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On conditional fault tolerance of hierarchical cubic networks ☆

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ABSTRACT

This paper considers the conditional fault tolerance, h-super connectivity κ^h and h-super edge-connectivity λ^h of the hierarchical cubic network HCN_n, an attractive alternative network to the hypercube, and shows $\kappa^h(HCN_n) = \lambda^h(HCN_n) = 2^h(n+1-h)$ for any h with $0 \le h \le n-1$. The results imply that at least $2^h(n+1-h)$ vertices or edges have to be removed from HCN_n to make it disconnected with no vertices of degree less than h, and generalize some known results.

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1. Introduction

It is well known that interconnection networks play an important role in parallel computing/communication systems. An interconnection network can be modeled by a graph $G = (V, E)$, where V is the set of processors and E is the set of communication links in the network.

The n-dimensional hypercube Q_n is a graph whose vertex-set consists of all binary vectors of length n, with two vertices being adjacent whenever the corresponding vectors differ in exactly one coordinate. For its regularity, symmetry, high connectivity, logarithmic diameter and simple routing, the hypercube becomes one of the most popular, versatile and efficient topological structures of interconnection networks [8].

However, the hypercube has been considered unsuitable for building large systems since the relatively high vertex-degree results in an additional difficulty in interconnection. To make up for these defects, as an alternative to the hypercube network, many variations of the hypercube network are proposed in the literature. One of them is the hierarchical cubic networks HCN_n proposed by Ghose and Desai [5], which is feasible to be implemented with thousands of or more processors, with retaining some good properties of the hypercubes, such as regularity, symmetry and logarithmic diameter.

Previous works relating to the HCN_n can be found in [1,3–5,25,26]. A shortest-path routing algorithm is presented in [1,25,26]. A broadcasting algorithm appears in [1]. Some parallel algorithms are designed in [5]. The diameter is computed in [25,26], which is about two-thirds the diameter of a comparable hypercube. Hamiltonian cycles are constructed in [1,3,25]. The wide-diameter and fault-diameter are computed in [4], which are also about two-thirds of those of a comparable hypercube.

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In real networks, since the fault of vertices and edges are inevitable, measuring the fault tolerance in networks are very important. The traditional connectivity is a good measurement for the fault tolerance of networks. The *connectivity* $\kappa(G)$ (resp. *edge-connectivity* $\lambda(G)$) of G is defined as the minimum number of vertices (resp. edges) whose removal from G results in a disconnected graph. The connectivity $\kappa(G)$ and edge-connectivity $\lambda(G)$ of a graph G are two important measurements for fault tolerance of the network since the larger $\kappa(G)$ or $\lambda(G)$ is, the more reliable the network is (see [22]).

However, the definitions of $\kappa(G)$ and $\lambda(G)$ are implicitly assumed that any subset of system components is equally likely to be faulty simultaneously, which may not be true in real applications, thus they underestimate the reliability of the network. To overcome such a shortcoming, Harary [6] introduced the concept of conditional connectivity by appending some requirements on connected components, Latifi *et al.* [7] specified requirements and proposed the concept of the restricted h-connectivity. These parameters can measure fault tolerance of an interconnection network more accurately than the classical connectivity. The concepts stated here are slightly different from theirs (see [24]).

For a graph G, $\delta(G)$ denotes its minimum vertex-degree. A subset $S \subset V(G)$ (resp. $F \subset E(G)$) is called an *h-vertex-cut* (resp. *edge-cut*), if $G - S$ (resp. $G - F$) is disconnected and $\delta(G - S) \geq h$. The *h-super connectivity* $\kappa^h(G)$ (resp. *h-super edge-connectivity* $\lambda^h(G)$) of G is defined as the cardinality of a minimum h-vertex-cut (resp. h-edge-cut) of G. It is clear that $\kappa^0(G) = \kappa(G)$ and $\lambda^0(G) = \lambda(G)$.

