Chapter 3

Plane Graphs and Planar Graphs

3.1 Plane Graphs and Euler's Formula

Surface Embedding of Graphs

Let S be a given surface such as the plane, the sphere, the torus and so on. If a graph G can be drawn in S such that its edges intersect only at their end-vertices, then G is said to be **embeddable on the surface S**. Such a drawing of G in S is called an **embedding** of G in S, denoted by $\widetilde{G}$.

Planar Graphs and Plane Graphs

If a graph G is embeddable on the plane (or the sphere), G is called a **planar graph**; otherwise G is called a **non-planar graph**. If G is a planar graph, then any embedding $\widetilde{G}$ of G on the plane can itself be regarded as a graph isomorphic to G. Therefore, we refer to an embedding $\widetilde{G}$ of G as a **plane graph**.

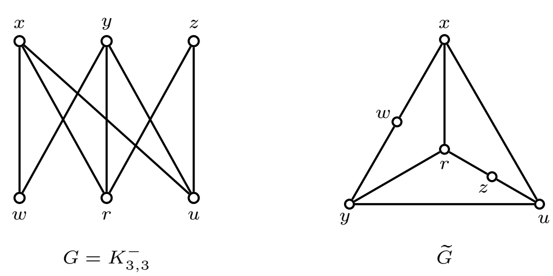

Figure 3.1: $K^-_{3,3}$ and its planar embedding

Figure 3.1 shows a planar graph $K^-_{3,3}$, obtained from $K_{3,3}$ by deleting any one edge, and its embedding on the boundary of a tetrahedron. Such an embedding of $K^-_{3,3}$ will be useful in the proof of Theorem 3.6 in the next section.

Since the concept of embedding of a graph has no relation to orientations of edges, in the following discussions, we will restrict ourselves to undirected graphs. Furthermore, we consider the surface S as the plane or the sphere in this chapter. In fact, we have the following result.

Theorem 3.1 A graph G is embeddable on the sphere S if and only if it is embeddable on the plane P.

Proof: To show this theorem we make use of a mapping known as stereographic projection. Consider a sphere S resting on a plane P, and denoted by z the point of S that is diagonally opposite the point of contact of S and P. The mapping

$$\phi: \ S \to P,$$

defined by $\phi(z) = \infty$ and

$$\phi(s) = p \in P \setminus \{\infty\} \text{ for any } s \in S \setminus \{z\} \Leftrightarrow \ z, s, p \text{ are collinear,}$$

see Figure 3.2, is bijective clearly.

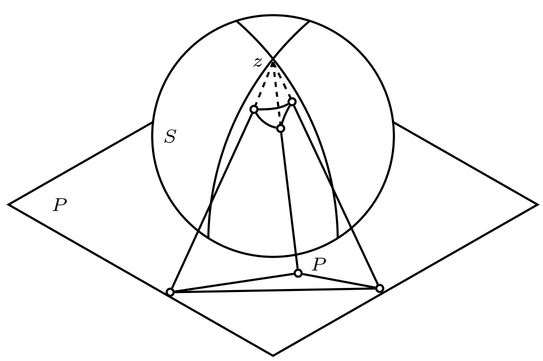

Figure 3.2: Stereographic projection

Suppose that G is embeddable on the plane P and $\widetilde{G}$ is its embedding in P. Then $\phi^{-1}(\widetilde{G})$ is an embedding of G in the sphere S. Conversely, suppose that $\widetilde{G'}$ is an embedding of G in S. Without loss of generality, suppose that z is not in $\widetilde{G'}$. Then $\phi(\widetilde{G'})$ is an embedding of G in P. Thus, G is embeddable in S if and only if it is embeddable on P. ∎

Faces of Plane Graphs

Let G be a nonempty plane graph. It can partition the plane into several connected regions, which are called **faces**. We use $F(G)$ and $\phi(G)$ to denote the set and the number of faces of G, respectively.

It is clear that $\phi(G) \geq 1$ for any plane graph G, and $\phi(G) = 1$ if and only if G is a forest.

