Theory and Application of Graphs

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Kluwer Academic Publishers
Dordrecht/Boston/London

Chapter 1

Basic Concepts of Graphs

1.1 Graph and Graphical Representation

Graph: Mathematically, a **graph** is a mathematical structure on a set of elements with a binary relation. Concretely speaking, a graph is an ordered triple (V, E, ψ), where V and E are two disjoint sets, ψ is a mapping from $E \to V \times V$.

Vertex-set: The set V is nonempty and called the **vertex-set** of the graph, an element in V called a **vertex**.

Edge-set: The set E is called the **edge-set** of the graph, an element in E called an **edge**.

Incidence function: The mapping ψ is called an **incidence function**, which maps an edge into a pair of vertices called **end-vertices** of the edge.

Digraph: If $V \times V$ is considered as a set of ordered pairs (x, y), then the graph is called a **directed graph**, or **digraph** for short. For an edge e of a digraph, sometimes, called a **directed edge** or **arc**, if $\psi(e) = (x, y)$, then the end-vertices x and y are called the **tail** and the **head** of the edge e, respectively; and the edge e is sometimes called an **out-going edge** of x or an **in-coming edge** of y.

Undirected Graph: If $V \times V$ is considered as a set of unordered pairs $\{x, y\}$, then the graph is called an **undirected graph**. Usually, it is customary to henceforth denote an unordered pair of vertices by either xy or yx instead of $\{x, y\}$. Edges of an undirected graph are sometimes called **undirected edges**.

Type of Edges: From definition, it is possible that two end-vertices of an edge are identical, such an edge is called a **loop**. It is also possible that more than one edges are mapped into the same element in $V \times V$ under the mapping ψ, these edges are called **parallel edges** or **multi-edges**. For $x, y \in V(G)$, set

$E_G(x,y) = \{e \in E(G) : \ \psi(e) = (x,y)\}$ and $\mu_G(x,y) = |E(x,y)|$. The parameter $\mu(G) = \max\{\mu_G(x,y) : \ \forall\ x,y \in V(G)\}$ is called the **multiplicity** of G.

Example 1.1.1 $D = (V(D), E(D), \psi_D)$ is a digraph, where

$$\begin{aligned} V(D) &= \{x_1, x_2, x_3, x_4, x_5\}, \\ E(D) &= \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9\} \end{aligned}$$

and ψ_D is defined by

$$\begin{array}{lll} \psi_D(a_1) = (x_1, x_2), & \psi_D(a_2) = (x_3, x_2), & \psi_D(a_3) = (x_3, x_3), \\ \psi_D(a_4) = (x_4, x_3), & \psi_D(a_5) = (x_4, x_2), & \psi_D(a_6) = (x_4, x_2), \\ \psi_D(a_7) = (x_5, x_2), & \psi_D(a_8) = (x_2, x_5), & \psi_D(a_9) = (x_3, x_5). \end{array}$$

In such a digraph D, two edges a_5 and a_6 are parallel edges, but two edges a_7 and a_8 are not. The edge a_3 is a loop.

Example 1.1.2 $H = (V(H), E(H), \psi_H)$ is a digraph, where

$$\begin{aligned} V(H) &= \{y_1, y_2, y_3, y_4, y_5\} \\ E(H) &= \{b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9\} \end{aligned}$$

and ψ_H is defined by

$$\begin{array}{lll} \psi_H(b_1) = (y_1, y_2), & \psi_H(b_2) = (y_3, y_2), & \psi_H(b_3) = (y_3, y_3), \\ \psi_H(b_4) = (y_4, y_3), & \psi_H(b_5) = (y_4, y_2), & \psi_H(b_6) = (y_4, y_2), \\ \psi_H(b_7) = (y_5, y_2), & \psi_H(b_8) = (y_2, y_5), & \psi_H(b_9) = (y_3, y_5). \end{array}$$