For an arbitrarily given graph G and any integer h, determining the exact values of $\kappa^h(G)$ and $\lambda^h(G)$ is quite difficult, no polynomial algorithm to compute them has been yet known so far. In fact, the existence of $\kappa^h(G)$ and $\lambda^h(G)$ is an open problem for a general graph G and $h \geq 1$. The main interest of the researchers is to determine the values of κ^h and λ^h for some well-known classes of networks and any h. For a long time, almost all of the research on this topics has been focused on some small h's, only the hypercube network, its κ^h and λ^h were determined [18,20,21] for any h with $0 \leq h \leq n-2$.

In recent years, some new methods and techniques have been discovered, from which κ^h and λ^h have been determined for some well-known classes of networks and for any h. For example, κ^h and λ^h were determined for star networks [14], (n,k)-star networks [9–11] and exchanged hypercubes [13]; κ^h was determined for (n,k)-arrangement networks [16], exchanged crossed cubes [17] and locally twisted cubes [19]; λ^h was determined for hypercube-like networks [12].

Since the h-super connectivity κ^h and h-super edge-connectivity λ^h can provide more accurate measure for the fault tolerance of the system, this paper is interested κ^h and λ^h in the hierarchical cubic network HCN_n, which has much attractive properties than hypercubes. Chiang and Chen [1] determined $\kappa(HCN_n) = \lambda(HCN_n) = n+1$, Zhou *et al.* [27] proved that $\kappa^1(HCN_n) = 2n$ and $\kappa^2(HCN_n) = 4(n-1)$. However, for any $h \geq 1$, $\kappa^h(HCN_n)$ and $\lambda^h(HCN_n)$ have not yet been considered as far now. This paper, we generalize the above results by proving that $\kappa^h(HCN_n) = 2^h(n+1-h)$ for any h with $0 \leq h \leq n-1$, and $\lambda^h(HCN_n) = 2^h(n+1-h)$ for any h with $0 \leq h \leq n$.

The rest of the paper is organized as follows. In Section 2, we recall the structure of HCN_n and some lemmas used in our proofs. The main proof of the result is in Section 3. Conclusions are in Section 4.

For graph terminology and notation not defined here we follow Xu [22]. For a subset X of vertices in G, we do not distinguish X and the induced subgraph $G[X]$.

2. Definitions and lemmas

Let V_n be the set of binary sequence of length n, i.e., $V_n = \{x_1x_2\cdots x_n : x_i \in \{0,1\}, 1 \leq i \leq n\}$. For $x = x_1x_2\cdots x_n \in V_n$, the element $\bar{x} = \bar{x}_1\bar{x}_2\cdots\bar{x}_n \in V_n$ is called the bitwise complement of x, where $\bar{x}_i = \{0,1\} \setminus \{x_i\}$ for each $i \in \{1,2,\ldots,n\}$.

A hypercube network Q_n is an n-dimensional cube, shortly n-cube, its vertex-set V_n, and two vertices being linked by an edge if and only if they differ exactly in one coordinate. For the sake of simplicity, we use xQ_n to denote the Cartesian product $\{x\} \times Q_n$ of a vertex x and a hypercube network Q_n.

Definition 2.1. ([5]) An n-dimensional hierarchical cubic network HCN_n with vertex-set $V_n \times V_n$ is obtained from 2^n n-cubes $\{xQ_n : x \in V_n\}$ by adding edges between two n-cubes, called *crossing edges*, according to the following rule: A vertex (x,y) in xQ_n is linked to

(1) (y,x) in yQ_n if $x \neq y$ or
(2) $(\bar{x},\bar{y})$ in $\bar{x}Q_n$ if $x = y$.

The vertex (y,x) in yQ_n or $(\bar{x},\bar{y})$ in $\bar{x}Q_n$ is called an external neighbor of (x,y) in xQ_n.

