1.8 Eulerian Graphs

Definitions: A (directed) trail that traverses every edge and every vertex of G is called an **Euler (directed) trail**. A closed Euler (directed) trail is called an **Euler (directed) circuit**. A (di)graph is **eulerian** if it contains an Euler (directed) circuit, and **noneulerian** otherwise.

Euler trails and Euler circuits are named after L. Euler (1707–1783), who in 1736 characterized those graphs which contain them in the earliest known paper on graph theory. The well-known Königsberg seven bridges problem is as follows.

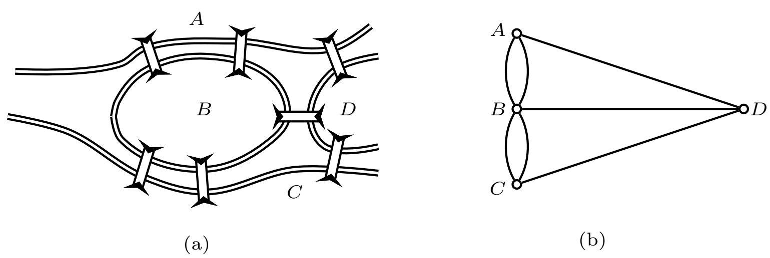


Figure 1.20: The Königsberg bridges and the corresponding graph

Euler abstracted the Königsberg seven bridges as the graph shown in Figure 1.20 (b), where vertices represent the four lands and edges represent the seven bridges, formalized the Königsberg seven bridges problem to the question whether such a graph contains an Euler circuit.

Characteristic Theorem: We now give a characterization of eulerian graphs.

Theorem 1.7 **A digraph is eulerian if and only if it is connected and balanced.**

Proof: Suppose that G is an Euler digraph and let C be an Euler directed circuit of G. Then G is connected since C traverses every vertex of G by the definition. Arbitrarily choose $x \in V(C)$. It each time occurs as an internal vertex of C, two edges are incident with x, one out-going edge and another in-coming edge. Thus, $d^+_G(x) = d^-_G(x)$.

Conversely, suppose to the contrary that a connected and balanced digraph is not eulerian. Choose such a digraph with the number of edges as few as possible. Then G contains directed cycle since $\delta^+ = \delta^- \neq 0$ (the exercise 1.7.3). Let C be a directed circuit of maximum length in G. By our assumption, C is not an Euler directed circuit of G, and so $G - E(C)$ contains a connected component G' with

$\varepsilon(G') > 0$. Since C is itself balanced, thus the connected graph D' is also balanced. Since $\varepsilon(G') < \varepsilon(G)$, it follows from the choice of G that G' contains an Euler directed circuit C'. Since G is connected, $V(C) \cap V(C') \neq \emptyset$. Thus, $C \oplus C'$ is a directed circuit of G with length larger than $\varepsilon(C)$, contradicting the choice of C. ∎

Corollary 1.7.1 **A connected digraph G contains an Euler directed trail from x to y if and only if G satisfies the conditions:**

$$\begin{array}{l} d^+_G(x) - d^-_G(x) = 1 = d^-_G(y) - d^+_G(y); \text{ and} \\ d^+_G(u) = d^-_G(u), \ \forall\ u \in V \setminus \{x, y\}. \end{array} \tag{1.8}$$

Corollary 1.7.2 **An undirected graph is eulerian if and only if it is connected and has no vertex of odd degree.**

Corollary 1.7.3 **An undirected graph G contains an Euler trail if and only if G is connected and contains at most two vertices of odd degree.**

Applications: We conclude this section with two important classes of eulerian digraphs, the de Bruijn digraphs and the Kautz digraphs, which occur in the literature and textbooks of graph theory frequently.

Example 1.8.1 For any integers d and n with $n \geq 1$ and $d \geq 2$, we have, in Example 1.4.4, defined the n-dimensional d-ary **de Bruijn digraph** $B(d, n)$ as $(n-1)$th iterated line digraph $L^{n-1}(K_d^+)$, where K_d^+ denotes the digraph obtained from a complete digraph K_d of order d by appending one loop at each vertex. We have known that it is a d-regular connected digraph and so an eulerian digraph by Theorem 1.7.