Example 1.1.3 $G = (V(G), E(G), \psi_G)$ is an undirected graph, where

$$\begin{aligned} V(G) &= \{z_1, z_2, z_3, z_4, z_5, z_6\} \\ E(G) &= \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\} \end{aligned}$$

and ψ_G is defined by

$$\begin{array}{lll} \psi_G(e_1) = z_1 z_2, & \psi_D(e_2) = z_1 z_4, & \psi_G(e_3) = z_1 z_6, \\ \psi_G(e_4) = z_2 z_3, & \psi_G(e_5) = z_3 z_4, & \psi_G(e_6) = z_3 z_6, \\ \psi_G(e_7) = z_2 z_5, & \psi_G(e_8) = z_4 z_5, & \psi_G(e_9) = z_5 z_6. \end{array}$$

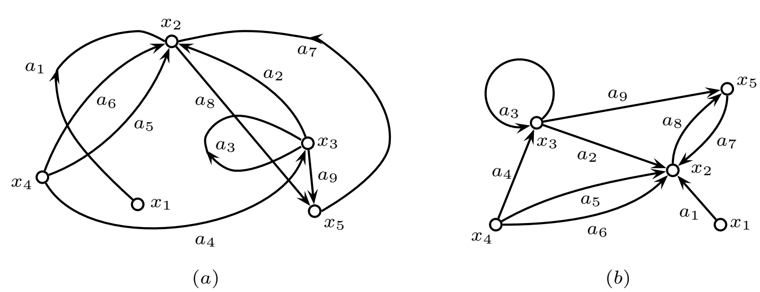
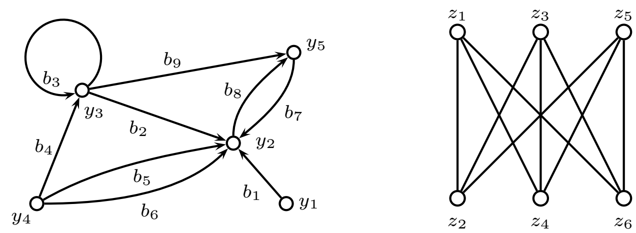


Figure 1.1: Two graphical representations of the digraph D

Graphical Representation A graph can be drawn on the plane. Each vertex x of the graph is indicated by a point, each edge is indicated by a directed line segment or an undirected line segment. Such a geometric diagram is called a **graphical representation** of the graph. It depicts the incidence relationship holding between its vertices and edges intuitively.

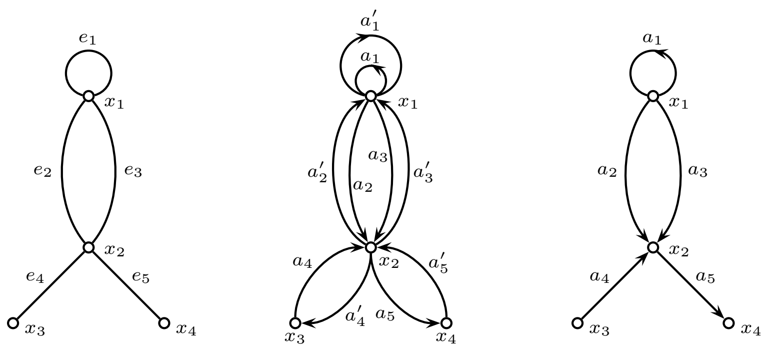


(a) H (b) G

Figure 1.2: Graphical representations of graphs H and G

Incident and Adjacent: The end-vertices of an edge are said to be **incident** with the edge, and vice versa. Two vertices which are incident with a common edge are **adjacent**, as are two edges which are incident with a common vertex.

Simple Graphs: A graph is said to be **loopless** if it contains no loops. A graph is said to be **simple** if it contains neither parallel edges nor loops. For a graph without parallel edges, the mapping ψ is injective. In other words, for each edge e there exists a unique pair of vertices corresponding to the edge. Thus it is convenience to use a subset of $V \times V$ instead of the edge-set E directly. In this case, we may simply write (V, E) for (V, E, ψ). For instance, the graph G defined in Example 1.1.3 is simple, which can be written as $G = (V(G), E(G))$, where $E(G) = \{z_1z_2, z_1z_4, z_1z_6, z_2z_3, z_3z_4, z_3z_6, z_2z_5, z_4z_5, z_5z_6\}$.



