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On fault tolerance of (n,k)-star networks ☆

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ABSTRACT

Fault tolerance of an (n,k)-star network is measured by its h-super connectivity $\kappa_s^{(h)}$ or h-super edge-connectivity $\lambda_s^{(h)}$. Li et al. (2014) [8], (2012) [6] determined $\kappa_s^{(h)}$ and $\lambda_s^{(h)}$ for $0 \le h \le n-k$. This paper determines that $\kappa_s^{(h)} = \lambda_s^{(h)} = \frac{(h+1)!(n-h-1)}{(n-k)!}$ for $n-k \le h \le n-2$.

1. Introduction

It is well known that interconnection networks play an important role in parallel computing/communication systems. An interconnection network can be modeled by a graph in which vertices correspond to processors and edges correspond to communication links.

Let G be a connected graph. A subset $T \subset V(G)$, if any, is called an *h-vertex-cut*, if $G-T$ is disconnected and has the minimum degree at least h. The *h-super connectivity* $\kappa_s^{(h)}(G)$ of G is defined as the minimum cardinality over all h-vertex-cuts of G. Similarly, a subset $F \subset E(G)$, if any, is called an *h-edge-cut*, if $G-F$ is disconnected and has the minimum degree at least h. The *h-super edge-connectivity* $\lambda_s^{(h)}(G)$ of G is defined as the minimum cardinality over all h-edge-cuts of G.

The h-super connectivity and h-super edge-connectivity are important measure of fault tolerance of networks and have been received considerable attention in the literature (see, for example, [6–8] and references cited therein).

For the n-dimensional star graph S_n, Li and Xu [7] proved that $\kappa_s^{(h)}(S_n) = \lambda_s^{(h)}(S_n) = (h+1)!(n-h-1)$ for any h with $0 \le h \le n-2$. As a generalization of S_n, the (n,k)-star graph $S_{n,k}$, where $2 \le k \le n-1$, when $0 \le h \le n-k$, Li and Xu [6,8] determined that

$$\kappa_s^{(h)}(S_{n,k}) = n + h(k-2) - 1 \tag{1.1}$$

and

$$\lambda_s^{(h)}(S_{n,k}) = \begin{cases} (n-h-1)(h+1) & \text{for } h \leqslant \min\{k-2, \frac{n}{2}-1\}, \\ (n-k+1)(k-1) & \text{otherwise.} \end{cases} \tag{1.2}$$

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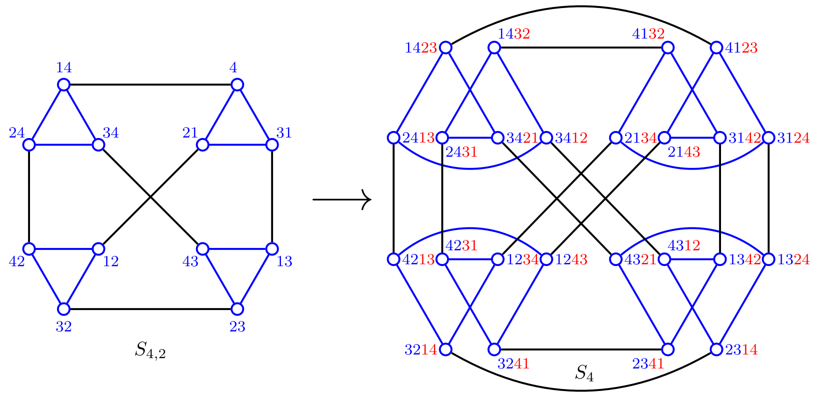

Fig. 1. A (4, 2)-star graph $S_{4,2}$ and its 2-split graph $S^2_{4,2}$, which is isomorphic to a star S_4.

When $n-k+1 \le h \le n-2$, in this paper, we prove that

$$\kappa_s^{(h)}(S_{n,k}) = \lambda_s^{(h)}(S_{n,k}) = \frac{(h+1)!(n-h-1)}{(n-k)!}$$

by using an $(n-k)!$-split graph of $S_{n,k}$.

The rest of the paper is organized as follows. In Section 2, we give definitions of a star graph S_n, an (n,k)-star graph $S_{n,k}$ and an $(n-k)!$-split graph of $S_{n,k}$, and some lemmas used in our proofs. The proof of our main result is in Section 3. Conclusions are in Section 4.

