P$  be the path from  $u$  determined by  $I'$ .

If  $h > k$  or if  $h \in [k]$  and  $0 \notin I$ , then  $u$  and  $v$  agree in position  $h$ , and  $P$  is the only path in the constructed set containing vertices that differ from them in position  $h$ . Furthermore, all internal vertices of  $P$  differ from  $u$  and  $v$  in position  $h$ .

If  $h = 0$ , then  $i \leq k - i$  (Case A). All internal vertices of  $P$  differ from  $u$  and  $v$  in positions  $k$  and  $k - 1$ , and no other path has any such vertices.

If  $h \in [k]$  and  $0 \in I$ , then  $1 \leq h \leq i$  (Case B). The first vertex of  $P$  after  $u$  differs from  $u$  only in position  $h$ . The next vertex disagrees with  $u$  on all of positions  $1, \dots, i$  except  $h$ , and this remains true of all other internal vertices of  $P$ , because  $I$  contains no element of  $\{1, \dots, i\}$ . All the other paths in the construction have no vertices satisfying either of these conditions.

Since  $r < d(Q_{n,k})$ , all the paths have length at most  $d(Q_{n,k}) + 1$ ; in fact, all have length at most  $d(Q_{n,k})$  unless  $r = d(Q_{n,k}) - 1$ . In this case there are  $r$  paths of length  $r$ , which suffices since  $r = d(Q_{n,k}) - 1 = n - \lfloor \frac{k}{2} \rfloor - 1$ .

*Case 2:  $r \geq d(Q_{n,k})$ .* Let  $s = d(Q_{n,k})$ . Since  $r = \min\{i + j, k - i + j + 1\}$ , we have  $i + j \geq s$  and  $k - i + j + 1 \geq s$ , so  $k + 2j + 1 \geq 2s$ . If  $j \leq n - k - 1$ , then  $2n - k - 1 \geq 2s = 2n - 2\lfloor \frac{k}{2} \rfloor$ , which is impossible. Hence  $j = n - k$ . With  $j = n - k$ , we have  $i \geq \lceil \frac{k}{2} \rceil$  and  $k + 1 - i \geq \lceil \frac{k}{2} \rceil$ , so  $\lceil \frac{k}{2} \rceil \leq i \leq \lfloor \frac{k}{2} \rfloor + 1$ . When  $i = \lceil \frac{k}{2} \rceil$ , we have  $r = i + j$ ; when  $i = \lfloor \frac{k}{2} \rfloor + 1$ , we have  $r = k - i + j + 1$ . Both cases apply when  $k$  is odd. In either case,  $r = n - \lfloor \frac{k}{2} \rfloor = d(Q_{n,k})$ , and we define three lists (Table 2).

**Table 2**Two cases for  $j = n - k$  and  $\lceil \frac{k}{2} \rceil \leq i \leq \lfloor \frac{k}{2} \rfloor + 1$ .

Case	$i$	$I$	$J$	$I'$
A	$\lceil \frac{k}{2} \rceil$	$(1, \dots, i, k+1, \dots, n)$	$(k+1, \dots, n)$	$(0, i+1, \dots, k)$
B	$\lfloor \frac{k}{2} \rfloor + 1$	$(0, i+1, \dots, k, k+1, \dots, n)$	$(k+1, \dots, n)$	$(1, \dots, i)$

Since  $j = n - k$ , in each case  $I$  has length  $r$ . By Lemma 3, the cyclic permutations of  $I$  yield  $r$  disjoint  $u, v$ -paths of length  $r$ . Since  $r = n - \lfloor \frac{k}{2} \rfloor = d(Q_{n,k})$ , this yields enough paths of length at most  $d(Q_{n,k})$ . Let  $T = \{i+1, \dots, k\}$  in Case A,  $T = \{1, \dots, i\}$  in Case B. Every vertex in each of these paths is constant in the positions of  $T$  (all-0 or all-1), and in fact all-0 in Case A.

Since  $r = n - \lfloor \frac{k}{2} \rfloor$ , we only need to find  $\lfloor \frac{k}{2} \rfloor + 1$  more paths of length at most  $d(Q_{n,k}) + 1$ . Note that  $I'$  has length  $\lfloor \frac{k}{2} \rfloor + 1$ . Form  $\lfloor \frac{k}{2} \rfloor + 1$  lists by inserting  $J$  after the first element of each cyclic permutation of  $I'$ . The first and last lists are  $(0, J, i+1, \dots, k)$  and  $(k, J, 0, i+1, \dots, k-1)$  in Case A,  $(1, J, 2, \dots, i)$  and  $(i, J, 1, \dots, i-1)$  in Case B. Each of these lists has length  $n - \lceil \frac{k}{2} \rceil + 1$ , which is at most  $d(Q_{n,k}) + 1$ .

Each of these lists is an ordering of a single set of elements. By an argument like that of Lemma 1, they have no common proper initial segments. Since they also do not contain all of  $\{0, 1, \dots, k\}$ , by Lemma 2 these paths are disjoint.

In Case A, each internal vertex on each of these paths is not all 0 in the positions of  $T$ . In Case B, each internal vertex on each of these paths has between 1 and  $|T| - 1$  nonzero positions in  $T$ . Hence these paths are disjoint from the earlier paths.  $\square$

#### 4. Consequences

From Theorem 1 and the definition of wide diameter, we immediately obtain an upper bound on  $d_\omega(Q_{n,k})$ .

