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Generalized measures of fault tolerance for bubble sort networks☆

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ABSTRACT

The κ^k and λ^k are two generalized measurements for fault tolerance of large-scale processing system. For the bubble sort networks B_n, this paper determines $\kappa^k(B_n) = \lambda^k(B_n) = 2^k(n-k-1)$ for $k \le n/2$. The results show that to disconnect B_n with each vertex in resulting graph has at least k fault-free neighbors, at least $2^k(n-k-1)$ faulty vertices or faulty edges have to occur. In particular, the results also settle affirmatively a conjecture proposed by Shi and Wu (Acta Math. Appl. Sin-E., 33 (4)(2017), 933–944).

1. Introduction

In large parallel computing and communication systems, processors are connected by communication links according to some interconnection networks, where the processors and communication links correspond to the vertices and edges in the networks respectively. In the large-scale interconnection networks, it can not avoid faulty vertex and faulty edge occurred, thus estimating the fault tolerance for networks is of crucial importance. The *connectivity* $\kappa(G)$ is the smallest size of vertices whose deletion disconnects the graph G. The *edge-connectivity* $\lambda(G)$ is defined similarly, with vertices replaced by edges (see [20]). But, these two definitions are based on any subset of interconnection network can be faulty with equal possibility, which just occur in the worst case. Thus κ and λ underrate the fault tolerance of the large-scale networks. Motivated by this weakness, Harary [5] introduced the concept of conditional connectivity by imposing a number of additional requirements on the remaining networks. Thereafter, Latifi et al. [8] generalized this concept in some sense and proposed restricted k-connectivity by restricting each vertex has at least k fault-free neighbors. These generalized measurements can more accurately estimate the fault tolerance of an interconnection network in real applications.

Suppose G is a connected graph, and T is a subset in $V(G)$. If $G-T$ is disconnected and $\delta(G-T) \ge k$, we call T a *k-vertex-cut* of G. The *k-connectivity* $\kappa^k(G)$ is defined as the cardinality of a minimum k-vertex-cut of G. The *k-edge-cut* and *k-edge-connectivity* $\lambda^k(G)$ can be defined similarly, with T a set of edges. It is obvious that $\kappa^0 = \kappa$ and $\lambda^0 = \lambda$.

For a graph G, to determine $\kappa^k(G)$ and $\lambda^k(G)$ for any integer k is not a easy task. For hypercubes Q_n, κ^k and λ^k were determined for any k about two decades ago (see [14,18,19]). For the other networks except Q_n, people concentrate upon on κ^k and λ^k for some small k's at long durations [4]. Recently, using some new methods, κ^k and λ^k have been established for

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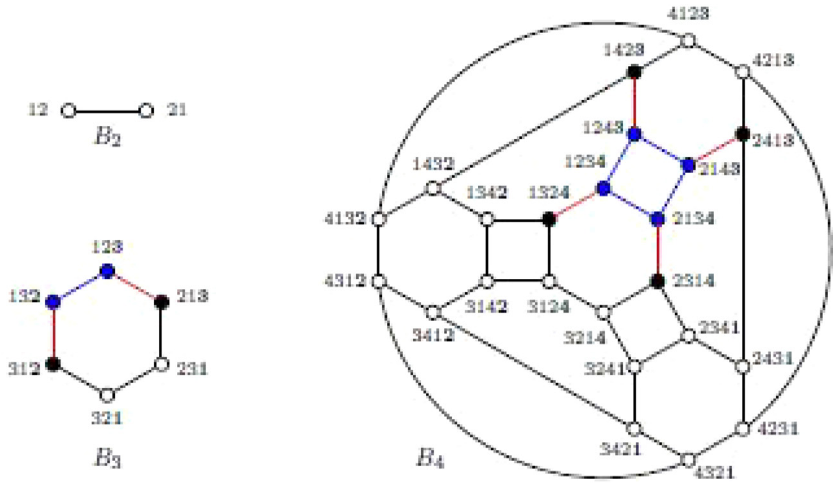

Fig. 1. 2,3,4 dimensional bubble sort networks.

some well-known networks and for any k [11,17]. For instance, κ^k and λ^k were given for exchanged hypercubes [12], star networks [13] and hierarchical cubic networks [10].