For G shown in Figure 3.3, for example, we have

$$F(G) = \{f_0, f_1, f_2, f_3, f_4, f_5\} \quad \text{and} \quad \phi(G) = |F(G)| = 6.$$

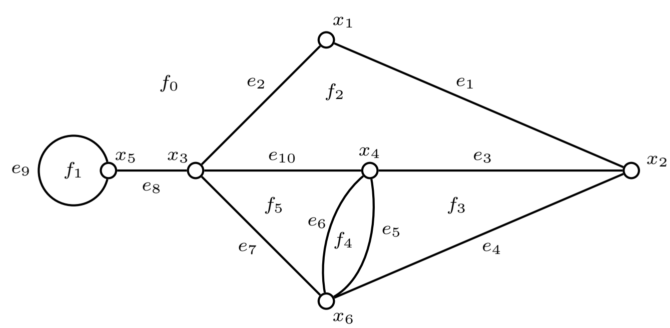

Figure 3.3: The faces of a plane graph

Denote by $B_G(f)$ the **boundary** of the face $f \in F(G)$, in general, **which consists of several edge-disjoint closed walks**.

For example, the face f_0 of G shown in Figure 3.3 has boundary

$$B_G(f_0) = x_1\, e_2\, x_2\, e_4\, x_6\, e_7\, x_3\, e_8\, x_5\, e_9\, x_5\, e_8\, x_3\, e_1\, x_1.$$

The number of edges in $B_G(f)$ is the **degree** of f, denoted by $d_G(f)$. For G shown in Figure 3.3, for example, we have $d_G(f_0) = 7$, $d_G(f_1) = 1$.

Any planar embedding of a planar graph has exactly one unbounded face, called the **exterior face**; in the plane graph of Figure 3.3, f_0 is the exterior face.

For any vertex x or any edge e of a planar graph G, G can be embedded in the plane in such a way that x or e is on the boundary of the exterior face of the embedding (the exercise 3.1.2).

For a given plane graph G, there is the following relations between the face degrees and $\varepsilon(G)$, similar to one between the vertex degrees and $\varepsilon(G)$ (see Corollary 1.1.1).

Theorem 3.2 **For any plane graph G,**

$$\sum_{f \in F(G)} d_G(f) = 2\varepsilon(G).$$

Proof: If G is empty, then the conclusion holds clearly. Suppose now that G is nonempty, and e is any edge of G. Then e either is on a common boundary of two distinct faces (for example, the edge e_1 of the graph shown in Figure 3.3 is on the boundary of f_0 and f_2) or appears in a boundary of some face twice (for example, the edge e_8 of the graph shown in Figure 3.3 appears on the boundary of f_0 twice). Thus the conclusion follows. ∎

There is a simple formula relating to the numbers of vertices v, the number of edges ε and the number of faces ϕ of a connected plane graph. It is the well-known **Euler's formula**.

Theorem 3.3 (Euler, 1753) **If G is a connected plane graph, then**

$$v - \varepsilon + \phi = 2.$$

Proof: Let G be a connected plane graph and T be a spanning tree of G. Then $\phi(T) = 1$ and $\varepsilon(\overline{T}) = \varepsilon - v + 1$. On the one hand, addition of each edge of $\overline{T}$ to T, the number of faces increases by at least one by Theorem 2.3, which implies $\phi(G) \geq \phi(T) + \varepsilon - v + 1$. On the other hand, to obtain a new face, one edge of $\overline{T}$ must be added to T, which implies $\phi(G) \leq \phi(T) + \varepsilon - v + 1$. Thus

$$\phi(G) = \phi(T) + \varepsilon - v + 1 = \varepsilon - v + 2,$$

as required. ∎

Corollary 3.3.1 **If G is a plane graph, then $v-\varepsilon+\phi=1+\omega$.**

Corollary 3.3.2 **All planar embeddings of a given connected planar graph have the same number of faces.**

Corollary 3.3.3 **If G is a simple connected planar bipartite graph of order $v\,(\geq 3)$, then $\varepsilon \leq 2v-4$.**

