1.7 Circuits and Cycles

A cycle of length \( k \) is called a \( k \)-cycle; a \( k \)-cycle is odd or even according as \( k \) is odd or even. A 3-cycle is often called a triangle.

We will use the word “cycle” to denote a graph or subgraph whose vertices and edges are the terms of a cycle. Usually, we denote a cycle of order \( n \) by \( C_n \), which is undirected or directed according to the context if it is not specified.

**Example 1.7.1** Let \( G \) be an undirected graph with \( \delta \geq 2 \). Then \( G \) contains a cycle certainly. Moreover, if \( G \) is simple, then \( G \) contains a cycle of length at least \( \delta + 1 \).

**Proof:** If \( G \) contains loops or parallel edges, then the conclusion holds clearly. Suppose that \( G \) is simple below and let \( P = (x_0, x_1, \ldots, x_k) \) be a longest path in \( G \). Then all neighbors of \( x_0 \) must lie on \( P \), that is \( N_G(x_0) \subseteq \{x_1, x_2, \ldots, x_k\} \).

Note that \( |N_G(x_0)| = d_G(x_0) \geq \delta(G) = \delta \geq 2 \).

Thus, there exists \( x_i \in N_G(x_0) \), \( \delta \leq i \leq k \). It follows that \( (x_0, x_1, \ldots, x_{i-1}, x_i, x_0) \) is a cycle of length at least \( \delta + 1 \) in \( G \).

The length of a shortest (directed) cycle in a (di)graph \( G \) is called the girth of \( G \), denoted by \( g(G) \).

**Example 1.7.2** If \( G \) be a \( k \)-regular undirected graph with girth \( g \) at least three, then

\[
v(G) \geq \begin{cases} 1 + k + k(k - 1) + \cdots + k(k - 1)^{(g-3)} & \text{if } g \text{ is odd;} \\ 2 \left(1 + (k - 1) + \cdots + (k - 1)^{(g-2)}\right) & \text{if } g \text{ is even.} \end{cases}
\]

**Proof:** It is clear that \( G \) is simple because of \( g \geq 3 \). We first consider the case that \( g \) is odd. Let \( g = 2d + 1 \), \( d \geq 1 \). Choose an \( x \in V(G) \) such that it has largest distance to other vertices. Let

\( J_i(x) = \{y \in V(G) : d_G(x, y) = i\}, 0 \leq i \leq d \).

Clearly, \( |J_0(x)| = 1 \), \( |J_1(x)| = k \). Since \( g = 2d + 1 \), there exists a unique \( xy \)-path of length \( i \) in \( G \) for any \( y \in J_i(x) \) \((1 \leq i \leq d)\), and any two vertices of \( J_i(x) \) are not adjacent in \( G \) for any \( i = 1, 2, \ldots, d - 1 \). This means that

\( |J_i(x)| = k(k - 1)^{i-1}, i = 1, 2 \cdots, d. \)
It follows that
\[
v(G) \geq |J_0(x)| + |J_1(x)| + \cdots + |J_d(x)|
\]
\[
= 1 + k + k(k - 1) + \cdots + k(k - 1)^{d-1}
\]
\[
= 1 + k + k(k - 1) + \cdots + k(k - 1)^{\frac{1}{2}(g-3)}
\]

The case that \( g \) is even can be similarly proved, and the details are left to the reader as an exercise (the exercise 1.7.10 (b)).

We have in Theorem 1.2 seen that every tournament contains a Hamilton directed path. It is perhaps surprising that if a tournament is strongly connected, then it must possess a significantly stronger properties. A (di)graph \( G \) of order \( v (\geq 3) \) is vertex-pancyclic if each vertex of \( G \) is contained in (directed) cycles of all length between 3 and \( v \).