2. Definitions and lemmas

For a given integer n with $n \ge 2$, let $I_n = \{1, 2, \ldots, n\}$, $I'_n = \{2, \ldots, n\}$. For $k \in I_n$, let $P(n,k)$ be the set of k-arrangements on I_n, that is, $P(n,k) = \{p_1 p_2 \ldots p_k : \ p_i \in I_n, p_i \ne p_j, 1 \le i \ne j \le k\}$. $P(n,n)$ will be shorted as $P(n)$. Clearly, $|P(n,k)| = \frac{n!}{(n-k)!}$. Usually, if $u = p_1 p_2 \ldots p_k \in P(n,k)$, we call p_i the *i-digit* of u for each $i \in I_k$. For simplicity, we write $up_{k+1} \ldots p_n$ for $p = p_1 p_2 \ldots p_k p_{k+1} \ldots p_n \in P(n)$, where u is called the prefix of p and $p_{k+1} \ldots p_n$ is called the suffix of p.

Definition 2.1. (Akers and Krishnamurthy, 1989 [1]) An n-dimensional star graph S_n is a graph with vertex-set $P(n)$, a vertex $p = p_1 p_2 \ldots p_i \ldots p_n$ being linked a vertex q if and only if $q = p_i p_2 \ldots p_{i-1} p_1 p_{i+1} \ldots p_n$ for some $i \in I'_n$.

Lemma 2.2. (Li and Xu, 2014 [7]) $\kappa_s^{(h)}(S_n) = \lambda_s^{(h)}(S_n) = (h+1)!(n-h-1)$ *for any* h *with* $0 \le h \le n-2$.

Definition 2.3. (Chiang et al., 1995 [3]) An (n,k)-star graph $S_{n,k}$ is a graph with vertex-set $P(n,k)$, a vertex $p = p_1 p_2 \ldots p_i \ldots p_k$ being linked a vertex q if and only if q is

(a) $p_i p_2 \cdots p_{i-1} p_1 p_{i+1} \cdots p_k$, where $i \in I'_k$ (swap p_1 with p_i), or
(b) $p'_1 p_2 p_3 \cdots p_k$, where $p'_1 \in I_n \setminus \{p_i : \ i \in I_k\}$ (replace p_1 by p'_1).

The vertices of type (a) are referred to as *swap-neighbors* of the vertex p and the edges between them are referred to as *swap-edges* or *i-edges*. The vertices of type (b) are referred to as *unswap-neighbors* of the vertex p and the edges between them are referred to as *unswap-edges*. Clearly, every vertex in $S_{n,k}$ has $(k-1)$ swap-neighbors and $(n-k)$ unswap-neighbors.

By definitions, it is clear that $S_{n,1} \cong K_n$, a complete graph with n vertices, and $S_{n,n-1} \cong S_n$.

Definition 2.4. Let G be a graph and t be a positive integer. A *t-split* graph G^t of G is a graph obtained from G by replacing each vertex x by a set V_x of t independent vertices, and replacing each edge $e = xy$ by a perfect matching E_e between V_x and V_y.

Fig. 1 shows a (4, 2)-star graph $S_{4,2}$ and its 2-split graph $S^2_{4,2}$, which is isomorphic to a star S_4.

Lemma 2.5. *Let* G *be a connected graph and* G^t *be a t-split graph of* G. *Then* $\kappa_s^{(h)}(G^t) \le t\,\kappa_s^{(h)}(G)$ *and* $\lambda_s^{(h)}(G^t) \le t\,\lambda_s^{(h)}(G)$.

Proof. Assume that T is a minimum h-vertex-cut and F is a minimum h-edge-cut in G. Then $\kappa_s^{(h)}(G) = |T|$ and $\lambda_s^{(h)}(G) = |F|$. Let $T^t = \{V_u : \ u \in T\}$ and $F^t = \{E_e : \ e \in F\}$. Then $|T^t| = t|T|$ and $|F^t| = t|F|$.

