5.5 The Travelling Salesman Problem

The Travelling Salesman Problem

A travelling salesman, starting in his own town, has to visit each of towns where he should go to precisely or at least once and return his home by the shortest route. This is known as the **travelling salesman problem**.

Contrast with the Chinese Postman Problem

At first glance, this problem is extremely similar to the Chinese postman problem. However, in contrast with the Chinese postman problem, no efficient algorithm for solving the travelling salesman problem is known as far. We will here describe an algorithm of Christofides (1976) for approximately solving the travelling salesman problem, following a comprehensive treatment of Gibbons (1985).

Graphic Models-Two Definitions

Use a connected undirected weighted graph $(G, \mathbf{w})$ to model the traffic system that the travelling salesman has to visit. We call a closed walk that contains each vertex of G at least once to be a **salesman route**. In the graph theoretic language, **the travelling salesman problem can be stated as finding a minimum weight Hamilton cycle or a minimum weight salesman route in $(G, \mathbf{w})$**. Such a Hamilton cycle, if exists, is called an **optimal cycle**; such a salesman route, which exists certainly, is called an **optimal route**.

In general, **the two definitions of the travelling salesman problem may have two different solutions even if optimal cycles exist**. For example, for the weighted graph $(G, \mathbf{w})$ in Figure 5.15, it clearly contains an optimal cycle (x, y, z, x) of weight 5 and an optimal route (x, z, x, y, x) of weight 4. **As his travelling route, any wise salesman would choose the latter rather than the former.**

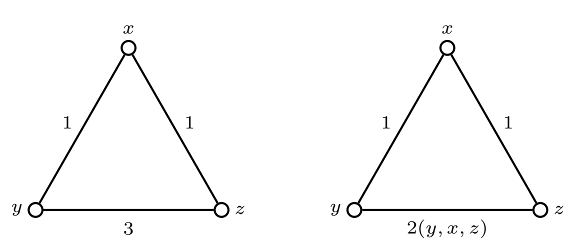


Figure 5.15: (a) An optimal cycle; (b) an optimal route

Generally speaking, **a connected weighted graph contains no optimal cycle perhaps, but contains an optimal route certainly**. In the following discussion of the travelling salesman problem, **we always adopt the second definition, that is, to find an optimal route in a given weighted graph**.

Let $(G, \mathbf{w})$ be a connected undirected weighted graph. If for any two distinct vertices x and y of G, the weighted distance $\mathbf{w}$(x,y), the minimum-weight of any xy-path in $(G, \mathbf{w})$, satisfies

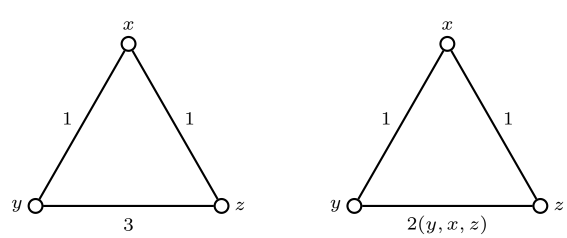
$$\mathbf{w}(x, y) \leq \mathbf{w}(x, z) + \mathbf{w}(z, y), \quad \forall\, z \in V(G) \setminus \{x, y\},$$

then we will say that **the triangle inequality is satisfied in $(G, \mathbf{w})$**.

Weighted Complete Graphs

For an undirected graph G, we can construct a weighted complete graph $(K_v, \mathbf{w}')$ with vertex-set as the same as G, where the weight $\mathbf{w}'(xy)$ of the edge xy of K_v is the weighted distance $\mathbf{w}(x, y)$ between x and y in $(G, \mathbf{w})$. It is clear that the triangle inequality is satisfied in $(K_v, \mathbf{w}')$, and each edge xy of K_v corresponds to an xy-path P in G with $\mathbf{w}(P) = \mathbf{w}'(xy)$ or an edge xy in G with $\mathbf{w}(xy) = \mathbf{w}'(xy)$.