**Theorem 1.5 (Moon, 1966)** Every strongly connected tournament of order \( v (\geq 3) \) is vertex-pancyclic.

**Proof:** We prove the theorem by induction on \( k \geq 3 \). Let \( G \) be a strongly connected tournament and \( u \) be any vertex of \( G \). For \( k = 3 \), let \( S = N^+_G(u) \), \( T = N^-_G(u) \). Then \( S \neq \emptyset \) and \( T \neq \emptyset \) since \( G \) is strongly connected. Moreover, \( T \cup \{u\} = \overrightarrow{S} \) since \( G \) is a tournament. Thus, \( (S,T) = (S,\overrightarrow{S}) \neq \emptyset \) by the exercise 1.5.4 (b). It follows that there exist \( x \in S \) and \( y \in T \) such that \( (x,y) \in E(G) \) and, hence, \( (u,x,y,u) \) is a directed 3-cycle containing \( u \) in \( G \) (see Figure 1.18 (a)).

![Figure 1.18: Two illustrations of the proof of Theorem 1.3](image)

Suppose that \( u \) is contained in directed cycles of all lengths between 3 and \( n \), where \( n < v \). We will prove that \( u \) is contained in a directed \((n+1)\)-cycle.

Let \( C = (u_0,u_1,\cdots,u_{n-1},u_0) \) be a directed \( n \)-cycle, where \( u_0 = u \). If there is some \( x \in V(G) \setminus V(C) \) such that \( N^+_G(x) \cap V(C) \neq \emptyset \) and \( N^-_G(x) \cap V(C) \neq \emptyset \), then there are adjacent vertices \( u_i \) and \( u_{i+1} \) on \( C \) such that \( (u_i,x),(x,u_{i+1}) \in E(G) \). In this case \( u \) is contained in the directed \((n+1)\)-cycle \((u_0,u_1,\cdots,u_i,x,u_{i+1},\cdots,u_{n-1},u_0)\).
Otherwise, let
\[ S = \{ x \in V(G) \setminus V(C) : N^+_G(x) \cap V(C) = \emptyset \} , \]
\[ T = \{ y \in V(G) \setminus V(C) : N^-_G(x) \cap V(C) = \emptyset \} . \]

It is clear that \( S \neq \emptyset \), \( T \neq \emptyset \) and \( (S,T) \neq \emptyset \). Let \( x \in S \) and \( y \in T \) such that \( (x,y) \in E(G) \) (see Figure 1.18 (b)). Thus \( u \) is contained in the directed \((n+1)\)-cycle \((u_0,x,y,u_2,\ldots,u_i,u_{i+1},\ldots,u_{n-1},u_0)\).

**Corollary 1.5** Let \( x \) and \( y \) be two vertices of a strongly connected tournament \( G \) of order \( v \geq 5 \). Then there exists an \((x,y)\)-walk of length exactly \((d+3)\), where \( d \) is diameter of \( G \).

**Proof:** Let \( P \) be a shortest \((x,y)\)-path in \( G \). Since \( 0 \leq d_G(x,y) \leq d \leq v - 1 \), it follows that \( 3 \leq d - d_G(x,y) + 3 \leq v + 2 \).

If \( m = d - d_G(x,y) + 3 \leq v \) then, by Theorem 1.5, there exists a directed \( m \)-cycle \( C_m \) containing \( y \). Thus, \( P \oplus C_m \) is an \((x,y)\)-walk of length \( d_G(x,y) + (d - d_G(x,y) + 3) = d + 3 \).

If \( d - d_G(x,y) + 3 = v + 1 \) then, by Theorem 1.5 and \( v \geq 5 \), there exist a directed 3-cycle \( C_3 \) and a directed \((v - 2)\)-cycle \( C_{v-2} \) containing \( y \). Thus, \( P \oplus C_{v-2} \oplus C_3 \) is an \((x,y)\)-walk of length \( d_G(x,y) + (v - 2) + 3 = d_D(x,y) + v - 2 + 3 = d + 3 \).

If \( d - d_G(x,y) + 3 = v + 2 \) then, by Theorem 1.5, there exist a directed 3-cycle \( C_3 \) and a directed \((v - 1)\)-cycle \( C_{v-1} \) containing \( y \). Thus, \( P \oplus C_{v-1} \oplus C_3 \) is an \((x,y)\)-walk of length \( d + 3 \).

Using the concepts of circuits and cycles, we can give a characterization of a bipartite graph in terms of the parity of cycles (or circuits).