Since $G - T$ (resp. $G - F$) is disconnected, then $G^t - T^t$ (resp. $G^t - F^t$) also is disconnected. Furthermore, it is easy to see that $(G - T)^t = G^t - T^t$ (resp. $(G - F)^t = G^t - F^t$).

Because T (resp. F) is an h-vertex-cut (resp. h-edge-cut) in G, each vertex in $G - T$ (resp. $G - F$) has at least h neighbors in $G - T$ (resp. $G - F$), and so each vertex in $(G - T)^t$ (resp. $(G - F)^t$) also has h neighbors in $(G - T)^t$ (resp. $(G - F)^t$), which implies that T^t is an h-vertex-cut (resp. F^t is an h-edge-cut) in G^t. Thus, we have

$$\kappa_s^{(h)}(G^t) \le |T^t| = t\,|T| = t\,\kappa_s^{(h)}(G),$$
$$\lambda_s^{(h)}(G^t) \le |F^t| = t\,|F| = t\,\lambda_s^{(h)}(G)$$

as required.

Lemma 2.6. *For any k with $2 \le k \le n-1$, there is an $(n-k)!$-split graph of $S_{n,k}$ that is isomorphic to a star graph S_n.*

Proof. Define an $(n-k)!$-split graph $S_{n,k}^{(n-k)!}$ of $S_{n,k}$ as follows.

For a vertex $u = p_1 p_2 \ldots p_k$ in $S_{n,k}$, it is replaced by $(n-k)!$ vertices

$$V_u = \{u p_{k+1} \ldots p_n \in P(n) : \ p_{k+i} \in I_n \setminus \{p_1, \ldots, p_k\} \text{ for } 1 \le i \le n-k\}.$$

For an edge uv in $S_{n,k}$, let $x = u p_{k+1} \ldots p_n \in V_u$, and define a matching E_{uv} between V_u and V_v as follows.

If uv is an i-edge in $S_{n,k}$ for some $i \in I'_k$, then $v = p_i p_2 \ldots p_{i-1} p_1 p_{i+1} \ldots p_k$. Let E_{uv} be the set of edges that link two vertices $x \in V_u$ and $y \in V_v$ with the same suffix.

If uv is an unswap-edge in $S_{n,k}$, then $v = p_{k+j} p_2 p_3 \cdots p_k$ for some $p_{k+j} \in I_n \setminus \{p_i : \ i \in I_k\}$. Let E_{uv} be the set of edges that link two vertices $x \in V_u$ and $y \in V_v$ with suffixes differing in exactly the $(k+j)$-digit.

Clearly, $S_{n,k}^{(n-k)!}$ has vertex-set $P(n)$, a vertex x is adjacent to a vertex y if and only if the label of y can be obtained from the label of x by swapping the first digit and the i-digit for some $i \in I'_n$. Therefore, by Definition 2.1, $S_{n,k}^{(n-k)!}$ is a star graph S_n. The Lemma follows.

3. Main results

In this section, we present our main results, that is, we determine the h-super connectivity and h-super edge connectivity of the (n,k)-star graph $S_{n,k}$. Since $S_{n,1} \cong K_n$, for which $\kappa_s^{(h)}$ and $\lambda_s^{(h)}$ do not exist for any h with $1 \le h \le n-1$, we only consider the case of $k \ge 2$ in the following discussion.

Lemma 3.1. *For $2 \le k \le n-1$ and $n-k \le h \le n-2$,*

$$\min\{\lambda_s^{(h)}(S_{n,k}), \kappa_s^{(h)}(S_{n,k})\} \ge \frac{(h+1)!(n-h-1)}{(n-k)!}.$$

Proof. For $2 \le k \le n-1$ and $n-k \le h \le n-2$, by Lemma 2.5, Lemma 2.6 and Lemma 2.2, we immediately have that

$$\kappa_s^{(h)}(S_{n,k})(n-k)! \ge \kappa_s^{(h)}(S_{n,k}^{(n-k)!}) = \kappa_s^{(h)}(S_n) = (h+1)!(n-h-1)$$
$$\lambda_s^{(h)}(S_{n,k})(n-k)! \ge \lambda_s^{(h)}(S_{n,k}^{(n-k)!}) = \lambda_s^{(h)}(S_n) = (h+1)!(n-h-1)$$

as required.