Proof: Let $\widetilde{G}$ be a planar embedding of G. If $\widetilde{G}$ is a tree, then by Theorem $\varepsilon = v-1 \leq 2v-4$ for $v \geq 3$. Suppose that $\widetilde{G}$ contains a cycle below. Since G is a simple bipartite graph, then by Corollary 1.6.2, G contains no odd cycle and so $d_{\widetilde{G}}(f) \geq 4$ for each face f of $\widetilde{G}$. It follows from Theorem 3.2 that

$$4\phi \leq \sum_{f\in F(\widetilde{G})} d_{\widetilde{G}}(f) = 2\varepsilon,$$

that is, $\varepsilon \geq 2\phi$. It follows from Euler's formula that $\varepsilon \leq 2v-4$. ∎

Corollary 3.3.4 **$K_{3,3}$ is non-planar.**

Proof: Since $K_{3,3}$ is simple and bipartite, $\varepsilon(K_{3,3})=9$ and $v(K_{3,3})=6$. Suppose to the contrary that $K_{3,3}$ is planar, then we can deduce from Corollary 3.3.3 a contradiction as follows.

$$9=\varepsilon(K_{3,3}) \leq 2\,v(K_{3,3})-4=8.$$

Therefore, $K_{3,3}$ is non-planar. ∎

Maximal Planar graphs

A simple planar graph G is called to be **maximal** if $G+xy$ is non-planar for any two nonadjacent vertices x and y of G. It is clear that each face of any planar embedding of a maximal planar graph is a triangle. A planar embedding of a maximal planar graph is called a **plane triangulation**.

Theorem 3.4 **Let G be a simple planar graph of order $v \geq 3$. Then G is maximal if and only if $\varepsilon = 3v - 6$.**

Proof: Let G be a simple planar graph of order $v \geq 3$ and $\widetilde{G}$ be a planar embedding of G. Then it is clear that $\widetilde{G}$ is maximal if and only if $d_{\widetilde{G}}(f) = 3$ for any $f \in F(\widetilde{G})$. It follows from Theorem 3.2 that

$$2\varepsilon = \sum_{f \in F(\widetilde{G})} d_{\widetilde{G}}(f) = 3\,\phi.$$

By Euler's formula, we have

$$v - \varepsilon + \frac{2}{3}\varepsilon = 2,$$

as desired. ∎

Corollary 3.4.1 **If G is a simple planar graph of order $v \geq 3$, then $\varepsilon \leq 3v - 6$.**

Corollary 3.4.2 **K_5 is non-planar.**

Proof: If K_5 is planar, then, by Corollary 3.4.1, we should have

$$10 = \varepsilon(K_5) \leq 3\,v(K_5) - 6 = 9.$$

But this is impossible. Thus, K_5 is non-planar. ∎

Corollary 3.4.3 **If G is a simple planar graph, then $\delta \leq 5$.**

Proof: The conclusion is clearly true for $v = 1$ or 2. For $v \geq 3$, by Corollary 1.1 and Corollary 3.4.1, we have

$$\delta\, v \leq \sum_{x \in V} d_G(x) = 2\varepsilon \leq 6v - 12,$$

which implies that $\delta \leq 5$. ∎

Planar Embedding with Straight Line Segments

The following feature of planar graphs is found by Wagner (1936) and, rediscovered by Fáry (1948).

Theorem 3.5 Any simple planar graph can be embedded in the plane so that each edge is a straight line segment.

Proof: Omitted.

Figure 3.4 shows a planar graph and its planar embedding with straight line segments.

