6 Super Connectivity of Networks

As we have known, it is NP-hard to compute the probability \( P_e(G, p_e) \) of vertex-failures

\[
P_e(G, p_e) = \sum_{i=\kappa}^{\ell} \ell_i p_e^i (1 - p_e)^{\mu - i}, \tag{6.1}
\]

where \( \ell_i \) is the number of vertex-cuts consisting of \( i \) vertices, and the probability \( P_e(G, p_e) \) of edge-failures

\[
P_e(G, p_e) = \sum_{i=\lambda}^{\varepsilon} c_i p_e^i (1 - p_e)^{\varepsilon - i}, \tag{6.2}
\]

where \( c_i \) are the number of edge-cuts consisting of \( i \) edges.

To minimize \( \ell_i \) in (6.1) and \( c_i \) in (6.2), Bauer and Boesch et al. [7,9,10] proposed the concept of super connected graphs.

Let \( G \) be a connected graph. \( G \) is said to be maximally connected if \( \kappa(G) = \delta(G) \); \( G \) is said to be maximally edge-connected if \( \lambda(G) = \delta(G) \). Let \( x \in V(G) \) with \( d_G(x) = \delta(G) \). If \( G \) is maximally connected (resp. maximally edge-connected), then \( N_G(x) \) is (resp. \( E_G(x) \)) a \( \kappa \)-cut (resp. \( \lambda \)-cut), its removal results in \( x \) to be an isolated vertex. Clearly, one expects that the number of \( \kappa \)-cuts (resp. \( \lambda \)-cuts) is as small as possible.

If the removal of every \( \kappa \)-cut (resp. \( \lambda \)-cut) results in an isolated vertex, then \( G \) is said to be super connected, super-\( \kappa \) for short, (resp. super edge-connected, super-\( \lambda \) for short).

The following theorem shows the relationships between super connectedness and super edge-connectedness. We first give a lemma.

Lemma 6.1 (Xu et al. [83], 2005) Let \( G \) be a connected graph with \( \kappa(G) = \lambda(G) > 0 \). If \( F \) is a \( \lambda \)-cut of \( G \), then either \( F = E_G(x) \) for some vertex \( x \) of \( G \) or \( F \) is a matching in \( G \).

Proof. Since \( F \) is a \( \lambda \)-cut in \( G \), there is a nonempty subset \( X \subset V(G) \) such that \( F = [X, \overline{X}] \). Let \( X_0 \) and \( Y_0 \) be sets of the end-vertices of all edges in \( F \) in \( X \) and \( \overline{X} \), respectively. Clearly, \( |X_0| \leq |F| \) and \( |Y_0| \leq |F| \).

Without loss of generality, assume \( |X_0| \leq |Y_0| \). Thus, we only need to prove that \( |X_0| = |F| \) if \( F \neq E_G(x) \) for any \( x \in V(G) \) since, in this case, every vertex in \( X_0 \) is matched by an edge in \( F \) with a unique vertex in \( Y_0 \). In fact, if \( X - X_0 \neq \emptyset \), then \( X_0 \) is a vertex-cut of \( G \), so \( |F| \geq |Y_0| \geq |X_0| \geq \kappa(G) = \lambda(G) = |F| \).

Assume \( X - X_0 = \emptyset \) below. It is clear that \( |X_0| \geq 2 \) since \( F \neq E_G(x) \) for any \( x \in V(G) \). Let \( |X_0| = t \) and \( E_x = \{ xy \in E(G) \mid y \in Y_0 \} \) for \( x \in X \). Since \( 2 \leq t \leq \lambda(G) \leq \delta(G) \), we have

\[
\begin{align*}
\delta(G) & \geq \lambda(G) = |F| = \sum_{x \in X_0} |E_x| = \sum_{x \in X_0} d(x) - 2|E(G[X_0])| \\
& \geq \delta(G)t - t(t - 1) = -t^2 + (\delta(G) + 1)t \\
& \geq \delta(G).
\end{align*}
\]

The last inequality holds because the function \( f(t) = -t^2 + (\delta(G) + 1)t \) is convex in the integer interval \([2, \delta(G)] \) and reaches the minimum value at the right end-point of the
interval, that is, \( f(t) \geq f(\delta(G)) = \delta(G). \) The equality is true if and only if \( t = |X_0| = \delta(G) = |F| \). The lemma follows.