**Corollary 1.** If  $n \geq 3$  and  $2 \leq k \leq n$ , then

$$d_\omega(Q_{n,k}) \leq \begin{cases} d(Q_{n,k}) & \text{for } 1 \leq \omega < n - \lfloor \frac{k}{2} \rfloor, \\ d(Q_{n,k}) + 1 & \text{for } n - \lfloor \frac{k}{2} \rfloor \leq \omega \leq n + 1. \end{cases}$$

**Proof.** When  $\omega < n - \lfloor \frac{k}{2} \rfloor$ , Theorem 1 provides at least  $\omega$  disjoint paths with length at most  $d(Q_{n,k})$  joining any two vertices in  $Q_{n,k}$ . When  $\omega \leq n + 1$ , it provides at least  $\omega$  such paths with length at most  $d(Q_{n,k}) + 1$ .  $\square$

We next give a lower bound on the fault diameter  $D_\omega(Q_{n,k})$ .

**Lemma 4.** Fix  $n \geq 3$ . If  $2 \leq k \leq n$  and  $n - \lfloor \frac{k}{2} \rfloor \leq \omega \leq n + 1$ , then

$$D_\omega(Q_{n,k}) \geq d(Q_{n,k}) + 1.$$

**Proof.** Since  $D_\omega(G)$  is nondecreasing in  $\omega$ , proving  $D_{n-\lfloor \frac{k}{2} \rfloor}(Q_{n,k}) \geq d(Q_{n,k}) + 1$  is sufficient. Let  $u = 0^n$  and  $v = 1^{n-k}0^{k-i}1^i$ , where  $i = \lceil \frac{k}{2} \rceil - 1$ . Note that  $v$  has 1s in  $n - \lfloor \frac{k}{2} \rfloor - 1$  positions. Let  $W$  be the set of neighbors of  $u$  whose single 1 occurs in a position where  $v$  has a 1, so  $|W| = n - \lfloor \frac{k}{2} \rfloor - 1$ . On any  $u, v$ -path in  $Q_{n,k} - W$ , the neighbor of  $u$  has a single 1 in a position among  $i+1, \dots, k$  or is  $0^{n-k}1^k$ .

By Proposition 2, the distance between  $v$  and a neighbor  $u'$  of  $u$  not in  $W$  is  $n - k + \min\{i+1, k-i\}$ . Since  $i = \lceil \frac{k}{2} \rceil - 1$ , the distance is  $n - \lfloor \frac{k}{2} \rfloor$ , which equals  $d(Q_{n,k})$ . Hence every  $u, v$ -path in  $Q_{n,k} - W$  has length at least  $d(Q_{n,k}) + 1$ .  $\square$

**Theorem 2.** For  $3 \leq n$  and  $2 \leq k \leq n$ ,

$$D_\omega(Q_{n,k}) = d_\omega(Q_{n,k}) = \begin{cases} d(Q_{n,k}) & \text{for } 1 \leq \omega < n - \lfloor \frac{k}{2} \rfloor; \\ d(Q_{n,k}) + 1 & \text{for } n - \lfloor \frac{k}{2} \rfloor \leq \omega \leq n + 1. \end{cases}$$

**Proof.** Since  $d(Q_{n,k}) \leq D_\omega(Q_{n,k}) \leq d_\omega(Q_{n,k})$  for  $1 \leq \omega \leq n + 1$ , Corollary 1 yields  $D_\omega(Q_{n,k}) = d_\omega(Q_{n,k}) = d(Q_{n,k})$  for  $1 \leq \omega < n - \lfloor \frac{k}{2} \rfloor$ .

For  $n - \lfloor \frac{k}{2} \rfloor \leq \omega \leq n + 1$ , Corollary 1 and Lemma 4 yield  $D_\omega(Q_{n,k}) = d_\omega(Q_{n,k}) = d(Q_{n,k}) + 1$ .  $\square$

## 5. Conclusions

The enhanced hypercube  $Q_{n,k}$  is a generalization of the hypercube  $Q_n$  and the folded hypercube  $FQ_n$  since  $Q_{n,0} = Q_n$  and  $Q_{n,n} = FQ_n$ . In this paper, we investigate the vulnerability of the diameter  $d(Q_{n,k})$  of the enhanced hypercubes by considering the fault diameter  $D_\omega(Q_{n,k})$  and the wide diameter  $d_\omega(Q_{n,k})$ . For  $n \geq 3$  and  $2 \leq k \leq n$  we prove that  $D_\omega(Q_{n,k}) = d_\omega(Q_{n,k})$ , which is equal to  $d(Q_{n,k})$  when  $1 \leq \omega < n - \lfloor \frac{k}{2} \rfloor$ , and equal to  $d(Q_{n,k}) + 1$  when  $n - \lfloor \frac{k}{2} \rfloor \leq \omega \leq n + 1$ . These results can provide more accurate measurements for fault tolerance of the system when the graphs are used to model the topological structure of large-scale parallel processing systems. We will further investigate the fault diameter of  $Q_{n,k}$  with hybrid vertex and edge faults.

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