A 2-dimensional hierarchical cubic network HCN_2 is shown in Fig. 1, where the red edges are the crossing edges in HCN_2.

Clearly, HCN_n is an $(n+1)$-regular graph. Chiang and Chen [1] determined its connectivity and edge-connectivity.

Lemma 2.2. *([1])* $\kappa(HCN_n) = \lambda(HCN_n) = n+1$.

From Definition 2.1, it is easy to obtain the following property about crossing edges in HCN_n.

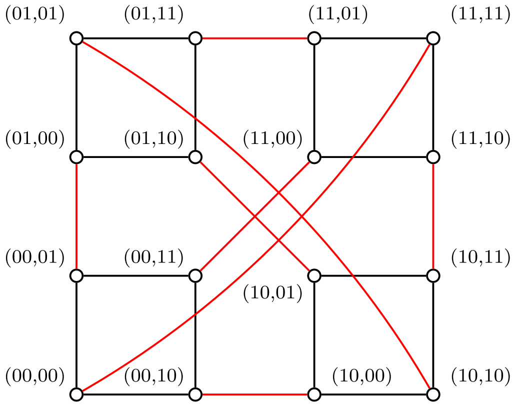

Fig. 1. 2-dimensional hierarchical cubic network HCN_2.

Lemma 2.3.

(1) *There are two crossing edges between two n-cubes xQ_n and yQ_n if and only if x and y are complementary; otherwise there is only one crossing edge.*
(2) *The set of crossing edges consists of a perfect matching of HCN_n.*

Since HCN_n is made up of 2^n n-cubes and a perfect matching, some properties on an n-cube Q_n are very useful for the proofs of our main results.

Lemma 2.4. *([18,20,21]) $\kappa^h(Q_n) = 2^h(n-h)$ for any h with $0 \le h \le n-2$, and $\lambda^h(Q_n) = 2^h(n-h)$ for any h with $0 \le h \le n-1$.*

Lemma 2.5. *([20]) If X is a subgraph in Q_n and $\delta(X) \ge h$, then $|X| \ge 2^h$.*

For a subgraph X in Q_n, $N_n(X)$ denotes the set of neighbors of X in $Q_n - X$.

Lemma 2.6. *If X is a subgraph in Q_n and $\delta(X) \ge h$, then $|X| + |N_n(X)| \ge 2^h(n-h)$ for any h with $0 \le h \le n-1$ and $n \ge 1$.*

Proof. For $n = 1$, $Q_1 \cong K_2$, the conclusion holds clearly. Assume $n \ge 2$ below. The proof proceeds by induction on $h \ge 0$ for a fixed n. Since Q_n is n-regular, for any non-empty subgraph X of Q_n, $|X| + |N_n(X)| \ge n + 1$, and so the conclusion is true for $h = 0$. Assume the induction hypothesis for $h - 1$ with $h \ge 1$.

It is well known that Q_n can be expressed as $Q_n = L \odot_i R$, where L and R are two $(n-1)$-cubes induced by the vertices with i-th coordinate is 0 and 1, respectively, the set of edges between L and R consists of a perfect matching in Q_n (see Xu [22]).

Let X be a subgraph in Q_n with $\delta(X) \ge h$. Then $E(X) \ne \emptyset$ since $h \ge 1$. Arbitrarily take an edge e of X, and assume that two end-vertices of e differ in only the i-th coordinate. Let $Q_n = L \odot_i R$. Then $X \cap L \ne \emptyset$ and $X \cap R \ne \emptyset$.