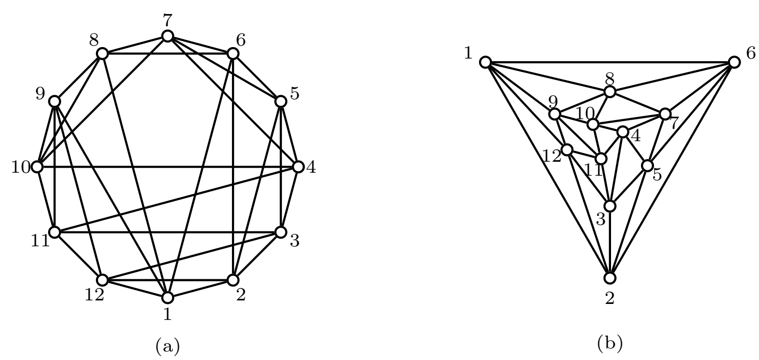

Figure 3.4: a planar graph and its planar embedding with straight line segments

3.2 Kuratowski's Theorem

It is clearly of importance to know which graphs are planar and which are not. In the preceding section we obtain some necessary conditions for a graph to be planar. Making use of these conditions we have already shown that, in particular, both K_5 and $K_{3,3}$ are non-planar. We will, in this section, see that these two non-planar graphs play an important role in the characterization of planarity of a graph.

A remarkably simple, useful criteria for graphs to be planar was found in 1930 by Kuratowski and Frink and Smith, independently. This criteria is called **Kuratowski's theorem** in the literature and textbooks on graph theory.

Before stating and proving Kuratowski's theorem, we need to describe other concepts on graphs.

An edge e is said to be **subdivided** when it is deleted and replaced by a single path of length two connecting its end-vertices of e, the internal vertex of this single path being a new vertex. This is illustrated in Figure 3.5.

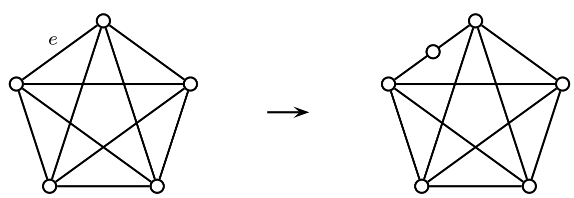

Figure 3.5: Subdivision of an edge e of K_5

A **subdivision** of a graph G is a graph obtained from G by a sequence of edge subdivisions.

Figure 3.6 illustrates two subdivisions of $K_{3,3}$.

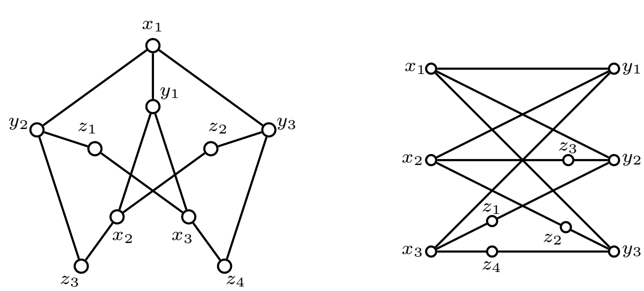

Figure 3.6: Two subdivisions of $K_{3,3}$

Theorem 3.6 A graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$ as its subgraph.

For this classical theorem, there are many simpler proofs than the original. The first relatively simple proof was given by Dirac and Schuster (1954), and some of other proofs have been given in Thomassen's paper(1981), Klotz (1989) and Makarychev (1997). A discussion of its history, the reader is referred to Kennedy, Quintas and Syslo (1985). The proof presented here is due to Tverberg (1989).

Proof: Omitted.

As a direct consequence of Theorem 3.6, we have immediately that Petersen graph is non-planar since it contains the subdivision of $K_{3,3}^-$ shown in Figure 3.6.

There are several other characterizations of planar graphs. For example (the exercise 3.2.4),

Wagner (1937) proved that **a graph is planar if and only if it contains no subgraph contractible to K_5 or $K_{3,3}$**;

McLane (1937) proved that **a graph is planar if and only if it has a fundamental cycles together with one additional cycle such that this collection of cycles contains each edge of the graph exactly twice**.

Another well-known characterization of planar graphs, due to Whitney(1932), concerns with the concept of dual graphs, which will be presented in the next section.