Lemma 3.2. *For $2 \le k \le n-1$ and $n-k \le h \le n-2$,*

$$\max\{\lambda_s^{(h)}(S_{n,k}), \kappa_s^{(h)}(S_{n,k})\} \le \frac{(h+1)!(n-h-1)}{(n-k)!}.$$

Proof. Since $n-k \le h$, $n-1-h \le k-1$. Let X be the set of k-arrangements on I_n whose the last $(n-1-h)$ digits are $12 \cdots (n-1-h)$. Then $|X| = \frac{(h+1)!}{(n-k)!}$. Let H be the subgraph of $S_{n,k}$ induced by X. Since $n \ge k+1$, $h+1-(n-k) \le h$ and H is an $(h+1, h+1-(n-k))$-star graph. Let T be the set of neighbors of X in $S_{n,k} - X$ and F be the set of edges between X and T. Since all the vertices with the last $(n-1-h)$ digits $12 \cdots (n-1-h)$ are in X, all the vertices in T are swap-neighbors of X and no two vertices in X share a common swap-neighbor in T, that is, $|F| = |T|$.

For a vertex of H, since it has h neighbors in X, it has exactly $(n-1-h)$ neighbors in T. It follows that

$$|F| = |T| = |X|(n-1-h) = \frac{(h+1)!(n-1-h)}{(n-k)!}.$$

We show that F is an h-edge-cut of $S_{n,k}$. To this end, we only need to show that any vertex v in $S_{n,k} - X$ has at least h neighbors in $S_{n,k} - F$. In fact, since $S_{n,k}$ is $(n-1)$-regular and v has at most one neighbor in X, v has at least $n-2\,(\geq h)$ neighbors in $S_{n,k} - X$, which implies that F is an h-edge-cut of $S_{n,k}$. It follows that

$$\lambda_s^{(h)}(S_{n,k}) \leq |F| = \frac{(h+1)!(n-1-h)}{(n-k)!}.$$

We now show that T is an h-vertex-cut of $S_{n,k}$. To this end, we only need to show that any vertex u in $S_{n,k} - (X \cup T)$ has at least h neighbors within.

We claim that at most one of neighbors of u is in T. Suppose to the contrary that u has two distinct neighbors v and w in T. Since all vertices in T are swap-neighbors of X, without loss of generality, we may assume

$$v = 1p_2 \ldots p_{h+1-(n-k)}p_1 23 \cdots (n-h-1), \tag{3.1}$$

$$w = 2p'_2 \ldots p'_{h+1-(n-k)} 1 p'_1 3 \cdots (n-h-1). \tag{3.2}$$

Since u and w are adjacent, their 1-digits are different, that is, the 1-digit of u is not 2. If v is an unswap-neighbor of u, then from (3.1) we should have

$$u = q_1 p_2 \ldots p_{h+1-(n-k)}p_1 23 \cdots (n-h-1), \tag{3.3}$$

where $q_1 \in I_n \setminus \{p_1, p_2, \ldots, p_{h+1-(n-k)}, 1, 2, \cdots, (n-h-1)\}$. Since $p_1 \neq 1$, $p'_1 \neq 2$ and $q_1 \neq 2$, comparing (3.2) and (3.3), we can easily find that u and w have different digits at least three positions. By Definition 2.3, w is not a neighbor of u, a contradiction.

If v is a swap-neighbor of u then, without loss of generality,

$$u = 3p_2 \ldots p_{h+1-(n-k)}p_1 21 \cdots (n-h-1). \tag{3.4}$$

Comparing (3.2) and (3.4), we can also easily find that u and w have different digits at least three positions, and so w is not a neighbor of u, a contradiction.

Since u has at most one neighbor in T, u has at least $(n-1)-1$ neighbors in $S_{n,k} - (X \cup T)$. Since $(n-1)-1 \geq h$, u has at least h neighbors in $S_{n,k} - (X \cup T)$. It follows that T is an h-vertex-cut of $S_{n,k}$, and so

$$\kappa_s^{(h)}(S_{n,k}) \leq |T| = \frac{(h+1)!(n-1-h)}{(n-k)!}.$$

The lemma follows.

By Lemma 3.1 and Lemma 3.2, we immediately obtain our main result.