**Theorem 1.6** A strongly connected digraph is bipartite if and only if it contains no odd directed circuit.

**Proof:** Suppose that \( G \) is a bipartite digraph with a bipartition \( \{X,Y\} \) and \( C = x_0e_1x_1 \cdots x_{k-1}e_kx_0 \) is a directed \( k \)-circuit in \( G \). We can, without loss of generality, suppose that \( x_0 \) is in \( X \). Then \( x_1 \in Y \), \( x_2 \in X \), \( x_3 \in Y \), \ldots, \( x_{k-1} \in Y \). Generally, \( x_{2i} \in X \) and \( x_{2i+1} \in Y \). Thus, there exists some \( i \) such that \( k - 1 = 2i + 1 \), which implies that \( k = 2i + 2 \), that is, \( C \) is even.

Conversely, let \( G \) be a strongly connected digraph that contains no odd directed
circuit. Then $G$ has no loops. We choose arbitrarily a vertex $u$ and define a partition $\{X, Y\}$ of $V(G)$ by setting

$$
X = \{x \in V(G) : d_G(u, x) \text{ is even}\}
$$
$$
Y = \{y \in V(G) : d_G(u, y) \text{ is odd}\}
$$

Clearly, $u \in X$. Since $G$ is strongly connected, $Y \neq \emptyset$ if $v \geq 2$. We only need to prove that $\{X, Y\}$ is a bipartition of $G$.

To the end, we first show that $G[Y]$ is empty. The assertion is true if $|Y| = 1$. Suppose $|Y| \geq 2$ below. Arbitrarily choose two distinct vertices $y$ and $z$ in $Y$. Let $P_1$ and $Q_1$ be shortest $(u, y)$-path and $(u, z)$-path, $P_2$ and $Q_2$ be shortest $(y, u)$-path and $(z, u)$-path, respectively, in $G$ (Strong connectedness of $G$ ensures the existence of these four directed paths). Then the length of $P_1$ is odd by definition of $Y$. Note that $P_1 \oplus P_2$ is a directed circuit on $G$ of even length. Therefore the length of $P_2$ is odd. Similarly, the length of $Q_2$ is odd too. Thus, if, to the contrary, there exists a directed edge $e$ joining from $z$ to $y$, then $P_2 \oplus Q_1 \oplus \{e\}$ contains an odd directed circuit, which contradicts to our hypothesis. Similarly, there exists no edges joining from $y$ to $z$. Therefore, no two vertices in $Y$ are adjacent, that is, $G[Y]$ is empty.

By the same argument, we can prove that $G[X]$ is empty too.

In the proof of Theorem 1.6, the strong connectedness of $G$ is no necessary for “only if”, but is necessary for “if”. For instance, the two tournaments of order three shown in Figure 1.6, the former is not strongly connected and contains no directed circuit, but it is not bipartite clearly.

**Corollary 1.6.1** A strongly connected digraph is bipartite if and only if it contains no odd directed cycle.

**Proof:** Suppose that $G$ is a strongly connected bipartite digraph. Then $G$ contains no odd directed circuit by Theorem 1.6. Thus, $G$ contains no odd directed cycle since any odd directed circuit certainly contains an odd directed cycle.

Conversely, suppose that $G$ contains no odd directed cycle. In order to prove that $G$ is bipartite, we only need to show that $G$ contains no odd directed circuit. Suppose to the contrary that $C$ is an odd directed circuit in $G$. Then $C$ is not a directed cycle and it can be expressed as the union of $k (\geq 2)$ edge-disjoint directed cycles $C_1, C_2, \ldots, C_k$, of which at least one is odd, contrary to the hypothesis.

**Corollary 1.6.2** (König, 1936) An undirected graph is bipartite if and only if it contains no odd cycle.
Corollary 1.6.3  A digraph is bipartite if and only if it contains no odd cycle.

The proofs of these two corollaries are left to the reader as exercises (the exercise 1.7.1).

We conclude this section with several examples.

**Example 1.7.3** An undirected graph $G$ has a balanced oriented graph if and only if $G$ contains no vertex of odd degree.