The bubble sort network B_n was first presented by Akers and Krishnamurthy [1], it has drawn considerable attention in recent years since it has high symmetry and simple hierarchical structure [1,7]. Much properties for B_n have been investigated, such as Hamiltonian laceability [2], bipancyclicity [6], embedded connectivity [9,22], subnetwork fault tolerance [16], conditional diagnosibility [25]. For $n \geq 3$, Cheng and Lipták [3] determined $\kappa^1(B_n) = 2n - 4$; afterwards, Yang et al. [21] shown that $\kappa^2(B_n) = 4(n-3)$ for $n \geq 4$; Shi and Wu [15] recently determined that $\kappa^3(B_n) = 8(n-4)$ for $n \geq 6$. Furthermore, it has been conjectured by Shi and Wu [15] that for $n \geq 3$ and $k \leq n/2$, $\kappa^k(B_n) = 2^k(n-k-1)$. In this paper, we prove that for $k \leq n/2$, $\kappa^k(B_n) = \lambda^k(B_n) = 2^k(n-k-1)$, which generalize the aforementioned results and also give an affirmative answer for the conjecture.

This paper proceeds as follows. Some structure properties of B_n and useful lemmas are given in Section 2, the main result and the proof are in Section 3. We conclude the paper in Section 4.

2. Preliminaries

Suppose integer $n \geq 2$, let $[n]$ denotes the set $\{1, 2, \ldots, n\}$, and $P(n)$ denotes all the permutations on $[n]$, that is, $P(n) = \{p_1 p_2 \ldots p_n : \ p_i \in [n], p_i \neq p_j, 1 \leq i \neq j \leq n\}$. For convenience, we do not distinguish the subset X of $V(G)$ and the induced subgraph $G[X]$. We follow Xu [20] for graph terminology and notation not defined here.

Definition 2.1 [1]**.** The n-dimensional bubble sort graph B_n has $n\,!$ vertices with each vertex represented by a permutation in $P(n)$. Two vertices u and v are linked in B_n if and only if u and v differ in exactly two consecutive digits.

It is clear that B_n is $(n-1)$-regular. The bubble sort graphs B_2, B_3 and B_4 are shown in Fig. 1. Akers and Krishnamurthy [1] shown that the bubble sort graph B_n is Cayley graph, bipartite, furthermore, B_n contains n disjoint subgraphs B_{n-1}. For each $i \in [n]$, all the vertices of the last digit fixed with i induced a subgraph B_{n-1} (see Fig. 1), denoted by B^i_{n-1}. Similarly, let B^{ij}_{n-2} denotes the subgraph of B_n induced by all the vertices whose the last two digits are ij. It is easy to see that $B^{ij}_{n-2} \cong B_{n-2}$ for each two distinct elements $i, j \in [n]$.

Lemma 2.2 [1]**.** *B_n can be partitioned into n disjoint subgraphs B^j_{n-1} isomorphic to B_{n-1} for each $j \in [n]$; moreover, for two distinct $j_1, j_2 \in [n]$, there are $(n-2)\,!$ independent edges between $B^{j_1}_{n-1}$ and $B^{j_2}_{n-1}$.*

We call the independent edges between $B^{j_1}_{n-1}$ and $B^{j_2}_{n-1}$ *crossing edges*. By Lemma 2.2, all the crossing edges form a perfect matching in B_n, we denote the matching as M_n. If u is linked with v by an edge in M_n, call u an *n-external neighbor* of v, shorted as *n-neighbor* of v. Then every vertex in B_n has just one n-neighbor.

Lemma 2.3 [1]**.** $\kappa(B_n) = \lambda(B_n) = n - 1$ *for* $n \geq 2$.

Lemma 2.4 [21]**.** *Two distinct vertices in B_n have at most two common neighbors, if they have any.*

The k-dimensional hypercube Q_k is a graph with 2^k vertices, each vertex is represented by a k-digits binary sequence. Two vertices u and v being adjacent in Q_k if the sequences of u and v differ in exactly one digit [20]. In the next, we will see that there exists Q_k as subgraph in the bubble sort networks B_n.