(a) (b) (c)

Figure 1.3: The symmetric digraph and oriented graph of an undirected graph

Symmetric Digraphs: An undirected graph can be thought of as a particular digraph, a **symmetric digraph**, in which there are two directed edges called **symmetric edges**, one in each direction, corresponding to each undirected edge. Thus, to study structural properties of graphs for digraphs is more general than for undirected graphs.

Underlying Graphs and Oriented Graph: There are many topics in graph theory that have no relations with direction of edges. The undirected graph obtained from a digraph D by removing the orientation of all edges is called an **underlying graph** of D. Conversely, the digraph obtained from an undirected graph G by specifying an orientation of each edge of G is called an **oriented graph** of G.

Figure 1.3 shows such graphs, where (a) is an undirected graph, (b) and (c) are its symmetric digraph and an oriented graph, respectively.

Others: Let (V, E, ψ) be a graph. The number of vertices, $\upsilon = |V|$, is called **order** of the graph; the number of edges, $\varepsilon = |E|$, is called **size** of the graph. A graph is called to be **empty** if $\varepsilon = 0$. An empty graph is called to be **trivial** if $\upsilon = 1$, and all other graphs **non-trivial**. A graph is **finite** if both υ and ε are finite. Throughout this book all graphs are always considered to be finite.

The letter G always denotes a graph, which is directed or undirected according to the context if it is not specially noted. Sometimes, to emphasize, we use the letter D to denote a digraph. When just one graph is under discussion, the letters υ and ε always denote order and size of the graph, respectively.

The symbols $\lfloor r \rfloor$ and $\lceil r \rceil$ denote the greatest integer not exceeding the real number r and the smallest integer not less than r, respectively. The symbol

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k\,!}$$

denotes the number of k-combinations of n distinct objects ($k \leq n$).

As an application of a graph, we give an example.

Example 1.1.4 In any group of six people, there must be three people who get to either know each other or not.

Proof: We use the points A, B, C, D, E, F on the plane to denote these six people, respectively. We draw a red line joining two points if two people have known each other, a blue line otherwise. Use G to denote the resulting diagram. We only need to prove that G certainly contains either a red triangle or a blue triangle. Consider a point, say F. There exist three lines of the same color which are incident with a common point F. Without loss of generality, we can suppose that they are three red lines FA, FB and FC. Consider the triangle ABC. If it has no red line, then it is a blue triangle; if it has a red line, say AB, then the triangle FAB is red.
$\blacksquare$

1.2 Graph Isomorphism

Isomorphism: Two graphs often have the same structure, differing only in the way their vertices and edges are labelled or the way they are drawn on the plane. To make this idea more exact, we introduce the concept of isomorphism. A graph $G = (V(G), E(G), \psi_G)$ is **isomorphic** to a graph $H = (V(H), E(H), \psi_H)$ if there exist two bijective mappings

$$\theta : V(G) \to V(H) \quad \text{and} \quad \phi : E(G) \to E(H)$$

such that for any $e \in E(G)$,

$$\psi_G(e) = (x, y) \Longleftrightarrow \psi_H(\phi(e)) = (\theta(x), \theta(y)) \in E(H). \tag{1.1}$$

The pair (θ, ϕ) of mappings is called an **isomorphic mapping** from G to H.

Since such two mappings θ and ϕ are bijective, H also isomorphic to G. Thus we often call that G and H are **isomorphic**, write $G \cong H$, the pair (θ, ϕ) of mappings is called an **isomorphism** between G and H.