**Theorem 6.2** (Xu et al. [83], 2005) Let \( G \) be a super-\( \kappa \) graph with \( \delta(G) \geq 4 \). Then \( G \) is super-\( \lambda \).

**Proof.** Since \( G \) is super-\( \kappa \), \( \kappa(G) = \delta(G) \), and so \( \kappa(G) = \lambda(G) = \delta(G) \) immediately from \( \kappa(G) \leq \lambda(G) \leq \delta(G) \).

Suppose to the contrary that \( G \) is not super-\( \lambda \). Then there exists a \( \lambda \)-cut \( F \) with \( F \neq E_G(x) \) for any vertex \( x \) of \( G \). By Lemma 6.1, \( F \) is a matching in \( G \). Let \( X, X_0 \) and \( Y_0 \) be subsets of vertices of \( G \) defined in the proof of Lemma 6.1. Let \( \lambda = |F| \) and

\[
X_0 = \{x_1, x_2, \ldots, x_\lambda\} \quad \text{and} \quad Y_0 = \{y_1, y_2, \ldots, y_\lambda\},
\]

where \( x_i \) is matched with \( y_i \) by an edge in \( F \) for \( i = 1, 2, \ldots, \lambda \). Consider the set of vertices \( S = \{x_1, x_2, y_3, y_4, \ldots, y_\lambda\} \) in \( G \). Clearly, \( S \) is a \( \kappa \)-cut of \( G \). Since \( G \) is super-\( \kappa \), \( S = N_G(u) \) for some \( u \in V(G) \). If \( u \in X, y_1, y_2u \in F \), contradicting to the fact that \( F \) is a matching. If \( u \in Y, x_1u, x_2u \in F \), a contradiction too. Thus, \( G \) is super-\( \lambda \), and so the theorem follows.

**Corollary 6.3** Let \( G \) be a connected graph and \( L \) be the line graph \( L(G) \). If \( \delta(G) \geq 3 \) and \( L \) is super-\( \kappa \), then \( L \) is super-\( \lambda \).

**Proof.** Since \( L \) is super-\( \lambda \) and \( \delta(L) = \xi(G) \geq 2\delta(G) - 2 \geq 4 \) for \( \delta(G) \geq 3 \), the corollary follows from Theorem 6.2.

In Theorem 6.2, the condition \( \delta(G) \geq 4 \) is necessary. For example, the graph \( K_3 \times K_2 \) is super-\( \kappa \) but not super-\( \lambda \). In 2009, Liu and Wang [36] generalized results in Lemma 6.1 and Theorem 6.2.

Characterizations of all non-super-\( \kappa \) or non-super-\( \lambda \) graphs have received considerable attention in the literature. In 1982, Tindell [61] characterized super-\( \lambda \) for vertex-transitive graphs, that is, a connected vertex-transitive graph of degree \( d \) which is neither a complete graph nor a cycle is super-\( \lambda \) if and only if it contains no \( K_4 \). In 1984, Boesch et al. [10] proved that the only connected edge-transitive graphs which are not super-\( \lambda \) are cycles \( C_n \). In 2003, Meng [45] characterized non-super-\( \kappa \) vertex- and edge-transitive graphs. In 2008, Zhang and Meng [89] characterized non-super-\( \kappa \) edge-transitive graphs; in 2007, Liang et al. [34] determined all non-super-\( \kappa \) vertex-transitive bipartite graphs. In 2009, Balbuena et al. [6] proved that for any \( k \)-regular graph of diameter \( d \) and odd girth \( g \), if \( d < g \) then the graph is super-\( \kappa \) when \( g \geq 5 \) and a complete graph otherwise. In 2011, Zhou and Feng [91] completely characterized all super-\( \kappa \) but not super-\( \lambda \) graphs.