Assume $n \geq 3$ and $1 \leq k \leq n/2$, K be a subset of $\{n-1, n-3, \dots, n-2k+1\}$. For a vertex $u = 12\dots(n-1)n$ in B_n, let u^K denotes the set of vertices obtained from u by exchanging i and $(i+1)$ for all $i \in K$. Note that K could be empty, and $u^{\emptyset}$ is u itself. Let

$$X_{n,k} = \{u^K : K \subseteq \{n-1, n-3, \dots, n-2k+1\}\}.$$

Define a mapping ϕ from $X_{n,k}$ to $V(Q_k)$:

$$\phi(u^K) = x_1 x_2 \dots x_k,$$

where for each $j \in [k]$,

$$x_j = \begin{cases} 1, & \text{if } n-2j+1 \in K; \\ 0, & \text{otherwise.} \end{cases}$$

By definition, the mapping ϕ is bijective. Let u^{K_1}, u^{K_2} be two vertices in $X_{n,k}$, by the definition, if u^{K_1} and u^{K_2} are linked in B_n, then $K_2 \subset K_1$ and $|K_1 - K_2| = 1$ or $K_1 \subset K_2$ and $|K_2 - K_1| = 1$, then $\phi(u^{K_1})$ and $\phi(u^{K_2})$ differ in exactly one coordinate, thus $\phi(u^{K_1})$ and $\phi(u^{K_2})$ are linked in Q_k, and vice versa. Then the mapping ϕ preserves adjacency. Therefore we have the subgraph induced by $X_{n,k}$ in B_n is isomorphic to Q_k, that is, $B_n[X_{n,k}] \cong Q_k$ and $|X_{n,k}| = 2^k$.

Lemma 2.5 [15]**.** *Let Q_k be a hypercube in B_n. Then any two distinct vertices in Q_k have no common neighbors in $B_n - Q_k$.*

For a subgraph X in B_n, $N_n(X)$ denotes the set of neighbors of X in $B_n - X$. By the definition of $X_{n,k}$, we have that $X_{n,k} \subset B_{n-2}^{(n-1)n} \cup B_{n-2}^{n(n-1)}$. Let $\overline{X_{n,k}} = V(B_n) - (X_{n,k} \cup N_n(X_{n,k}))$ and

$$f(n,k) = \max\{|N(v) \cap N_n(X_{n,k})| : v \in \overline{X_{n,k}}\}.$$

Lemma 2.6. *If $n \geq 3$ and $1 \leq k \leq n/2$, then $f(n, k) \leq 2$.*

Proof. We apply induction on $n \geq 3$. Note that for $k = 1$, $X_{n,1}$ is an edge xy. It is easy to see that each vertex in $\overline{X_{n,1}}$ can not have neighbors both in $N(x)$ and $N(y)$, otherwise there exists a C_5 in B_n, contradicts to B_n is bipartite. Then by Lemma 2.4, we have $f(n, 1) \leq 2$. And it is straight to know that $f(4,2) = 1$ (see Fig. 1). So for $n \leq 4$, the conclusion holds. In the next, we assume that $n \geq 5$ and $2 \leq k \leq n/2$. Assume the assertion holds for $n-2$ with $k-1 \leq (n-2)/2$, thus we have

$$f(n-2, k-1) \leq 2. \tag{2.1}$$

By Lemma 2.2, B_n can be partitioned into n disjoint B_{n-1}'s, and B_{n-1} can be partitioned into $(n-1)$ disjoint B_{n-2}'s similarly. For an arbitrary vertex v in $\overline{X_{n,k}}$, we consider the neighbors of v and let

$$\begin{aligned} U_0 &= V(B_{n-2}^{(n-1)n} \cup B_{n-2}^{n(n-1)}) - X_{n,k} - N_n(X_{n,k}), \\ U_1 &= V(B_{n-2}^{(n-2)n} \cup B_{n-2}^{(n-3)n} \cup B_{n-2}^{(n-2)(n-1)} \cup B_{n-2}^{(n-3)(n-1)}) - N_n(X_{n,k}), \\ U_2 &= \overline{X_{n,k}} - U_0 - U_1. \end{aligned}$$