Theorem 3.3. *For $2 \leq k \leq n-1$ and $n-k \leq h \leq n-2$,*

$$\kappa_s^{(h)}(S_{n,k}) = \lambda_s^{(h)}(S_{n,k}) = \frac{(h+1)!(n-h-1)}{(n-k)!}. \tag{3.5}$$

Remark 3.4. We would like to make some remarks on our result.

When $h = n-k$, the results in (1.1), (1.2) and (3.5) are consistent, that is, $\kappa_s^{(n-k)}(S_{n,k}) = \lambda_s^{(n-k)}(S_{n,k}) = (n-k+1)(k-1)$.

The alternating group network $AN_n (n \geq 3)$, proposed by Ji [5], is a Cayley graph on the alternating group A_n with respect to the generating set $S = \{(123), (132), (12)(3i) \ : \ 4 \leq i \leq n\}$. Cheng et al. [2] proved that $AN_n \cong S_{n,n-2}$. When $h \in \{0, 1, 2\}$, $\kappa_s^{(h)}(AN_n)$ and $\lambda_s^{(h)}(AN_n)$ can be obtained by (1.1) and (1.2). Very recently, Feng et al. [4] have determined $\lambda_s^{(3)}(AN_n) = 12(n-4)$ for $n \geq 5$. When $h \geq 2$, the following result is obtained from Theorem 3.3 immediately.

Corollary 3.5. $\kappa_s^{(h)}(AN_n) = \lambda_s^{(h)}(AN_n) = \frac{1}{2}\,(h+1)!(n-h-1)$ *for $2 \leq h \leq n-2$.*

4. Conclusions

This paper considers the refined measure, k-super connectivity $\kappa_s^{(h)}$ and k-super edge-connectivity $\lambda_s^{(h)}$ for the fault-tolerance of a network, and the (n,k)-star graph $S_{n,k}$ $(2 \leq k \leq n-1)$, which is an attractive alternative network to the hypercube. In early articles [6,7], we determined $\kappa_s^{(h)}(S_{n,k})$ and $\lambda_s^{(h)}(S_{n,k})$ for $0 \leqslant h \leqslant n-k$, which are two different values. In this paper, we proved $\kappa_s^{(h)}(S_{n,k}) = \lambda_s^{(h)}(S_{n,k}) = \frac{(h+1)!(n-h-1)}{(n-k)!}$ for $n-k \leq h \leq n-2$. This result implies that at least $\frac{(h+1)!(n-h-1)}{(n-k)!}$ vertices or edges have to be removed from an (n,k)-star $S_{n,k}$ to make it disconnected and no vertices of degree less than h. When the (n,k)-star graph is used to model the topological structure of a large-scale parallel processing system, this result can provide a more accurate measure for the fault tolerance of the system.

References

[1] S.B. Akers, B. Krishnamurthy, A group theoretic model for symmetric interconnection networks, IEEE Trans. Comput. 38 (4) (1989) 555–566.
[2] E. Cheng, K. Qiu, Z. Shen, A note on the alternating group network, J. Supercomput. 59 (1) (2012) 246–248.
[3] W.-K. Chiang, R.-J. Chen, The (n,k)-star graphs: a generalized star graph, Inform. Process. Lett. 56 (1995) 259–264.
[4] Y.-Q. Feng, R.-X. Hao, J.-X. Zhou, On computing of a conditional edge connectivity of alternating group network, Linear Multilinear Algebra (2017), http://dx.doi.org/10.1080/03081087.2016.1277689.
[5] Y.-H. Ji, A new class of Cayley networks based on the alternating groups, Appl. Math. J. Chinese Univ. Ser. A 14 (1998) 235–239 (in Chinese).
[6] X.-J. Li, J.-M. Xu, Generalized measures of edge fault tolerance in (n,k)-star graphs, Math. Sci. Lett. 1 (2) (2012) 133–138.
[7] X.-J. Li, J.-M. Xu, Generalized measures for fault tolerance of star networks, Networks 63 (3) (2014) 225–230.
[8] X.-J. Li, J.-M. Xu, Fault-tolerance of (n,k)-star networks, Appl. Math. Comput. 248 (2014) 525–530.