Let $X_0 = X \cap L$, $X_1 = X \cap R$. Since $\delta(X) \ge h$ in Q_n and the set of edges between L and R is a matching, $\delta(X_0) \ge h - 1$ in L and $\delta(X_1) \ge h - 1$ in R. Using the induction hypothesis in L and R respectively, we have

$$|X_i| + |N_{n-1}(X_i)| \ge 2^{h-1}(n-h) \text{ for each } i \in \{0,1\}.$$

It follows that

$$|X| + |N_n(X)| \ge |X_0| + |N_{n-1}(X_0)| + |X_1| + |N_{n-1}(X_1)| \ge 2^h(n-h).$$

By the induction principle, the lemma follows. □

3. Main results

Lemma 3.1. *For $n \ge 1$, $\kappa^h(HCN_n) \le 2^h(n+1-h)$ for any h with $0 \le h \le n-1$, and $\lambda^h(HCN_n) \le 2^h(n+1-h)$ for any h with $0 \le h \le n$.*

Proof. For $n = 1$, $HCN_1 \cong C_4$, a cycle of length 4, the conclusion holds clearly. Assume $n \ge 2$ below. Let $x_1Q_n, x_2Q_n, \ldots, x_{2^n}Q_n$ be 2^n n-cubes in HCN_n. For a fixed h with $0 \le h \le n-1$, let x_1Q_h be a subgraph in x_1Q_n induced by the vertices with the rightmost $(n-h)$ bits 0s of the second component, S be the neighbors of x_1Q_h in $HCN_n - x_1Q_h$. Then $HCN_n - S$ is disconnected.

On the one hand, by the choice of Q_h, S must contain all vertices with exactly one 1 in the rightmost $(n-h)$ coordinates of the second component, such vertices have exactly $2^h(n-h)$. On the other hand, S must contain all external neighbors of x_1Q_h, such external neighbors have exactly 2^h. Thus, $|S| = 2^h(n-h) + 2^h = 2^h(n+1-h)$.

We now need to prove that S is an h-vertex-cut, i.e., each vertex in $HCN_n - S$ has at least h neighbors.

We first show that $|S \cap V(x_jQ_n)| \le 1$ for each $j \ne 1$. On the contrary, suppose that $|S \cap V(x_{j_0}Q_n)| = 2$ for some $j_0 \ne 1$. Then there are two crossing edges, say e_1 and e_2, between x_1Q_h and $x_{j_0}Q_n$, and so $j_0 = \bar{x}_1$ by Lemma 2.3. By Definition 2.1, two of end-vertices of $\{e_1, e_2\}$ in x_1Q_h is certainly (x_1, x_1) and $(x_1, \bar{x}_1)$. Since the distance between (x_1, x_1) and $(x_1, \bar{x}_1)$ is n, we have $n \le h$, a contradiction. It follows that $|S \cap V(x_jQ_n)| \le 1$ for each $j \ne 1$.

For any $j \ne 1$, let z be a vertex in $x_jQ_n - S$. Then z has at most one neighbor in $S \cap V(x_jQ_n)$ since $|S \cap V(x_jQ_n)| \le 1$. By Definition 2.1, z has at most one neighbor in $HCN_n - x_jQ_n$. Thus, the vertex z has at least $(n+1) - 2 \ge h$ neighbors in $HCN_n - S$.

Let $S_1 = V(x_1Q_n) \cap S$ and $T_1 = V(x_1(Q_n - Q_h) - S_1)$. All that's left is to prove that each vertex in $x_1Q_n - S_1$ has at least h neighbors in $HCN_n - S$. It is clear that each vertex in x_1Q_h has h neighbors in $HCN_n - S$ by the choice of x_1Q_h.

If T_1 is empty then we have done. Assume $T_1 \ne \emptyset$ and let $w \in T_1$. Then $h \le n-2$. If w has no neighbors in S_1, then it has at least n neighbors in $HCN_n - S$. Suppose that w has neighbors in S_1. By the choice of x_1Q_h, there is exactly one 1 in the rightmost $(n-h)$ coordinates of the second component of each vertex in S_1, and so there are exactly two 1s in the rightmost $(n-h)$ coordinates of the second component of w, which implies that w has at most two neighbors in S_1. Thus w has at least $(n-2) \ge h$ neighbors in $HCN_n - S$.