To show that two graphs are isomorphic, one must indicate an isomorphism between them. For instance, two digraphs D and H defined in Example 1.1.1 and Example 1.1.2, respectively, are isomorphic since the pair of mappings (θ, ψ) is an isomorphism between them, where $\theta : V(D) \to V(H)$ and $\psi : E(D) \to E(H)$ are defined by

$$\begin{array}{ll} \theta(x_i) = y_i, & \text{for each } i = 0, 1, 2, \cdots, 5; \quad \text{and} \\ \psi(a_j) = b_j, & \text{for each } i = 0, 1, 2, \cdots, 9. \end{array}$$

The concept of isomorphism for simple graphs is simple. Two simple graphs G and H are isomorphic if and only if there is a bijection $\theta : V(G) \to V(H)$ such that $(x, y) \in E(G)$ if and only if $(\theta(x), \theta(y)) \in E(H)$. In this case, the condition (1.1) is usually called the **adjacency-preserving condition**.

It is clear that if G and H are isomorphic, then $\upsilon(G) = \upsilon(H)$ and $\varepsilon(G) = \varepsilon(H)$. But the converse is not always true. Generally speaking, to judge whether or not two graphs are isomorphic is quite difficult.

It is easy to see that "to be isomorphic" is an equivalent relation on graphs; hence, this relation divides the collection of all graphs into equivalence classes. Two graphs in the same equivalence classes have the same structure, and differ only in the labels of vertices and edges. Since we are primarily interested in structural properties of graphs, we will identify two isomorphic graphs, and often write $G = H$

for $G \cong H$. We often omit labels when drawing them on the plane; an unlabelled graph can be thought of as a representative of the equivalence class of isomorphic graphs. We assign labels to vertices and edges in a graph mainly for the purpose of referring to them.

Some Special Classes of Graphs: Next, we introduce some special classes of graphs, which frequently occur in our discussion later on.

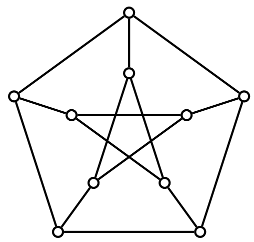
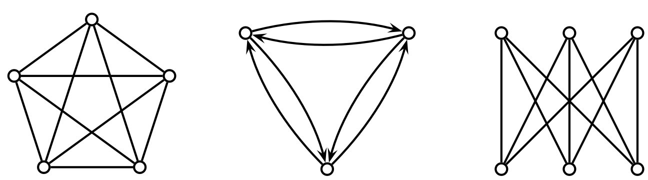


Figure 1.4: Petersen graph

The graph shown in Figure 1.4 is called **Petersen graph**, an interesting graph, which often occurs in the literature and any textbook on graph theory as various counterexamples.



(a) K_5 (b) K_3 (c) $K_{3,3}$

Figure 1.5: (a) K_5, (b) K_3, (c) $K_{3,3}$

A **complete graph** is one in which each ordered pair of distinct vertices is linked by exactly one edge. Up to isomorphism, there is just one complete graph on υ vertices, denoted by K_υ. The graphs shown in Figure 1.5 (a) and (b) are a complete undirected graph K_5 and a complete digraph K_3, respectively. It is clear that

$$\varepsilon(K_\upsilon) = \begin{cases} \upsilon(\upsilon-1) & \text{if } K_\upsilon \text{ is directed;} \\ \frac{1}{2}\,\upsilon(\upsilon-1) & \text{if } K_\upsilon \text{ is undirected.} \end{cases}$$

An oriented graph of a complete undirected graph is called a **tournament**. The reason why we call it the name is that it can be used to indicate the results of games in a round-robin tournament between υ players. A directed edge (x, y) means that the player x has won the player y. Up to isomorphism, the tournament of order one is a trivial graph; there is just one tournament of order two; two tournaments of order three; four tournaments of order four. These not isomorphic tournaments are shown in Figure 1.6.