**Proof:** The condition is necessary clearly. By induction on the number of edges $\varepsilon$, we prove the condition is sufficient. If $\varepsilon = 0$ there is nothing to do, and so suppose $\varepsilon > 0$ and the assertion holds for any undirected graph with the number of edges $\varepsilon \leq m$. Let $G$ be an undirected graph without vertices of odd degree and $\varepsilon(G) = m + 1$. Let

$$S = \{x \in V(G) : d_G(x) = 0\},$$

and let $G_1 = G - S$. Then $G_1$ contains no vertex of odd degree and $\delta(G_1) \geq 2$. Thus $G_1$ contains a cycle by Example 1.7.1. Let $C$ be a cycle in $G_1$. Then we obtain a directed cycle $C'$ by assigning each edge in $C$ an orientation whose direction agrees with one of $C$. Let $G_2 = G - E(C)$, then $G_2$ contains no vertex of odd degree and

$$\varepsilon(G_2) = \varepsilon(G) - \varepsilon(G_1) < \varepsilon(G) = m + 1.$$

By the induction hypothesis, there exists a balanced oriented graph $D'$ of $G_2$. It follows that $D = D' \oplus C'$ is a balanced oriented graph of $G$. \hfill $\blacksquare$

**Example 1.7.4** Any strongly connected digraph with odd circuits contains odd directed circuits, and, hence contains odd directed cycles.

**Proof:** The assertion can be easily deduced from Corollary 1.6.3, the detail is left to the reader. We here give a direct proof.

Suppose that $G$ is a strongly connected digraph and

$$C = x_1e_1x_2e_2\cdots x_ie_ixe_{i+1}\cdots x_{2k+1}e_{2k+1}x_1$$

be an odd circuit in $G$, where $x_i \in V(G)$, $e_i \in E(G)$. Use $P_i$ to denote a shortest $(x_i, x_{i+1})$-path for $i = 1, 2, \cdots, 2k$; and $P_{2k+1}$ a shortest $(x_{2k+1}, x_1)$-path in $G$. Strong connectedness of $G$ ensures existence of these directed paths.

If there is some $P_i$ with even length, then $\psi_G(e_i) = (x_{i+1}, x_i)$ by the shortness of $P_i$. Thus, $P_i + e_i$ is an odd directed cycle in $G$.  

Suppose that the length of $P_i$ is odd for every $i = 1, 2, \ldots, 2k + 1$. Let $W = P_1 \oplus P_2 \cdots \oplus P_{2k+1}$. Then $W$ is a closed directed walk of odd length, and it can be expressed as the union of several edge-disjoint directed circuits, of which at least one is odd.

**Example 1.7.5** Let $G$ be a non-bipartite undirected graph. If $G$ is simple and $\varepsilon > \frac{1}{4} (v - 1)^2 + 1$, then $G$ contains a triangle.

![Figure 1.19: An illustration in the proof of Example 1.7.5](image)

**Proof:** Since $G$ is non-bipartite, $G$ contains odd cycle by Corollary 1.6.2. Let $C$ be a shortest cycle of odd length with the set $S$ of $k$ vertices in $G$. Suppose to the contrary that $k$ is at least five since $G$ is simple. We first show that

$$|(S, \overline{S})| \leq 2(v - k). \quad (1.7)$$

Otherwise, there exists some $u \in \overline{S}$ such that $|N_G(u) \cap S| \geq 3$. Let $x, y, z \in N_G(u) \cap S$. Since $G$ contains no triangle, there are $a, b, c \in S \setminus \{x, y, z\}$ and three cycles (see Figure 1.19) $C_1 = (u, x, \cdots, a, \cdots, y, u)$, $C_2 = (u, y, \cdots, b, \cdots, z, u)$ and $C_3 = (u, z, \cdots, c, \cdots, x, u)$, all of their lengths are less than $k$ and at least one is odd, contrary to the choice of $C$. Therefore the inequality (1.7) holds.

By the choice of $C$ we have $G[S] = C$ and, hence, $\varepsilon(G[S]) = k \geq 5$. It follows from (1.7) and Example 1.3.1 that

$$\varepsilon(G) = \varepsilon(G[S]) + |(S, \overline{S})| + \varepsilon(G[S])$$

$$\leq k + 2(v - k) + \frac{1}{4} (v - k)^2 \leq \frac{1}{4} (v - 1)^2 + 1,$$

which contradicts our hypothesis. Therefore, $G$ contains a triangle.

**Exercises:** 1.7.4, 1.7.6, 17.7;