For a vertex v in U_0, by symmetry, assume $v \in V(B_{n-2}^{(n-1)n})$. Note that $B_{n-2}^{(n-1)n} \cong B_{n-2}$, by Definition 2.1, we have the subgraph of $B_{n-2}^{(n-1)n}$ induced by $X_{n,k}$ is isomorphic to the subgraph of B_{n-2} induced by $X_{n-2,k-1}$, that is $B_{n-2}^{(n-1)n}[X_{n,k}] \cong B_{n-2}[X_{n-2,k-1}] \cong Q_{k-1}$. Similarly, $B_{n-2}^{(n-1)n}[N_n(X_{n,k})] \cong B_{n-2}[N_{n-2}(X_{n-2,k-1})]$. Using induction in $B_{n-2}^{(n-1)n}$, by (2.1) we have $|N(v) \cap N_{n-2}(X_{n-2,k-1})| \leq f(n-2, k-1) \leq 2$. Now we consider the $(n-1)$-neighbor and n-neighbor of the vertex v. Note that there is a perfect matching between $B_{n-2}^{(n-1)n}$ and $B_{n-2}^{n(n-1)}$, which are the whole $(n-2)!$ edges between B_{n-1}^{n} and $B_{n-1}^{(n-1)}$. By the definition of U_0, for the vertex v in U_0, the unique n-neighbor of v should also be in U_0. In addition, the vertex v has no $(n-1)$-neighbor in $N_n(X_{n,k})$, otherwise there exists a vertex w in $N_n(X_{n,k})$ has two $(n-1)$-neighbors, a contradiction. Thus the vertex v has no $(n-1)$-neighbor or n-neighbor in $N_n(X_{n,k})$ (see Fig. 2). Therefore, we have

$$|N(v) \cap N_n(X_{n,k})| = |N(v) \cap N_{n-2}(X_{n-2,k-1})| \leq 2.$$

For a vertex v in U_1, by symmetry, assume $v \in V(B_{n-2}^{(n-2)n})$. By Lemma 2.5, since the vertices of $B_{n-2}^{(n-2)n} \cap N_n(X_{n,k})$ induced a Q_{n-2} in $B_{n-2}^{(n-2)n}$, $|N(v) \cap B_{n-2}^{(n-2)n} \cap N_n(X_{n,k})| \leq 1$. Note that the n-external neighbor of v should not be in $B_{n-1}^{(n-1)}$, then v has at most one $(n-1)$-external neighbor in $N_n(X_{n,k})$, therefore we have $|N(v) \cap N_n(X_{n,k})| \leq 2$.

For a vertex v in U_2, v has at most one n-external or $(n-1)$-external neighbor in $N_n(X_{n,k})$, then $|N(v) \cap N_n(X_{n,k})| \leq 1$.

That is to say, $|N(v) \cap N_n(X_{n,k})| \leq 2$ for arbitrary vertex v in $\overline{X_{n,k}}$, then $f(n, k) \leq 2$. The lemma follows. □

Lemma 2.7. *If $n \geq 3$ and $0 \leq k \leq n/2$, then $\kappa^k(B_n) \leq 2^k(n-k-1)$ and $\lambda^k(B_n) \leq 2^k(n-k-1)$.*

Proof. When $k = 0$, the assertion is correct by Lemma 2.3, we assume $k \geq 1$ below. Let H denotes the induced subgraph $B_n[X_{n,k}]$, thus $H \cong Q_k$. Let F denotes all the edges between $X_{n,k}$ and $N_n(X_{n,k})$.

Assume v is a vertex in $X_{n,k}$, $|N(v) \cap N_n(X_{n,k})| = (n-k-1)$ since $H \cong Q_k$. Moreover, by Lemma 2.5, for a vertex $v \in N_n(X_{n,k})$, we have $|N(v) \cap X_{n,k}| \leq 1$. Therefore, we have

$$|F| = |N_n(X_{n,k})| = 2^k(n-k-1).$$

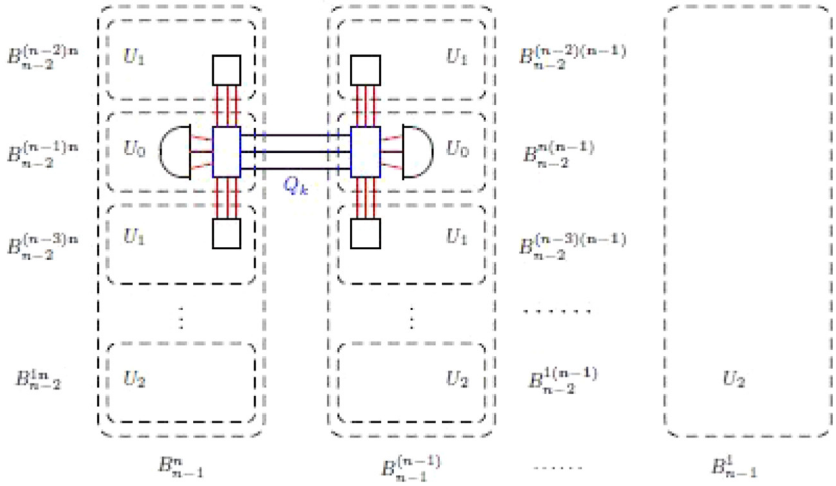

Fig. 2. The neighbors of a Q_k in B_n.