3.3 Dual Graphs

Geometric Dual

Let G be a plane graph with the edge-set $\{e_1, e_2, \cdots, e_\varepsilon\}$ and the face-set $F(G) = \{f_1, f_2, \cdots, f_\phi\}$. We can define a graph G^* with vertex-set $V(G^*) = \{f_1^*, f_2^*, \cdots, f_\phi^*\}$ and the edge-set $\{e_1^*, e_2^*, \cdots, e_\varepsilon^*\}$, and two vertices f_i^* and f_j^* are linked by an undirected edge e_i^* if and only if e_i is on a common boundary of two faces f_i and f_j of G. The graph G^* is called the **geometric dual** of G.

A plane graph G and its geometric dual G^* are shown in Figure 3.7, where G is depicted by the light lines and G^* by the heavy lines.

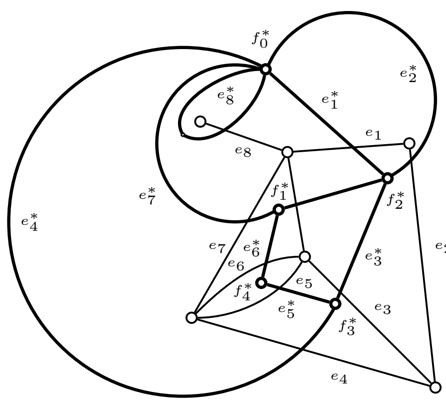

Figure 3.7: A plane graph and its geometric dual

It is a simple observation that the geometric dual G^* of a plane graph G is planar and satisfies the following relations:

$$\begin{cases} v(G^*) = & \phi(G), \\ \varepsilon(G^*) = & \varepsilon(G), \\ d_{G^*}(f^*) = & d_G(f), \quad \forall\ f \in F(G). \end{cases} \tag{3.1}$$

It should be noted that isomorphic plane graphs may have non-isomorphic geometric duals.

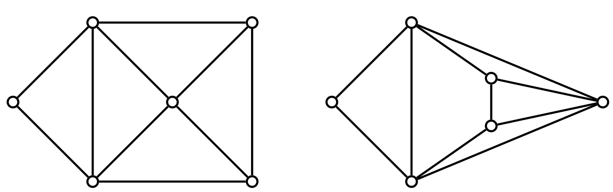

Figure 3.8: Two isomorphic plane graphs with non-isomorphic geometric duals

It is easy to prove that G^* **is a connected plane graph, and** $G \cong G^{*^*}$ $\Leftrightarrow$ **if and only if** G **is connected.** (the exercise 3.3.1)

Theorem 3.7 **Let** G **be a plane graph and** G^* **the geometric dual of** G**,** $B \subseteq E(G)$ **and** $B^* = \{e^* \in E(G^*) :\ e \in B\}$**. Then**

(a) $G[B]$ **is a cycle of** G **if and only if** B^* **is a bond of** G^***;**

(b) B **is a bond of** G **if and only if** $G^*[B^*]$ **is a cycle of** G^***.**

Proof: Omitted.

Combinatorial Dual

Motivated by the facts in Theorem 3.7, Whitney (1932) formulated an abstract notion of duality for general graphs, combinatorial dual of a graph.

Let G and G' be two graphs. If there is a bijective mapping $\varphi :\ E(G) \to E(G')$ such that for any $B \subseteq E(G)$, $G[B]$ is a cycle of G if and only if $\varphi(B) = \{e' \in E(G') :\ \varphi(e) = e',\ e \in B\}$ is a bond of G', then G' is called the **combinatorial dual** of G.