By the lexicographic product \(^5\) and line graphs, Meng [46] showed that a connected vertex- and edge-transitive graph \( G \) is not super-\( \kappa \) if and only if \( G \cong C_n(K_m^c) \) or \( G \cong L(Q_3)(K_m^c) \) for \( n \geq 6 \) and \( m \geq 1 \). Zhang and Meng [89] showed that all connected

\(^5\)The lexicographic product of \( G \) by \( H \), denoted by \( G(H) \), is the graph with vertex set \( V(G) \times V(H) \) and, two vertices \( x_1x_2 \) and \( y_1y_2 \) of \( G(H) \) are adjacent if and only if either \( x_1y_1 \in E(G) \) or \( x_1 = y_1 \) and \( x_2y_2 \in E(H) \).
irreducible edge-transitive graphs are super-$\kappa$ except for $C_n$ ($n \geq 6$) and $L(Q_3)$, where irreducible means the graph has no two distinct vertices with the same neighbor set. Other related results, the interested reader is referred to Meng et al. [35, 60, 88].

In addition, Shieh [59] showed that the Cartesian product of two regular graphs with maximum connectivity (resp. edge-connectivity) is super-$\kappa$ (resp. super-$\lambda$) except for $K_n \times K_2$ with $n \geq 4$ (resp. $n \geq 2$).

A quite natural question can be proposed from the concept of super connectedness. Let $G$ be a super connected or super edge-connected graph. How many vertices or edges can be removed to obtain a disconnected graph without isolated vertices? To answer this question, the concepts of the super connectivity and the super edge-connectivity of a graph are proposed.

Let $G = (V, E)$ be a graph and $S \subset V$. If $G - S$ is disconnected and $\delta(G - S) > 0$ (equivalently, $G - S$ has no isolated vertices), then $S$ is said a super vertex-cut of $G$.

In general, super vertex-cuts do not always exist in a graph. For example, a complete graph $K_n$ has no super vertex-cuts. If a graph $G$ has super vertex-cuts, then the super connectivity of $G$ is defined as the minimum cardinality of all super vertex-cuts in $G$, denoted by $\kappa_s(G)$. If $G$ has no super vertex-cuts, then we define $\kappa_s(G) = \infty$, that is, $\kappa_s(G)$ does not exist. A vertex-cut $S$ of $G$ is said to be a $\kappa_s$-cut if $|S| = \kappa_s(G)$.

Analogously, we can define the concept of the super edge-connectivity of $G$. Let $F \subset E$. If $G - F$ is disconnected and $\delta(G - F) > 0$ (equivalently, $G - F$ has no isolated vertices), then $F$ is a super edge-cut of $G$. Super edge-cuts do not always exist in a graph. If a graph $G$ has super edge-cuts, then the super edge-connectivity is defined as the minimum cardinality of all super edge-cuts in $G$, denoted by $\lambda_s(G)$. If $G$ has no super edge-cuts, then we define $\lambda_s(G) = \infty$, that is, $\lambda_s(G)$ does not exist. A super edge-cut $S$ of $G$ is said to be a $\lambda_s$-cut if $|S| = \lambda_s(G)$.

Although as a graph-theoretical concept, the super connectivity was proposed by Boesch and Tindell [10] in 1984, as graph-theoretical parameters, the super connectivity and the super edge-connectivity were formally proposed and noted the difference from the restricted connectivity the restricted edge-connectivity by Balbuena et al. [5] and Xu et al. [85] in recent years.

![Figure 1: $\kappa_r(G) = \infty$ and $\kappa_s(G) = 3$](image)

By definitions, it is easy to see that $F$ is a super edge-cut if and only if $F$ is a restricted edge-cut in a graph $G$. Thus, $\lambda_s(G) = \lambda_r(G)$ for any connected graph $G$.

However, super vertex-cuts and restricted vertex-cuts are different from each other. For example, the graph $G$ shown in Figure 1 has the only super vertex-cut $S = \{x_2, x_5, x_7\}$, and so $\kappa_s(G) = 3$. A unique possible restricted vertex-cut is also $S$, however, it contains $N_G(x_7)$ as a subset. Thus, $\kappa_r(G)$ does not exists.
Indeed, the existence of $\kappa_s(G)$ and $\kappa_r(G)$ has not been yet solved for a general graph $G$. However, for a graph $G$ if $\kappa_r(G)$ exists then $\kappa_s(G)$ exist and $\kappa_s(G) \leq \kappa_r(G)$ since any restricted vertex-cut is a super vertex-cut. Conversely, if $\kappa_s(G)$ does not exist then $\kappa_r(G)$ does not exist. The following result holds obviously, which shows relationships between the super connectivity and the restricted connectivity.