If $(n, k) \notin \{(3, 1), (4, 2)\}$, by Lemma 2.6, then $\delta(\overline{X_{n,k}}) \geq (n-1) - 2 \geq k$. If $(n, k) \in \{(3, 1), (4, 2)\}$, it can be check straightly that $f(3, 1) = 1$ and $f(4, 2) = 1$ (see Fig. 1), thus $\delta(\overline{X_{n,k}}) \geq (n-1) - 1 \geq k$. Therefore we have $N_n(X_{n,k})$ is a k-vertex-cut in B_n, thus $\kappa^k(B_n) \leq |N_n(X_{n,k})| = 2^k(n-k-1)$.

For a vertex $v \in N_n(X_{n,k})$, since $|N(v) \cap X_{n,k}| \leq 1$, we have $|N(v) \cap (B_n - H)| \geq n-2 \geq k$. Then F is a k-edge-cut, thus $\lambda^k(B_n) \leq |F| = 2^k(n-k-1)$.

This completes the proof. □

In the following, we give some properties about the subgraph X of B_n with $\delta(X) \geq k$. By the recursive structure of B_n, we partition B_n into $B_{n-1}^1, B_{n-1}^2, \ldots, B_{n-1}^n$, let $X_i = X \cap V(B_{n-1}^i)$ for $i \in [n]$.

Lemma 2.8. *If X is a subgraph with minimum degree at least k in B_n, then $|X| \geq 2^k$.*

Proof. We use induction on $n (\geq 2)$, the conclusion is trivial for $k = 0$ or $n = 2$, assume $k \geq 1$ and $n \geq 3$ below.

If there is an $i \in [n]$ such that $X \subseteq B_{n-1}^i$, note that B_{n-1}^i is isomorphic to B_{n-1}, then we have $k \leq \delta(X) \leq n-2$. By the induction hypothesis, we have $|X| \geq 2^k$.

If there exist at least two $i \in [n]$ such that $X_i \neq \emptyset$. Since $\delta(X) \geq k$ and each vertex in B_{n-1}^i has only one n-external neighbor, $\delta(X_i) \geq k-1$ if $X_i \neq \emptyset$. By the induction hypothesis, $|X_i| \geq 2^{k-1}$. It follows that $|X| = \sum_{i=1}^n |X_i| \geq 2 \cdot 2^{k-1} = 2^k$.

This completes the proof by the induction principles. □

Let $g_n(X) = |X| + |N_n(X)|$.

Lemma 2.9. *Suppose $0 \leq k \leq n-1$ and X is a subgraph of B_n, if $\delta(X) \geq k$, then $g_n(X) \geq 2^k(n-k)$.*

Proof. We can straightly check the conclusion holds for $n \leq 2$ or $k = 0$, assume $n \geq 3$ and $k \geq 1$ below. The proof is by induction on n with $k \leq n-1$. If $\delta(X) \geq n-1$, X must be B_n, then $|X| = n! \geq 2^{n-1}$, therefore, the conclusion also holds for $k = n-1$. We assume the induction hypothesis for $n-1$ with $k \leq n-2$.

If there exists an $i \in [n]$ such that $X \subseteq B_{n-1}^i$, note that B_{n-1}^i is isomorphic to B_{n-1}, then we have $k \leq \delta(X_i) \leq n-2$. By the induction hypothesis, we have $g_{n-1}(X) \geq 2^k(n-1-k)$. Note that each vertex has just one n-external neighbor, together with Lemma 2.8, it follows that $|N_n(X) - N_{n-1}(X)| = |X| \geq 2^k$. Then

$$g_n(X) = g_{n-1}(X) + |N_n(X) - N_{n-1}(X)| \geq 2^k(n-k-1) + 2^k = 2^k(n-k).$$