Figure 3.9 shows a graph G and its combinatorial dual G', where

$$\begin{aligned} \varphi :\ E(G) &\to E(G') \\ e_i &\mapsto \varphi(e_i) = e'_i,\ i = 1, 2, \cdots, 9. \end{aligned}$$

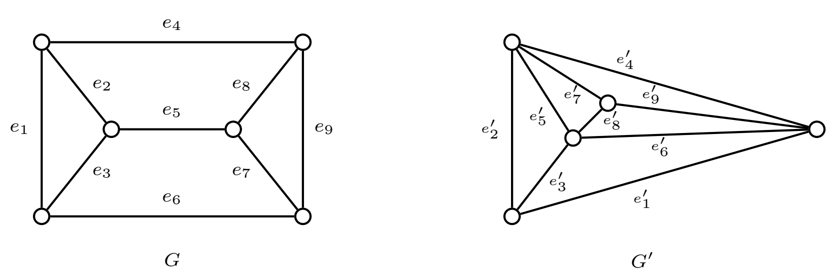

Figure 3.9: A graph and its combinatorial dual

Although, in general, it is difficult to find the combinatorial dual of a given graph, the combinatorial definition coincides with the geometric definition for plane graphs

Theorem 3.8 **Let G be a plane graph and G^* is its geometric dual. Then G^* is the combinatorial dual of G. Moreover, G is the combinatorial dual of G^*.**

Proof: Omitted

We have noticed the definition of combinatorial dual makes no reference to planarity of a graph. With this concept, however, Whitney (1932) obtained another characterization of planar graphs.

Theorem 3.9 (Whitney's theorem) **A graph G is planar if and only if it has combinatorial dual.**

Proof: Omitted.

Applications

3.4 Regular Polyhedra

Theorem 3.11 There are exactly five regular polyhedra. (see Figure 3.10)

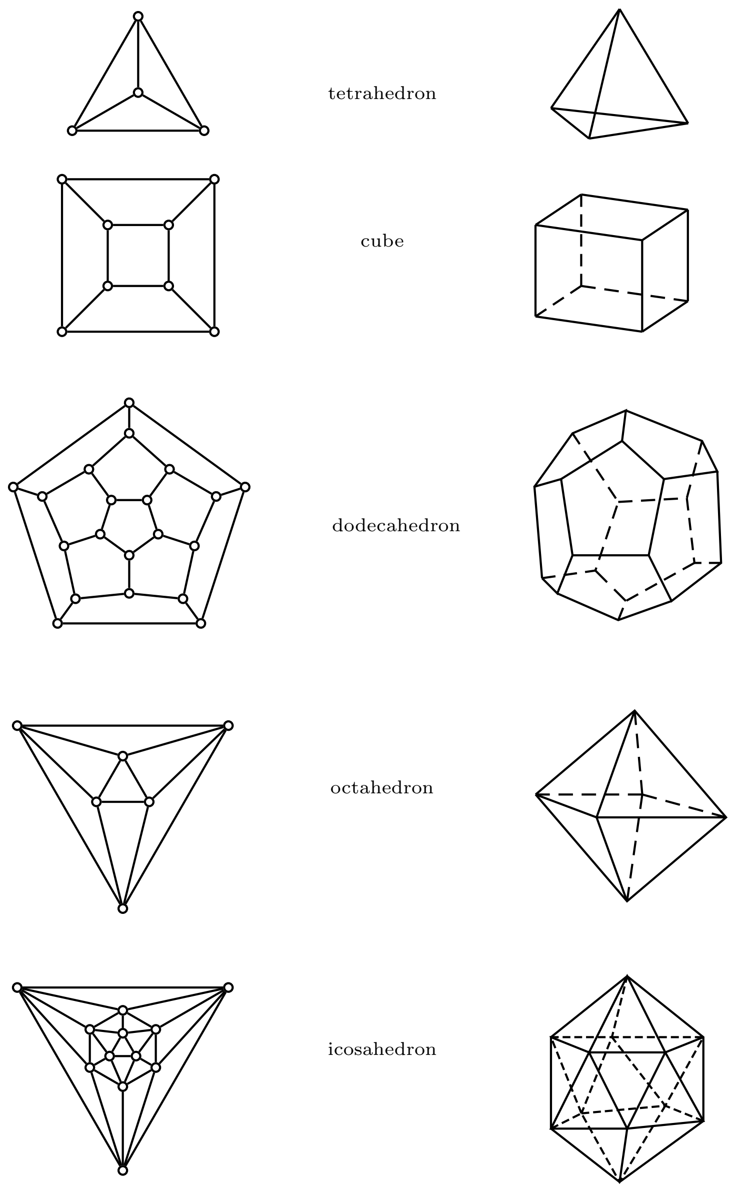

Figure 3.10: The regular polyhedra and the corresponding plane graphs

The five regular polyhedra were known to the ancient Greeks, and were described by Plato in his **Timaeus ca.** 350 BC, and so are called **platonic solids**