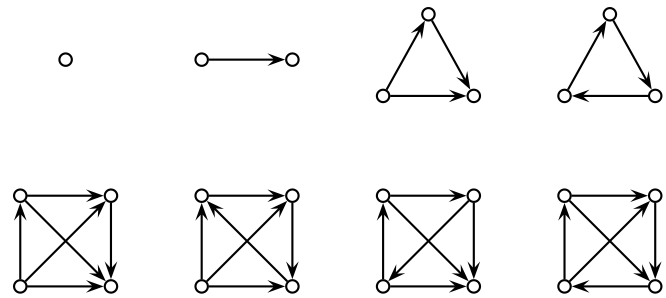


Figure 1.6: Nonisomorphic tournaments of order v for $v = 1, 2, 3, 4$

A **bipartite graph** is one whose vertex-set can be partitioned into two subsets X and Y, so that each edge has one end-vertex in X and another in Y, such a partition $\{X, Y\}$ is called a **bipartition** of the graph. We call a graph to be **equally bipartite** if it is bipartite and has a bipartition with the same number of vertices in each part. We often use the symbol $G(X \cup Y, E)$ to denote a bipartite simple graph $G = (V, E)$ with bipartition $\{X, Y\}$. Similarly, we can define a **k-partite graph** and an equally k-partite graph.

A **complete bipartite graph** is a bipartite simple graph $G(X \cup Y, E)$ in which each vertex of X is joined by exactly one edge to each vertex of Y; if $|X| = m$ and $|Y| = n$, up to isomorphism, such a complete bipartite undirected graph is unique and denoted by $K_{m,\,n}$. The graph shown in Figure 1.5 (c) is $K_{3,\,3}$. It is customary to call $K_{1,\,n}$ a **star**. Usually, write $K_n(2)$ for $K_{n,\,n}$.

Similarly, we can define **complete k-partite graph** and $K_n(k)$.

$$\varepsilon(K_{m,\,n}) = mn \quad \text{and} \quad \varepsilon(K_n(k)) = \tfrac{1}{2}\, k(k-1)n^2.$$

It is also easy to verify that for any bipartite simple graph G of order n,

$$\varepsilon(G) \leq \begin{cases} \frac{1}{4}\, n^2 & \text{if } n \text{ is even;} \\ \frac{1}{4}\, (n^2 - 1) & \text{if } n \text{ is odd.} \end{cases}$$

Bipartite graphs are an important class of graphs. In fact, every digraph corresponds a bipartite undirected graph. Let $D = (V, E, \psi)$ be a digraph, where

$$V(D) = \{x_1, x_2, \cdots, x_v\} \quad \text{and} \quad E(D) = \{a_1, a_2, \cdots, a_\varepsilon\}.$$

Construct an equally bipartite undirected graph $G = (X \cup Y, E(G), \psi_G)$ with

$$\begin{aligned} X &= \{x_1', x_2', \cdots, x_v'\}, \quad Y = \{x_1'', x_2'', \cdots, x_v''\}, \\ E(G) &= \{e_1, e_2, \cdots, e_\varepsilon\}, \text{ where } \psi_G(e_l) = x_i' x_j'' \\ &\iff \text{there is } a_l \in E(D) \text{ such that } \psi_D(a_l) = (x_i, x_j) \\ &\qquad (l = 1, 2, \cdots, \varepsilon). \end{aligned}$$

Such a constructed bipartite undirected graph G is called an **associated bipartite graph** with the digraph D. For instance, the graph G shown in Figure 1.7 (b) is an associated bipartite graph with the digraph D shown in (a). It is clear that

$$\upsilon(G) = 2\,\upsilon(D) \quad \text{and} \quad \varepsilon(G) = \varepsilon(D). \tag{1.2}$$