**Theorem 6.4** Let $G$ be a connected graph.

1. If $\kappa_r(G)$ exists, then $\kappa(G) \leq \kappa_s(G) \leq \kappa_r(G)$, and if $\delta(G) = \kappa(G) < \kappa_s(G)$ then $G$ is super-connected.

2. If $G$ is neither $K_{1,n}$ nor $K_3$, then $\lambda(G) \leq \lambda_r(G) = \lambda_s(G)$, and if $\delta(G) = \lambda(G) < \lambda_s(G)$ then $G$ is super edge-connected.

By the definitions, super connectedness and super edge-connectedness of $G$ can be measured by $\kappa_s(G)$ and $\lambda_s(G)$, respectively. Therefore, studying the super connectivity and super edge-connectivity of $G$ is more significant than super connectedness and super edge-connectedness of $G$.

By Theorem 6.4, it always holds that $\lambda_r(G) = \lambda_s(G)$. However, in the literature, most authors like to use the term “restricted” instead of “super”. In our discussion, we will not differentiate $\lambda_r(G)$ and $\lambda_s(G)$. Thus, all results studied in Section 2, Section 3 and Section 4 are valid for $\lambda_s(G)$.

By Theorem 2.1, if $G$ is $\lambda_s$-optimal, then there is a $\lambda_s$-cut whose removal isolates an edge. If the removal of every $\lambda_s$-cut results in an isolated edge, then $G$ is said to be super-$\lambda_s$. Similarly, if the removal of every $\kappa_s$-cut results in an isolated edge, then $G$ is said to be super-$\kappa_s$ for short.

As mentioned in the beginning of this section, to compute the probability $P_v(G, p_v)$ of vertex-failures or the probability $P_e(G, p_e)$ of edge-failures in a graph $G$, we only need to calculate every coefficient $\ell_i$ in (6.1) with $i = \kappa, \kappa + 1, \ldots, v$ or every coefficient $c_i$ in (6.2) with $i = \lambda, \lambda + 1, \ldots, \varepsilon$. It is clear that if $G$ is super-$\kappa$ or super-$\lambda$ then $\ell_\kappa$ in (6.1) or $c_\lambda$ in (6.2) is the number of vertices of degree $\delta$ in $G$. Furthermore, if $G$ is $\delta$-regular and super-$\kappa$ or super-$\lambda$ then

$$\ell_\kappa = c_\lambda = v.$$  

It is also easy to see that if $G$ is $\delta$-regular and super-$\kappa_s$ or super-$\lambda_s$ then

$$\ell_i = v \binom{\nu - \delta}{i - \delta}, \quad \forall \ i = \kappa + 1, \kappa + 2, \ldots, \kappa_s - 1;$$
$$c_i = v \binom{\nu - \delta}{i - \delta}, \quad \forall \ i = \lambda + 1, \lambda + 2, \ldots, \lambda_s - 1. \quad (6.3)$$

This also shows that determining $\kappa_s(G)$ and $\lambda_s(G)$ are of important significance for a given graph $G$. However, we have not known any result on $\kappa_s$. Several authors studied super-$\lambda_s$ graphs and determined the value of $c_{\lambda_s}$ for some classes of regular graphs, for example, Boesch and Wang [11], Li and Li [31, 32] for circulant graphs, Wang and Li [62], Ou and Zhang [56] for general regular graphs according to the value of $\lambda_s$.

**Theorem 6.5** (Ou and Zhang [56]) Let $G$ be a connected graph of order $n$ ($\geq 4$). If $\delta(G) > n/2$, then $G$ is super-$\lambda_s$.  

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Proof. According to Corollary 2.9, \( G \) is \( \lambda_s \)-optimal. If \( G \) is not super-\( \lambda_s \), then, by Corollary 3.3, there is a \( \lambda_s \)-atom \( X \) such that \( |X| \geq \delta(G) \). Thus, we have
\[
n/2 \geq |X| \geq \delta(G) > n/2,
\]
a contradiction, and so the theorem follows.