From the above discussions, each vertex of $HCN_n - S$ has at least h neighbors within. Therefore, S is an h-vertex-cut in HCN_n, and so $\kappa^h(HCN_n) \le |S| = 2^h(n+1-h)$.

Let F be the set of edges between x_1Q_h and S. Then $HCN_n - F$ is disconnected. From the above discussions, it is easy to see that F is an h-edge-cut in HCN_n and $|F| = |S|$. Thus $\lambda^h(HCN_n) \le |F| = |S| = 2^h(n+1-h)$.

The lemma follows. □

Theorem 3.2. *For $n \ge 1$, $\kappa^h(HCN_n) = 2^h(n+1-h)$ for any h with $0 \le h \le n-1$, and $\lambda^h(HCN_n) = 2^h(n+1-h)$ for any h with $0 \le h \le n$.*

Proof. For $n = 1$, $HCN_1 \cong C_4$, a cycle of length 4, the conclusion holds clearly. Assume $n \ge 2$ below. By Lemma 2.2 and Lemma 3.1, we only to show $\kappa^h(HCN_n) \ge 2^h(n+1-h)$ for any h with $1 \le h \le n-1$, and $\lambda^h(HCN_n) \ge 2^h(n+1-h)$ for any h with $1 \le h \le n$.

To the end, let F be a minimum h-vertex-cut (or h-edge-cut) of HCN_n, X be the vertex-set of the minimum connected component of $HCN_n - F$, and let

$$Y = \begin{cases} V(HCN_n - X \cup F) & \text{if } F \text{ is a vertex-cut;} \\ V(HCN_n - X) & \text{if } F \text{ is an edge-cut.} \end{cases}$$

Let $H_1, H_2, \ldots, H_{2^n}$ be 2^n n-cubes in HCN_n. For any $i \in \{1, 2, \ldots, 2^n\}$, let

$$\begin{aligned} &X_i = X \cap V(H_i), Y_i = Y \cap V(H_i), \\ &F_i = \begin{cases} F \cap V(H_i) & \text{if } F \text{ is a vertex-cut;} \\ F \cap E(H_i) & \text{if } F \text{ is an edge-cut.} \end{cases} \\ &F_C = \begin{cases} \emptyset & \text{if } F \text{ is a vertex-cut;} \\ F \cap \left(\bigcup_{i \ne j} E(H_i, H_j)\right) & \text{if } F \text{ is an edge-cut.} \end{cases} \end{aligned}$$

where $E(H_i, H_j)$ denotes the set of edges between H_i and H_j for $i \ne j$. Let

$$\begin{aligned} &J_X = \{i \in \{1, 2, \ldots, 2^n\} : X_i \ne \emptyset\}, \\ &J_Y = \{i \in \{1, 2, \ldots, 2^n\} : Y_i \ne \emptyset\} \quad \text{and} \\ &J_0 = J_X \cap J_Y. \end{aligned}$$

Clearly, if $J_0 \ne \emptyset$ then $X_i \ne \emptyset$ and $Y_i \ne \emptyset$ for each $i \in J_0$. By the choice of F, every vertex in $X_i \cup Y_i$ has at least h neighbors in $HCN_n - F$, at most one of them is an external neighbor. This fact implies that F_i is an $(h-1)$-vertex-cut of H_i if F is a vertex-cut, or an $(h-1)$-edge-cut of H_i if F is an edge-cut. Since H_i is an n-cube and $h-1 \ge 0$, by Lemma 2.4 we have

$$|F_i| \ge 2^{h-1}(n+1-h) \text{ for each } i \in J_0, \tag{3.1}$$

and by Lemma 2.5 we have

$$|X_i| \ge 2^{h-1} \text{ and } |Y_i| \ge 2^{h-1} \text{ for each } i \in J_0. \tag{3.2}$$

If $h = n$ then F is an n-edge-cut. We will prove $|F| \geq 2^n$.