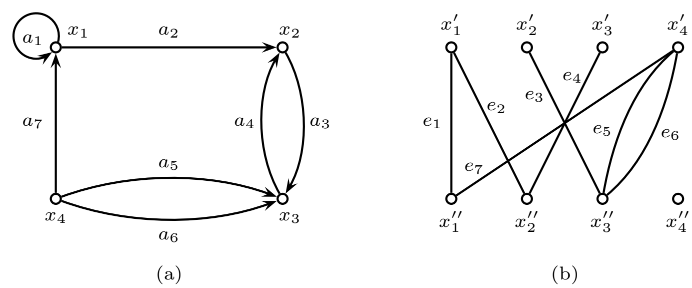


Figure 1.7: A digraph D and its associated bipartite graph G

Example 1.2.1 We construct an equally bipartite simple graph, called ***n*-cube**, or **hypercube**, denoted by $Q_n = (V(Q_n), E(Q_n))$, where

$$V(Q_n) = \{x_1x_2\cdots x_n : \ x_i \in \{0,1\},\ i = 1, 2, \cdots, n\},$$

and two vertices $x = x_1x_2\cdots x_n$ and $y = y_1y_2\cdots y_n$ are linked by an undirected edge if and only if they differ in exactly one coordinate, i.e.,

$$xy \in E(Q_n) \Longleftrightarrow \sum_{i=1}^{n} |x_i - y_i| = 1.$$

The graphs shown in Figure 1.8 are Q_1, Q_2, Q_3 and Q_4.

By definition, Q_n is a simple undirected graph, and has 2^n vertices. We show that Q_n is bipartite. To the end, let

$$\begin{array}{l} X = \{x_1x_2\cdots x_n : \ x_1 + x_2 + \cdots + x_n \equiv 0 \pmod 2\}; \\ Y = \{y_1y_2\cdots y_n : \ y_1 + y_2 + \cdots + y_n \equiv 1 \pmod 2\}. \end{array}$$

Then, by definition, $X \cup Y = V(Q_n)$, $X \cap Y = \emptyset$. Therefore, $\{X, Y\}$ is a bipartition of $V(Q_n)$. We can claim that there is no edge between any two vertices in X. Suppose to the contrary that there exist $x = x_1x_2\cdots x_n$, $x' = x_1'x_2'\cdots x_n' \in X$ such that $xx' \in E(Q_n)$. Then $\sum\limits_{i=1}^{n} |x_i - x_i'| = 1$, namely,

$$|(x_1 + x_2 + \cdots + x_n) - (x_1' + x_2' + \cdots + x_n')| = 1.$$

This contradicts the fact that x and x' are in X. There, therefore, is no edge between any two vertices in X.

Similarly, there is no edge between any two vertices in Y. Therefore, Q_n is a bipartite graph with the bipartition $\{X, Y\}$.