By Theorem 6.5, Ou and Zhang gave the values of \( c_i \) in (6.3) for a \( \delta \)-regular connected graph of order \( n \) and \( \delta > n/2 \) as follows.

\[
c_i = \begin{cases} 
0 & \text{if } 0 \leq i \leq \delta - 1; \\
n \left( \frac{n\delta/2 - \delta}{\delta - i} \right) & \text{if } \delta \leq i \leq 2\delta - 3; \\
n \left( \frac{n\delta/2 - \delta}{\delta - 2} \right) + n\delta/2 & \text{if } i = 2\delta - 2. 
\end{cases}
\]
7 Lower Bounds of Super Edge-Connectivity

By Theorem 2.3, for a $\lambda_s$-graph $G$, $\lambda_s(G) = \lambda(G)$ if and only if there is a $\lambda$-cut $B = E_G(X)$ such that both $G[X]$ and $G[\overline{X}]$ are nontrivial and connected. This result is not always true for a $\kappa_s$-connected graph, and the necessity is valid for a claw-free graph.

A graph $G$ is said to be a claw-free graph if $G$ contains no vertex-induced subgraphs isomorphic to $K_{1,3}$. It is easy to verify that the line graph of a undirected graph is a claw-free graph.

**Theorem 7.1** Let $G$ be a connected claw-free graph and $S$ a $\kappa_s$-cut of $G$. Then $G - S$ contains exactly two components.

**Proof.** Let $G_1, G_2, \ldots, G_\omega$ ($\omega \geq 2$) be connected components of $G - S$. Then $G_i$ contains at least two vertices for $i = 1, 2, \ldots, \omega$ and there are no edges between $G_i$ and $G_j$ for any $i \neq j$. Since $S$ is a $\kappa_s$-cut, for any $x \in S$, it is adjacent to every component, otherwise $S' = S \setminus \{x\}$ is also a super vertex-cut, contradicting to the minimality of $S$. If $\omega \geq 3$, the vertex $x$ is adjacent to at least three components. This is impossible for $G$ is a claw-free graph. Therefore, $\omega = 2$, and so the theorem follows.

The concepts of the super connectivity and the super edge-connectivity are valid for digraphs providing we substitute “strongly connected” for “connected”.

We now give a lower bound for the super edge-connectivity of a class of strongly connected digraphs. Use $G(n, d, k)$ to denote a $d$-regular connected digraph with order $n$ and diameter $k$, without loops at the end-vertices of any pair of symmetric edges.

If $d = 1$, $G(n, 1, k)$ is a directed cycle $C_n$ and $\lambda_s(C_n) = \infty$. If $k = 1$, $G(n, d, k)$ is a complete digraph $K_{d+1}$, and $\lambda_s(K_{d+1}) = \infty$ for $d \leq 2$ and $\lambda_s(K_{d+1}) = 2d - 2$ for $d \geq 3$. Thus, in the following discussion, we assume $d \geq 2$ and $k \geq 2$, and so $n \geq 4$.

**Theorem 7.2** (Xu and Lü [80]) For a connected digraph $G = G(n, d, k)$, if $\lambda_s(G)$ exists, then

$$\lambda_s(G) \geq \begin{cases} \min \left\{ \frac{(n - d^{k-1})(d - 1)}{d^{k-1} + d - 2}, 2d - 2 \right\} & \text{for } k \neq 3; \\ \min \left\{ \frac{n}{2d + 2}, 2d - 2 \right\} & \text{for } k = 3. \end{cases}$$

**Proof.** Since $G$ is $d$-regular and connected, $G$ is strongly connected. Let $\lambda_s = \lambda_s(G)$ since $\lambda_s(G)$ exists. To prove the theorem, it is sufficient to show that if $\lambda_s < 2d - 2$ then

$$n \leq \begin{cases} \lambda_s \frac{d^{k-1} + d - 2}{d - 1} + d^{k-1} & \text{for } k \neq 3; \\ 2 \lambda_s \frac{d}{d + 1} & \text{for } k = 3. \end{cases} \quad (7.1)$$