We here give another equivalent definition of de Bruijn digraph $B(d, n)$. Its vertex-set

$$V = \{x_1 x_2 \cdots x_n : \ x_i \in \{0, 1, \cdots, d-1\}, \ i = 1, 2, \cdots, n\},$$

and edge-set E, where for $x, y \in V$, if $x = x_1 x_2 \cdots x_n$, then

$$(x, y) \in E \Longleftrightarrow y = x_2 x_3 \cdots x_n \alpha, \quad \alpha \in \{0, 1, \cdots, d-1\}.$$

Three smaller de Bruijn digraphs $B(2, n)$ are depicted in Figure 1.21 according to this kind of definition. ∎

Example 1.8.3 For any given integers d and n with $n \geq 1$ and $d \geq 2$, we have, also in Example 1.4.4, defined the n-dimensional d-ary **Kautz digraph** $K(d, n)$ as $(n-1)$th iterated line digraph $L^{n-1}(K_{d+1})$, where K_{d+1} is a complete

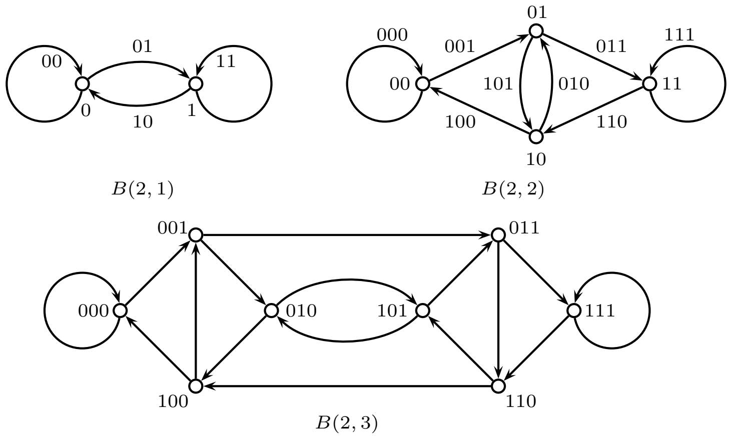


Figure 1.21: de Bruijn digraphs $B(2,1)$, $B(2,2)$ and $B(2,3)$

digraph of order $d+1$. We have known that it is a d-regular connected digraph and so an eulerian digraph by Theorem 1.7.

We here give another equivalent definition of Kautz digraph $K(d,n)$. Its vertex-set

$$V = \{x_1 x_2 \cdots x_n : \ x_i \in \{0,1,\cdots,d\},\ x_{i+1} \neq x_i,\ i = 1,2,\cdots,n-1\},$$

and the edge-set E, where for $x, y \in V$, if $x = x_1 x_2 \cdots x_n$, then

$$(x,y) \in E \Longleftrightarrow y = x_2 x_3 \cdots x_n \alpha, \quad \alpha \in \{0,1,\cdots,d\} \setminus \{x_n\}.$$

Three smaller Kautz digraphs $K(d,n)$ are depicted in Figure 1.22.