For example, see Figure 5.15, $(K_3, \mathbf{w}')$ in (b) is constructed from $(G, \mathbf{w})$ in (a) by the above way.



Basic Theory-Relationships

Lemma 5.12 (i) For any Hamilton cycle C in $(K_v, \mathbf{w}')$, there exists a salesman route R in $(G, \mathbf{w})$ with $\mathbf{w}(R) = \mathbf{w}'(C)$.

(ii) For any optimal route R in $(G, \mathbf{w})$, there exists a Hamilton cycle C in $(K_v, \mathbf{w}')$ with $\mathbf{w}'(C) = \mathbf{w}(R)$.

Proof: (i) A required salesman route R in $(G, \mathbf{w})$ can be constructed as follows. For $xy \in E(C)$,

if $xy \in E(G)$, then $\mathbf{w}(xy) = \mathbf{w}'(xy)$, and set $xy \in R$;

if $xy \notin E(G)$, then G contains an xy-path P with $\mathbf{w}(P) = \mathbf{w}'(xy)$, and set $P \subseteq R$.

(ii) A required Hamilton cycle in K_v can be constructed as follows. Trace R, starting at a vertex x, and delete the vertex that has visited before in turn. Then remaining vertices, in the original order in R, institute a Hamilton cycle C in $(K_v, \mathbf{w}')$, which satisfies $\mathbf{w}'(C) = \mathbf{w}(R)$. ∎

Theorem 5.12 An optimal route R in $(G, \mathbf{w})$ corresponds to an optimal cycle C in $(K_v, \mathbf{w}')$ with $\mathbf{w}'(C) = \mathbf{w}(R)$, and vice versa.

Proof: Suppose that R is an optimal route in $(G, \mathbf{w})$. By Lemma 5.12(ii), there exists a Hamilton cycle C in $(K_v, \mathbf{w}')$ with $\mathbf{w}'(C) = \mathbf{w}(R)$. If C is not optimal, let C^* be an optimal cycle in $(K_v, \mathbf{w}')$, then $\mathbf{w}'(C^*) < \mathbf{w}'(C)$. Thus, by Lemma 5.12(i), there exists a salesman route R' in $(G, \mathbf{w})$ such that $\mathbf{w}(R') = \mathbf{w}'(C^*)$. Thus,

$$\mathbf{w}(R) \leq \mathbf{w}(R') = \mathbf{w}'(C^*) < \mathbf{w}'(C) = \mathbf{w}(R).$$

This contradiction implies that C is an optimal cycle in $(K_v, \mathbf{w}')$.

Conversely, suppose that C is an optimal cycle in $(K_v, \mathbf{w}')$. Then, by Lemma 5.12(i), there exists a salesman route R in $(G, \mathbf{w})$ with $\mathbf{w}(R) = \mathbf{w}'(C)$. If R is not optimal, and let R' be an optimal route in $(G, \mathbf{w})$. Then $\mathbf{w}(R') < \mathbf{w}(R)$. By Lemma 5.12(ii), there is a Hamilton cycle C' in $(K_v, \mathbf{w}')$ with $\mathbf{w}'(C') = \mathbf{w}(R')$. Thus,

$$\mathbf{w}'(C') = \mathbf{w}(R') < \mathbf{w}(R) = \mathbf{w}'(C) \leq \mathbf{w}'(C').$$

This contradiction implies that R be an optimal route in $(G, \mathbf{w})$. ∎

Transform of Problem

By Theorem 5.12, **finding an optimal route in $(G, \mathbf{w})$ can be referred to finding an optimal cycle in $(K_v, \mathbf{w}')$ within which the triangle inequality is satisfied**.

If the triangle inequality is satisfied in $(G, \mathbf{w})$, then G is a spanning subgraph of K_v, and $\mathbf{w} = \mathbf{w}'|E(G)$.