If $J_0 = \emptyset$, then F is only consists of crossing edges. Let G be a contracting graph of HCN_n, obtained by contracting each n-cube H_i in HCN_n as a single vertex x_i and by removing all loops. It is easy to see that G is a complete graph K_{2^n} plus a perfect matching, and F is an edge-cut of G. Thus, $|F| \geq \lambda(G) = 2^n$.

If $J_0 \neq \emptyset$ then, $|F_i| \geq 2^{n-1}$ for $i \in J_0$ by (3.1). Combining (3.2) with $H_i \cong Q_n$, we have $|F_i| = |X_i| = |Y_i| = 2^{n-1}$ and X_i is $(n-1)$-regular for each $i \in J_0$. Without loss of generality, assume $1 \in J_0$. Since $\delta(X) \geq n \geq 2$ and X_1 is $(n-1)$-regular, all external neighbors of X_1 are certainly in $X \setminus X_1$. So $|J_X| \geq |X_1| + 1 = 2^{n-1} + 1$. Since $|X_i| = |Y_i|$ for each $i \in J_0$, by the minimality of X, we have $|J_Y| \geq |J_X| \geq 2^{n-1} + 1$. Since $|J_X \cup J_Y| = 2^n$, we have $|J_0| = |J_X| + |J_Y| - |J_X \cup J_Y| \geq 2$. Thus, $|F| \geq \sum_{i \in J_0} |F_i| \geq 2 \times 2^{n-1} = 2^n$.

In the following discussion, we assume $1 \leq h \leq n-1$ and need to show that

$$|F| \geq 2^h(n+1-h) \text{ for } 1 \leq h \leq n-1. \tag{3.3}$$

If $|J_0| \geq 2$ then, by (3.1), we have that

$$|F| \geq \sum_{i \in J_0} |F_i| \geq 2 \times 2^{h-1}(n+1-h) = 2^h(n+1-h).$$

Thus, (3.3) holds if $|J_0| \geq 2$. Assume $0 \leq |J_0| \leq 1$ below.

Let $a = |J_X \setminus J_0|, b = |J_Y \setminus J_0|, c = |\{1, \ldots, 2^n\} \setminus (J_X \cup J_Y)|$. By the choice of X with minimum cardinality, we have $a \leq b$. If $c \geq 1$, then there exists some i such that $V(H_i) \subseteq F$ and F is a vertex-cut, therefore $|F| \geq 2^n \geq 2^h(n+1-h)$ for $h \leq n-1$, and so (3.3) holds. Next, assume $c = 0$, that is, $a + b + |J_0| = 2^n$.

If $a \geq 1$ and $b \geq 1$ then, by Lemma 2.3, for $j_1 \in J_X \setminus J_0, j_2 \in J_Y \setminus J_0$, there is at least one crossing edge between H_{j_1} and H_{j_2}, and so there are at least ab crossing edges between $\cup_{j_1 \in J_X \setminus J_0} H_{j_1}$ and $\cup_{j_2 \in J_Y \setminus J_0} H_{j_2}$. Each of these crossing edges must be in F if F is an edge-cut, or one of its end-vertices must be in F if F is a vertex-cut. Therefore, we have

$$\sum_{i \in J_X \cup J_Y \setminus J_0} |F_i| + |F_C| \geq \sum_{i \in J_X \setminus J_0, j \in J_Y \setminus J_0} |E(H_i, H_j)| \geq ab. \tag{3.4}$$

We consider two cases depending on $|J_0| = 0$ or $|J_0| = 1$.

Case 1. $|J_0| = 0$.