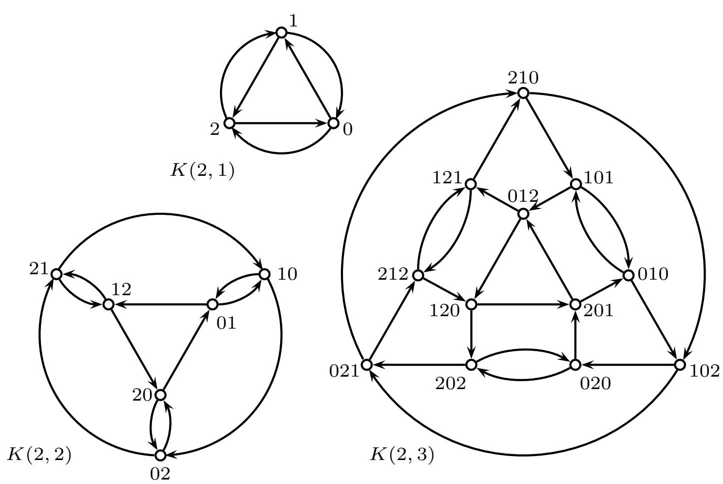


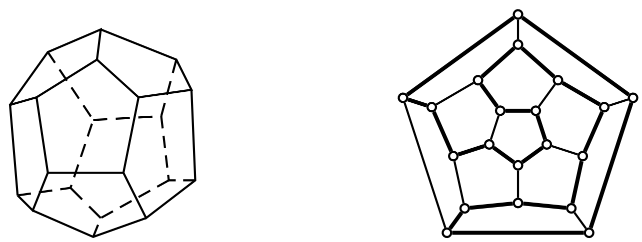
Figure 1.22: Kautz digraphs $K(2,1)$, $K(2,2)$ and $K(2,3)$

Both de Bruijn digraph $B(d,n)$ and Kautz digraph $K(d,n)$ are d-regular and have diameter n. Moreover, $B(d,n)$ has d^n vertices, while $K(d,n)$ has $d^n + d^{n-1}$ vertices. ∎

1.9 Hamiltonian Graphs

Definitions: A (directed) cycle that contains every vertex of a (di)graph G is called a **Hamilton (directed) cycle**. A (di)graph is **hamiltonian** if it contains a Hamilton (directed) cycle, and **non-hamiltonian** otherwise. The problem determining whether a given graph is hamiltonian is called the **Hamilton problem**.

The name "hamiltonian" is derived from Sir William Rowan Hamilton (1805-1865), a well-known Irish mathematician. In 1857, Hamilton introduced a game known as "A Voyage Round the World".



(a) (b)

Figure 1.23: The dodecahedron and the corresponding graph

The dodecahedron can be expressed as a graph shown in Figure 1.23 (b). A solution of the Hamilton's game is illustrated by the heavy edges in Figure 1.23 (b).

Although the Hamilton problem is quite difficult to solve, it is easy to show the following result.

Example 1.9.1 The Petersen graph is non-hamiltonian.

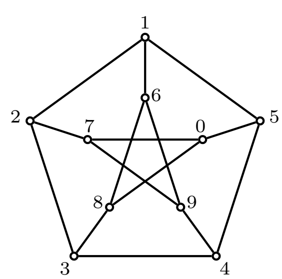


Figure 1.24: The Petersen graph is non-hamiltonian

Proof: Suppose that G is the Petersen graph, and suppose to the contrary that G is hamiltonian. We label the vertices of G with the digits $1, 2, \cdots, 9, 0$ shown as Figure 1.24. Let $T = \{16, 27, 38, 49, 50\}$ be the subset of edges of G. Then $G - T$ is disconnected. Thus, any Hamilton cycle of G must contain even edges in T. It is not difficult to see that any cycle containing exactly two edges in T is not hamiltonian.

It follows that every Hamilton cycle of G must contain four edges in T. Without loss of generality, suppose that C is a Hamilton cycle containing the edges $27, 38, 49, 50$. Then C must contain the edges $12, 15, 68, 69$, and does not contain the edges $23, 45$. Since the vertices 3 and 4 are contained in C, thus, the edge 34 must be contained in C. It follows that the set of edges $\{34, 49, 96, 68, 83\}$ can form a cycle that is contained in C, which is impossible. Therefore, Petersen graph is non-hamiltonian.

∎

A Relation to Line Graphs:

A digraph **G** is Eulerian $\Leftrightarrow$ **L(G)** is hamiltonian.

$\Leftarrow$ does not hold for undirected graphs, for example, a star $K_{1,3}$.