If there does not exist $i \in [n]$ such that $X \subseteq B_{n-1}^i$, then there exist two distinct $i_1, i_2 \in [n]$ such that $X_i \neq \emptyset$ for $i \in \{i_1, i_2\}$. Using the induction hypothesis in B_{n-1}^i respectively, we have

$$g_{n-1}(X_i) \geq 2^{k-1}(n-k) \quad \text{for each } i \in \{i_1, i_2\}.$$

Hence,

$$g_n(X) \geq \textstyle\sum_{i \in \{i_1, i_2\}} g_{n-1}(X_i) \geq 2^k(n-k).$$

The lemma follows by the induction principle. □

3. Main results

Theorem 3.1. *If $n \geq 3$ and $0 \leq k \leq n/2$, then $\kappa^k(B_n) = \lambda^k(B_n) = 2^k(n-k-1)$.*

Proof. By Lemma 2.7, it suffices to prove for any k with $k \leq n/2$,

$$\kappa^k(B_n) \geq 2^k(n-k-1) \text{ and } \lambda^k(B_n) \geq 2^k(n-k-1). \tag{3.1}$$

We use induction on n to prove (3.1).

If $k=0$, by Lemma 2.3, we have $\kappa^0(B_n) = \kappa(B_n) \geq n-1$ and $\lambda^0(B_n) = \lambda(B_n) \geq n-1$, that is the conclusion holds for $k=0$. If $n=3$, then B_n is C_6, it is easy to see that $\kappa^1(C_6) \geq \kappa(C_6) = 2$. Similarly, we have $\lambda^1(B_3) \geq 2$. Then the conclusion is correct for $n=3$, we suppose $n \geq 4$ and $k \geq 1$ below.

Assume (3.1) holds for $n-1$ with $k-1 \leq (n-1)/2$. Thus for any k with $1 \leq k \leq n/2$,

$$\kappa^{k-1}(B_{n-1}) \geq 2^{k-1}(n-k-1) \text{ and } \lambda^{k-1}(B_{n-1}) \geq 2^{k-1}(n-k-1). \tag{3.2}$$

Let T be a k-vertex-cut (or k-edge-cut) of B_n with minimum cardinality, it suffices to show that

$$|T| \geq 2^k(n-k-1) \quad \text{for } 1 \leq k \leq n/2. \tag{3.3}$$

Suppose X be the minimum connected component of $B_n - T$, $Y = B_n - X - T$.

By Lemma 2.2, we partition B_n into $B^1_{n-1}, B^2_{n-1}, \ldots, B^n_{n-1}$. For any $i \in [n]$, let

$$X_i = X \cap V(B^i_{n-1}), Y_i = Y \cap V(B^i_{n-1}),$$
$$T_i = \begin{cases} T \cap V(B^i_{n-1}), & \text{if } T \text{ is a vertex-cut;} \\ T \cap E(B^i_{n-1}), & \text{if } T \text{ is an edge-cut.} \end{cases}$$

Let

$$J_X = \{i \in [n] : \ X_i \neq \emptyset\}, J_Y = \{i \in [n] : \ Y_i \neq \emptyset\} \quad \text{and } J_0 = J_X \cap J_Y.$$

It is easy to see, if $i \in J_0 \neq \emptyset$ then $X_i \neq \emptyset$ and $Y_i \neq \emptyset$. By the definition of T, note that every vertex has unique external neighbor, we have $\delta(X_i) \geq k-1$ and $\delta(Y_i) \geq k-1$. Therefore, T_i disconnects B^i_{n-1} and $\delta(B^i_{n-1} - T_i) \geq k-1$. Note that B^i_{n-1} is isomorphic to B_{n-1}, by induction hypothesis (3.2), we have

$$|T_i| \geq 2^{k-1}(n-k-1) \quad \text{for each } i \in J_0. \tag{3.4}$$

If $|J_0| \geq 2$ then, by (3.4), we obtain that

$$|T| \ \geq \sum_{i \in J_0} |T_i| \geq 2^k(n-k-1).$$

Thus, (3.3) holds, and assume that $|J_0| \leq 1$ below.