To the end, let $F$ be a $\lambda_s$-cut of $G$ such that $|F| = \lambda_s$. Then, $V(G)$ can be partitioned into two disjoint nonempty sets $X$ and $Y$ such that $F = E_G(X, Y)$. Let $X_0$ and $Y_0$ be the sets of the initial and terminal vertices of the edges of $F$, respectively. Let

$$d_G(x, X_0) = \min\{d_G(x, u) : u \in X_0\},$$

$$d_G(Y_0, y) = \min\{d_G(v, y) : v \in Y_0\},$$

$$m = \max\{d_G(x, X_0) : x \in X\},$$

$$m' = \max\{d_G(Y_0, y) : y \in Y\}.$$
For any \( x_0 \in X_0 \) and \( y_0 \in Y_0 \), let
\[
X^\ell_\ell(x_0) = \{ x \in X : d_G(x, x_0) = \ell \}, \quad 0 \leq \ell \leq m;
\]
\[
Y^\ell'_\ell(y_0) = \{ y \in Y : d_G(y_0, y) = \ell' \}, \quad 0 \leq \ell' \leq m'.
\]

Noting that \( |X_0| \leq |F| \) and \( |Y_0| \leq |F| \), we have that
\[
|X| \leq \sum_{x_0 \in X_0} \sum_{\ell=0}^{m} |X^\ell_\ell(x_0)| \leq |F|(1 + d + d^2 + \cdots + d^m);
\]
\[
|Y| \leq \sum_{y_0 \in Y_0} \sum_{\ell'=0}^{m'} |Y^\ell'_\ell(y_0)| \leq |F|(1 + d + d^2 + \cdots + d^{m'}).
\]

We now consider the relationship between \( m \) and \( m' \). Choose \( x \in X \) and \( y \in Y \) such that \( d_G(x, X_0) = m \) and \( d_G(Y_0, y) = m' \). Since any \( (x, y) \)-path in \( G \) passes through \( F \), there exists an edge \( e = x_0y_0 \in F \), \( x_0 \in X_0, y_0 \in Y_0 \), such that
\[
d_G(x, x_0) + 1 + d_G(y_0, y) = d_G(x, y) \leq k.
\]
By the choices of \( x \) and \( y \), we have \( d_G(x, x_0) \geq m \) and \( d_G(y_0, y) \geq m' \). Thus,
\[
m' \leq d_G(y_0, y) \leq k - d_G(x, x_0) - 1 \leq k - m - 1.
\]
It follows from (7.2) that
\[
n = |X| + |Y| \leq |F| \frac{d^{m+1} + d^{k-m} - 2}{d - 1}, \tag{7.3}
\]
where \( 0 \leq m \leq k - 1 \).

Since \( G \) is \( d \)-regular, \( |E_G(X, Y)| = |E_G(Y, X)| \). Without loss of generality, we can suppose \( m \leq m' \) in the following discussion. There are two cases.

**Case 1** \( m \geq 1 \). Then \( m' \geq 1 \), so \( m \leq k - m' - 1 \leq k - 2 \) which implies \( k \geq 3 \). Define a function
\[
f(m) = \frac{d^{m+1} + d^{k-m} - 2}{d - 1}.
\]
It is a convex function in the integer interval \([1, k-2]\) and reaches the maximum value at an end-point of the interval. Since \( f(1) = f(k-2) \), it follows from (7.3) that
\[
n \leq |F| f(m) \leq |F| f(1) = \lambda s \frac{d^{k-1} + d^2 - 2}{d - 1}. \tag{7.4}
\]

**Case 2** \( m = 0 \). This case indicates \( X = X_0 \) and \( m' = k - 1 \). Let \( A(x) = \{(x, v) : v \in Y\} \). If \( 2 \leq |X| \leq d - 1 \), then we can deduce a contradiction as follows (Note that no loops are at the end-vertices of any pair of symmetric edges.).
\[
2d - 3 \geq |F| = \sum_{x \in X} |A(x)| \geq d|X| - |X||X| - 1 \geq 2d - 2.
\]
Thus, \(|X| \geq d\), and so there is a vertex \(x \in X\) which is adjacent to exactly one vertex in \(Y_0\). Since \(d_G(x, y) \leq k\) for any \(y \in Y\), the number of the farthest vertices in \(Y\) that can be reached from any vertex in \(Y_0\) is at most \(d^k - 1\), that is,

\[
\sum_{y \in Y_0} |Y^+_m(y)| \leq d^k - 1.
\]