Necessary Conditions: An obvious and simple necessary condition is that any hamiltonian digraph must be strongly connected; any hamiltonian undirected graph must contains no cut-vertex.

Theorem 1.8 **If G is hamiltonian, then**

$$\omega(G - S) \leq |S|, \quad \forall\ S \subset V. \tag{1.9}$$

Proof: Let C be a Hamilton cycle of G. Then

$$\omega(C - S) \leq |S|, \quad \forall\ S \subset V(C) = V(G).$$

Since $C - S$ is a spanning subgraph of $G - S$,

$$\omega(G - S) \leq \omega(C - S) \leq |S|, \quad \forall\ S \subset V$$

as required. ∎

Example 1.9.2 **The Petersen graph shows that the condition (1.9) is not sufficient.** To the end, let G be Petersen graph. It is not difficult to verify that

$$\omega(G - S) \begin{cases} = 1 \leq |S|, & \text{for } |S| \leq 2; \\ \leq 2 < |S|, & \text{for } |S| = 3; \\ \leq 3 < |S|, & \text{for } |S| = 4; \\ \leq 5 \leq |S|, & \text{for } |S| \geq 5, \end{cases} \qquad \forall\ S \subset V(G).$$

The implies that the Petersen graph satisfies the condition (1.9), however, it has been prove to be non-hamiltonian in Example 1.9.1. ∎

Sufficient Conditions: Since parallel edges and loops do not affect whether a graph is hamiltonian, it suffices to limit our discussion to simple graphs.

Theorem 1.9 **Let G be a simple undirected graph of order $v \geq 3$. If**

$$d_G(x) + d_G(y) \geq v, \text{ for any } x, y \in V(G), xy \notin E(G), \tag{1.10}$$

then G is hamiltonian.

Proof: By contradiction. Suppose that the theorem is false, and let G be a maximal non-hamiltonian simple graph satisfying the condition (1.10). Since $v \geq 3$, G is not a complete graph. Let x and y be nonadjacent vertices in G. By the choice of G, $G + xy$ is hamiltonian. Moreover, since G is non-hamiltonian, each Hamilton cycle of $G + xy$ must contain the edge xy. Thus there is a Hamilton path connecting x and y in G. Let $P = (x_1, x_2, \cdots, x_v)$ be a Hamilton path, where $x = x_1$ and $y = x_v$. Set

$$S = \{x_i \in V(P) : \; xx_{i+1} \in E(G), \; 1 \leq i \leq v - 2\},$$
$$T = \{x_j \in V(P) : \; yx_j \in E(G), \; 2 \leq j \leq v - 1\}.$$

Since G is simple and $y \notin S \cup T$, we have

$$|S| = d_G(x), \; |T| = d_G(y) \quad \text{and} \quad |S \cup T| \leq v - 1.$$

We now show that $|S \cap T| = 0$. In fact, if there is some $x_i \in S \cap T$, then $C = (x, x_{i+1}, x_{i+2}, \cdots, x_{v-1}, y, x_i, \cdots, x_2, x)$ is a Hamilton cycle in G (see Figure 1.25), contrary to our assumption. It follows that

$$d_G(x) + d_G(y) = |S| + |T| = |S \cup T| \leq v - 1.$$

But this contradicts the condition (1.10), and so G is hamiltonian. ∎

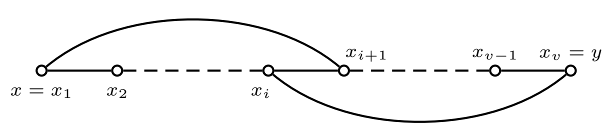


Figure 1.25: An illustration the proof of Theorem 1.9

Corollary 1.9 (Dirac, 1952) **Every simple graph with $v \geq 3$ and $\delta \geq \frac{1}{2} v$ is hamiltonian.** ∎

Intractability: The problem determining if a given graph is hamiltonian is of NP-completeness.

Exercises: 1.8.2, 1.8.3; 1.9.3, 1.9.7