If there exists some $i \in [n]$ such that $V(B^i_{n-1}) \subseteq T$, thus

$$|T| \geq |B^i_{n-1}| = (n-1)! \geq 2^k(n-k-1). \tag{3.5}$$

The last inequality in (3.5) follows since $(n-1)! = (n-1)(n-2)! \geq (n-1)(2k-2)! = 2^{k-1}(k-1)!(n-1) \geq 2^k(n-k-1)$ for $k \geq 3$, and it is also easy to check the truth directly for $k \leq 2$.

If $(X_i \cup Y_i) \neq \emptyset$ for each $i \in [n]$. Let $a = |J_X - J_0|$, $b = |J_Y - J_0|$, then $a + b = n - |J_0|$. Note that $|X| \leq |Y|$, hence $b \neq 0$.

Let E_C denotes all the edges between $\cup_{j_1 \in J_X - J_0} B^{j_1}_{n-1}$ and $\cup_{j_2 \in J_Y - J_0} B^{j_2}_{n-1}$. All those edges are independent since they are in M_n. By Lemma 2.2, for $j_1 \in J_X$ and $j_2 \in J_Y$, there are $(n-2)!$ independent crossing edges between $B^{j_1}_{n-1}$ and $B^{j_2}_{n-1}$. Hence $|E_C| \geq ab(n-2)!$. All the edges in E_C must be in T provided T is an edge-cut; every edge in E_C has at least one of its end-vertices in T provided T is a vertex-cut. Therefore, we always have

$$|T| \geq |E_C| \geq ab(n-2)!. \tag{3.6}$$

If $a \geq 1$, note that $b = n - a - |J_0| \neq 0$ and $|J_0| \leq 1$, by (3.6), we have

$$|T| \geq (n-2)(n-2)! \geq 2^k(n-k-1). \tag{3.7}$$

If $a = 0$, then $|J_0| = 1$ and $J_X = J_0$. We assume $J_0 = \{1\}$ without loss of generality, thus $X \subset B^1_{n-1}$. Since $\delta(X) \geq k$ and B^1_{n-1} is isomorphic to B_{n-1}, using Lemma 2.9, we have $|X| + |N_{B^1_{n-1}}(X)| = g_{n-1}(X) \geq 2^k(n-k-1)$.

First assume T is a k-vertex-cut, then $N_{B_n}(X) \subset T$. Note that each vertex in X has a neighbor linked by M_n, T must contain this neighbor, we have

$$|T| \geq |N_{B_n}(X)| \geq |X| + |N_{B^1_{n-1}}(X)| \geq 2^k(n-k-1). \tag{3.8}$$

Now assume T is a k-edge-cut, thus any edge between X and $N_{B^1_{n-1}}(X)$ should be contained in T_1, and hence we have $|T_1| \geq |N_{B^1_{n-1}}(X)|$. Recall that $X \subset B^1_{n-1}$, thus the n-external neighbor of each vertex in X can not be in X, therefore $|E_C| \geq |X|$. Hence

$$|T| \geq |E_C| + |T_1| \geq |X| + |N_{B^1_{n-1}}(X)| \geq 2^k(n-k-1). \tag{3.9}$$

By (3.5),(3.7)–(3.9), we have that the inequality (3.3) holds. Therefore we complete the proof by induction principles. □

In Theorem 3.1, let $k = 1, 2$ and 3, we have the following.

Corollary 3.2 [3,15,21]**.** $\kappa^1(B_n) = 2n - 4$ *for* $n \geq 3$; $\kappa^2(B_n) = 4n - 12$ *for* $n \geq 4$; $\kappa^3(B_n) = 8(n-4)$ *for* $n \geq 6$.

4. Concluding remarks

Two generalized measures of fault tolerance, κ^k and λ^k for the bubble sort graphs B_n have been investigated. We show that $\kappa^k(B_n) = \lambda^k(B_n) = 2^k(n-k-1)$ provided $0 \leq k \leq n/2$, which also resolves a conjecture proposed by Shi and Wu [15]. When $n/2 < k \leq n-2$, to determine the κ^k and λ^k for the bubble sort networks B_n is quite interesting, and is worth studying further. It should be mention that there does not exists a Q_k in B_n provided that $k > n/2$, to study $\kappa^k(B_n)$ and $\lambda^k(B_n)$, we need to find a new technique.

In addition, there are other generalized measures of fault-tolerance for networks, such as generalized r-connectivity $\kappa_r(G)$, r-component connectivity $c\kappa_r(G)$(see [23,24]). To study this two generalized measures for bubble sort networks also is of interest.

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