It follows that

\[
n \leq |X| + \sum_{y \in Y_0} \sum_{i=0}^{m'-1} |Y_i^+(y)| + \sum_{y \in Y_0} |Y_{m'}^+(y)| \leq |X| + |Y_0| \sum_{i=0}^{m'-1} d^i + d^{k-1} \leq |F| + |F| \frac{dm' - 1}{d - 1} + d^{k-1} = |F| + |F| \frac{d^{k-1} - 1}{d - 1} + d^{k-1},
\]

from which we obtain that

\[
n \leq \lambda_s \frac{d^{k-1} + d - 2}{d - 1} + d^{k-1}. \tag{7.5}
\]

Note that (7.4) is valid for \(k \geq 3\) and that (7.5) is valid for \(k \geq 2\). Comparing (7.4) and (7.5), we obtain (7.1) for \(k \neq 3\). When \(k = 3\), (7.4) is always valid and (7.5) is valid only for \(|X| \geq d\). Note that the values of the right hand of (7.4) and (7.5) are \(2\lambda_s(d + 1)\) and \(\lambda_s(d + 2) + d^2\), respectively. If (7.5) is valid, since \(\lambda_s = |F| \geq |X| \geq d\), then

\[
2\lambda_s(d + 1) \geq \lambda_s(d + 2) + d^2,
\]

which means \(n \leq 2\lambda_s(d + 1)\). Thus, we obtain (7.1) for \(k \geq 2\), and so the theorem follows.

By Theorem 7.2, we can obtain the following corollary immediately.

**Corollary 7.3** For a connected digraph \(G = G(n, d, k)\), if \(\lambda_s(G)\) exists and

\[
n \geq \begin{cases} 
3d^{k-1} + 2d - 4, & \text{for } k \neq 3; \\
4(d^2 - 1), & \text{for } k = 3.
\end{cases}
\]

then \(\lambda_s \geq 2d - 2\).

By Corollary 7.3, we can determine the values of \(\lambda_s\) for the de Bruijn digraph \(B(d, k)\) and the Kautz digraph \(K(d, k)\).

**Corollary 7.4** (Xu and Lü [42, 81]) For the de Bruijn digraph \(B(d, k)\) with \(d \geq 4\) and \(k \geq 1\),

\[
\lambda_s(B(d, k)) = \begin{cases} 
2d - 4 & \text{for } k = 1 \text{ and } d \geq 4; \\
2d - 2 & \text{otherwise}.
\end{cases}
\]

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Proof. Note that $B(d, k)$ contains $d$ pairs of symmetric arcs with no loops at their end-vertices. Choose a pair of symmetric arcs between two vertices, say $x$ and $y$. Since $B(d, k)$ is $(d - 1)$-connected, thus, $B(d, k) - \{x, y\}$ is strongly connected for $d \geq 4$, which implies that $\lambda_s(B(d, k))$ exists and that

$$\lambda_s(B(d, k)) \leq |E^+({\{x, y\}})| = 2d - 2.$$  

On the other hand, since the number of vertices is $d^k$, which satisfies the conditions of Corollary 7.3 for $d \geq 4$ and $k \geq 2$,

$$\lambda_s(B(d, k)) \geq 2d - 2.$$

Thus, $\lambda_s(B(d, k)) = 2d - 2$.

Corollary 7.5 (Fan, Lü and Xu [18, 80]) For the Kautz digraph $K(d, k)$ with $d \geq 3$ and $k \geq 1$, $\lambda_s(K(d, k)) = 2d - 2$.

Proof. Note that $K(d, k)$ contains no loops and that $K(d, k)$ contains $(d + 1)$ pairs of symmetric arcs with no loops at their end-vertices. Choose a pair of symmetric arcs between two vertices, say $x$ and $y$. Since $K(d, k)$ is $d$-connected, thus, $K(d, k) - \{x, y\}$ is strongly connected for $d \geq 3$, which implies that $\lambda_s(K(d, k))$ exists and that

$$\lambda_s(K(d, k)) \leq 2d - 2.$$  

On the other hand, since the number of vertices is $d^k + d^{k-1}$, which satisfies the conditions of Corollary 7.3 for $d \geq 3$ and $k \geq 2$,

$$\lambda_s(K(d, k)) \geq 2d - 2.$$

Thus, $\lambda_s(K(d, k)) = 2d - 2$.