In this case, $a \geq 1$. If $a \geq 2$, by (3.4) we have

$$|F| \geq \sum_{i \in J_X \cup J_Y} |F_i| + |F_C| \geq ab = a(2^n - a) \geq 2^n \geq 2^h(n+1-h).$$

If $a = 1$, without loss of generality assume $J_X = \{1\}$, then $X_1 \subseteq V(H_1)$ if F is a vertex-cut or $X_1 = V(H_1)$ if F is an edge-cut. If F is a vertex-cut, then all external neighbors of X_1 and all vertices in $V(H_1 - X_1)$ are contained in F, and so $|F| \geq |V(H_1)| = 2^n$. If F is an edge-cut, then all crossing edges incident with H_1 are contained in F, and so $|F| \geq |V(H_1)| = 2^n$. Whether F is a vertex-cut or an edge-cut, we have $|F| \geq 2^n \geq 2^h(n+1-h)$ for $1 \leq h \leq n-1$.

Case 2. $|J_0| = 1$.

In this case, $a \geq 0$ and $b = 2^n - a - 1$. Without loss of generality, we assume $J_0 = \{1\}$.

If $a \geq 1$, combining (3.1) and (3.4), we have

$$\begin{aligned}
|F| &\geq |F_1| + \textstyle\sum_{i \in J_X \cup J_Y \setminus J_0} |F_i| + |F_C| \\
&\geq 2^{h-1}(n+1-h) + a(2^n - a - 1) \\
&\geq 2^{h-1}(n+1-h) + 2^{n-1} \\
&\geq 2^{h-1}(n+1-h) + 2^{h-1}(n+1-h) \\
&\geq 2^h(n+1-h).
\end{aligned}$$

If $a = 0$, then $J_X = J_0 = \{1\}$. Since $\delta(X) \geq h$ and H_1 is an n-cube, by Lemma 2.6 $|X| + |N_{H_1}(X)| \geq 2^h(n+1-h)$. If F is a vertex-cut, then $N_{HCN_n}(X) \subset F$, and so

$$|F| \geq |N_{HCN_n}(X)| \geq |X| + |N_{H_1}(X)| \geq 2^h(n+1-h).$$

If F is an edge-cut then F_1 is the set of edges between X and $N_{H_1}(X)$, and so $|F_1| \geq |N_{H_1}(X)|$. Note that $|F_C| \geq |X|$ since $a = 0$. It follows that

$$|F| \geq |F_C| + |F_1| \geq |X| + |N_{H_1}(X)| \geq 2^h(n+1-h).$$

The theorem follows. □

Zhou *et al.*[27] determined $\kappa^1(HCN_n)$ and $\kappa^2(HCN_n)$, which can be obtained from Theorem 3.2 by setting $h = 1, 2$ respectively.

Corollary 3.3. *(Zhou et al. [27]) $\kappa^1(HCN_n) = 2n$ and $\kappa^2(HCN_n) = 4(n-1)$ for $n \geq 3$.*

4. Conclusions

In this paper, we investigate the refined measure, k-super connectivity κ^h and k-super edge-connectivity λ^h for the fault tolerance of a network. For the hierarchical cubic network HCN_n, which is an attractive alternative network to the hypercube, we prove $\kappa^h(HCN_n) = 2^h(n+1-h)$ for any h with $0 \leq h \leq n-1$, and $\lambda^h(HCN_n) = 2^h(n+1-h)$ for any h with $0 \leq h \leq n$, which implies that at least $2^h(n+1-h)$ vertices or edges have to be removed from HCN_n to make it disconnected with no vertices of degree less than h. When the hierarchical cubic networks HCN_n is used to model the topological structure of a large-scale parallel processing system, these results can provide a more accurate measure for the fault tolerance of the system.

There is an other important measure for the fault tolerance in networks G, g-extra connectivity, which defined as the minimum cardinality of vertex-cut T such that every component of $G-T$ has at least $g+1$ vertices. In this direction, g-extra connectivity in some regular networks have received much attention in recent years, such as hypercube-like graphs [2], split-stars [15], arrangement graphs [23]. The g-extra connectivity of the hierarchical cubic networks HCN_n will be a problem worth studying in the future.

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