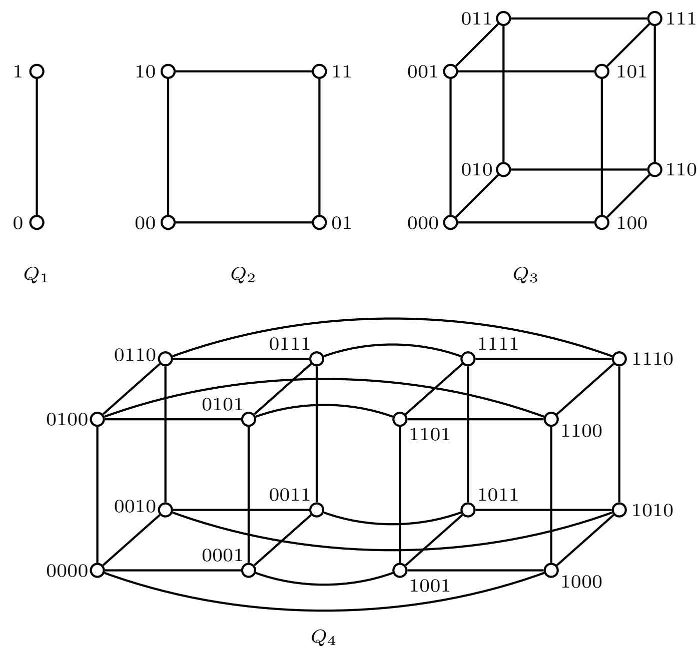


Figure 1.8: The n-cubes Q_n for $n = 1, 2, 3, 4$

Arbitrarily choose $x = x_1x_2 \cdots x_n \in V(Q_n)$. For a vertex $y = y_1y_2 \cdots y_n$ of Q_n, it is adjacent to x if and only if they differ in exactly one coordinate. This means that vertices adjacent to the vertex x have exactly n, that is, edges incident with x have n since Q_n is simple. Let us use E_X (resp. E_Y) to denote the set of edges incident with vertices in X (resp. Y). Then

$$\begin{gathered} n|X| = |E_X| = \varepsilon(Q_n) = |E_Y| = n|Y|, \\ |X| = |Y| = \tfrac{1}{2}\ \upsilon(Q_n) = 2^{n-1} \quad \text{and} \\ \varepsilon(Q_n) = n2^{n-1}. \end{gathered}$$

Example 1.2.2 The symbol $T_{k,\upsilon}$ denotes a complete k-partite graph of order υ in which each part has either $m = \left\lfloor \frac{\upsilon}{k} \right\rfloor$ or $n = \left\lceil \frac{\upsilon}{k} \right\rceil$ vertices. Prove that

(a) $\varepsilon(T_{k,\upsilon}) = \binom{\upsilon - m}{2} + (k-1)\binom{m+1}{2}$;

(b) $\varepsilon(G) \leq \varepsilon(T_{k,\upsilon})$ for any complete k-partite graph G, and the equality holds if and only if $G \cong T_{k,\upsilon}$.

Proof: (a) Let $\upsilon = km + r$, $0 \leq r < k$. Then $r = \upsilon - km$. By the definition of $T_{k,\upsilon}$, we have that

$$\begin{aligned} \varepsilon(T_{k,\upsilon}) = &\ \binom{\upsilon}{2} - r\binom{m+1}{2} - (k-r)\binom{m}{2} \\ = &\ \tfrac{1}{2}\ [\upsilon(\upsilon-1) - rm(m+1) - (k-r)m(m-1)] \\ = &\ \tfrac{1}{2}\ [\upsilon(\upsilon-1) - 2m(\upsilon - km) - km(m-1)] \\ = &\ \tfrac{1}{2}\ (\upsilon - m)(\upsilon - m - 1) + \tfrac{1}{2}\ (k-1)m(m+1) \\ = &\ \binom{\upsilon - m}{2} + (k-1)\binom{m+1}{2}. \end{aligned}$$

(b) Suppose that $G = K_{n_1,\cdots,n_k}$ is a complete k-partite graph with the largest number of edges. Then

$$\varepsilon(G) = \binom{\upsilon}{2} - \sum_{l=1}^{k} \binom{n_l}{2}.$$

If G is not isomorphic to $T_{k,\upsilon}$, then there must exist some i and j $(i < j)$ such that $n_i - n_j > 1$. Consider another complete k-partite graph G', the number of vertices in its k-partition are, respectively,

$$n_1, n_2, \cdots, n_{i-1}, (n_i - 1), n_{i+1}, \cdots, n_{j-1}, (n_j + 1), n_{j+1}, \cdots, n_k.$$

Then

$$\begin{aligned}
\varepsilon(G') &= \binom{\upsilon}{2} - \sum_{l=1\neq i,j}^{k} \binom{n_l}{2} - \binom{n_i - 1}{2} - \binom{n_j + 1}{2} \\
&= \binom{\upsilon}{2} - \sum_{l=1}^{k} \binom{n_l}{2} + (n_i - n_j) - 1 \\
&> \binom{\upsilon}{2} - \sum_{l=1}^{k} \binom{n_l}{2} = \varepsilon(G),
\end{aligned}$$

which contradicts to the choice of G. Thus, $G \cong T_{k,\upsilon}$. ∎

Exercises: 1.1.1, 1.2.4, 1.2.5, 1.2.6