3.5 Layout of Printed Circuits

There are many practical situations in which it is important to decide whether a given graph is planar, and if so, to then find a planar embedding of the graph. For example, a **VLSI (very large scale integrated)**-designer has to place the cells on printed circuit boards according to several designing requirements. One of these requirements is to avoids crossings since crossings lead to undesirable signals. One is, therefore, interested in knowing if the graph corresponding to a given electrical network is planar, where the vertices correspond to electrical cells and the edges correspond to the conductor wires connecting the cells.

Several different $O(v)$-algorithms for solving this problem have been proposed by different authors, for example, Hopcroft, Tarjan (1974) and Liu (1988) who used different techniques. These algorithms require lengthy explanations and verification. We therefore in this section describe a much simpler but nevertheless fairly efficient algorithm due to Demoucron, Malgrange and Pertuiset (1964), **DMP algorithm** for short.

3.1.16 The **thickness** $\vartheta(G)$ **of** G, is the minimum number of planar graphs into which the edges of G can be partitioned. **It is clear that** $\vartheta(G) = 0$ **if and only if** G **is planar.**

3.1.17 The **crossing number** $r(G)$ **of** G, is the minimum number of pairwise intersections of its edges when G is drawn in the plane. **Obviously,** $r(G) = 0$ **if and only if** G **is planar**.

Exercises: 3.1.4; 3.1.6; 3.2.3

Thank You !

平面图小结：

一、平面图的研究起源于四色猜想。

1852 年，英国青年学生 Francis Guthrie 在给英国地图染色时发现 4 种颜色就足够了。地图可以看成平图，行政区域就是这个平图的面。给地图上的每个行政区域染色，就是给这个平图的每个面染色。于是，形成四色猜想：任何平图的面是 4 色可染的。

平面图是拓扑图论的重要研究内容。L. Euler 于 1753 年发现了 Euler 公式而成为拓扑图论的奠基人。接着中断了 170 多年。1930 年，当波兰数学家 C. Kuratowski 和美国数学家 O. Frink & P. A. Smith 发现了平面图判定准则(定理 3.2)后，这方面的研究才开始复苏。

上世纪 50 年代，我国著名数学家吴文俊(1955) 基于代数拓扑学中的上同调理论发现了图的平面性判定准则。W. Tutte(1970)基于实域上链群的理论也发现一个判定准则。我国刘彦佩(1988)证明这两个判定准则从二元域 GF(2)上的空间理论来看是同一的。

二、掌握平面图与平图的概念和基本结果。

Euler公式，面数是平面图不变量，最大边数为$3\nu-6$，最小度不超过5。平面图的判断（Kuratowski定理）。

三、平面图判断算法。

平面图用于大规模集成电路板的设计激发了对平面图的研究。值得指出的是，吴文俊(1973，1974)利用代数拓扑的方法把平面性判定问题转化成模 2 代数方程组的求解问题，得到判定平面性的$O(v^6)$算法。刘彦佩(1978, 1979, 1988)把判定图的平面性问题转化为在辅助图上求支撑树问题，改进了吴文俊的结果，也得到一个线性算法。由吴文俊和刘彦佩所创立的方法，被欧洲组合学杂志三主编之一的 P. Rorenstiehl(1980)称为“吴--刘判别准则”和“吴--刘定理”。

非平面图和图的曲面嵌入是拓扑图论研究的重要内容之一. 本章没有涉及它，有兴趣的读者可参阅刘彦佩的专著《图的可嵌入性理论》(1995)和《Topological Theory on Graphs》（2008）。