Thus, if C is an optimal cycle in $(G, \mathbf{w})$, then C is certainly a Hamilton cycle in $(K_v, \mathbf{w}')$, and $\mathbf{w}(C) = \mathbf{w}'(C)$. Conversely, suppose that C is an optimal cycle in $(K_v, \mathbf{w}')$. If $C \subseteq G$, then C is an optimal cycle in $(G, \mathbf{w})$; if $C \nsubseteq G$, then, by Theorem 5.12, there is an optimal route R in $(G, \mathbf{w})$ such that $\mathbf{w}(R) = \mathbf{w}'(C)$.

This shows that **if the triangle inequality is satisfied in $(G, \mathbf{w})$, then an optimal cycle in $(K_v, \mathbf{w}')$ corresponds to either an optimal cycle or an optimal route in $(G, \mathbf{w})$**.

However, **if the triangle inequality is not satisfied in $(G, \mathbf{w})$, then an optimal cycle in $(K_v, \mathbf{w}')$ would correspond to an optimal route rather than an optimal cycle in $(G, \mathbf{w})$**.

Thus, **we only need to find an optimal cycle in $(K_v, \mathbf{w}')$ within which the triangle inequality is satisfied**.

Intractability: NP-completeness

An immediately obvious method is to enumerate all Hamilton cycles and then by comparison to find the minimum. This approach, although straightforward, presents us with an unacceptably large amount of computation. For a complete undirected graph K_v, there are $\frac{1}{2}(v-1)!$ essentially different Hamilton cycles. Unfortunately, as far no efficient algorithm is known for finding an optimal cycle in a weighted complete graph. In fact, **it has been proved to be an NP-hard problem** (see, for example, Garey and Johnson (1979)), even if weight of every edge is restricted to one and two, see Papadimitriou and Yannakakis (1993).

Necessarity of Approximation Algorithms

For the travelling salesman problem, as indeed for any other intractable problem, it is useful to have a polynomial-time algorithm which will produce, within known bound, an approximation to the required result. Such algorithms are called **approximation algorithms**. We now describe the best one in these known algorithms for solving the travelling salesman problem, discovered by Christofides in 1976.

Christofides' Approximation Algorithm

1. Find the weighted distance matrix $\mathbf{W}'$ of $(G, \mathbf{w})$ and construct $(K_v, \mathbf{w}')$.

2. Find a minimum tree T in $(K_v, \mathbf{w}')$.

3. Find the set V' of vertices of odd degree in T and a minimum weight perfect matching M in $G' = K_v[V']$.

4. Find an Euler circuit $C_0 = (x, y, z, \cdots, x)$ in $G^* = T \oplus M$.

5. Starting at vertex x, we trace C_0 and delete the vertex that has visited before in turn. Then remaining vertices, in the original order in C_0, determine a Hamilton cycle C in K_v, which is a required approximation optimal cycle.

In the algorithm,

the Dijsktra's (see Section 2.5) algorithm can be used in Step 1;

the Prim's algorithm (see Section 2.4) can be used in Step 2.

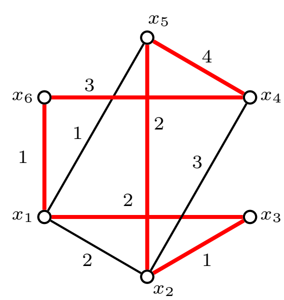
In Step 3, V' is nonempty certainly, and $|V'|$ is even by Corollary 1.1. Since $G' = K_v[V']$ is a complete graphs of even order, it must contain a perfect matching M by Corollary 5.2.1. Edmonds and Johnson (1970) have presented an efficient algorithm for finding minimum weight perfect in any weighted graph.

In Step 4, every vertex of G^* is even degree and, hence, G^* is eulerian by Corollary 1.7.2. Using the Edmonds and Johnson's algorithm can find an Euler circuit C_0 in G^*.

All of the above-mentioned algorithms are efficient, thus, Christofides' approximation algorithm is efficient.

Example 5.5.1 Consider the weighted graph $(G, \mathbf{w})$ in Figure 5.16 (a), within which the triangle inequality is satisfied. (b) shows its weighted distance matrix $\mathbf{W}'$; (c) shows $(K_6, \mathbf{w}')$ and a minimum tree T. $V' = \{x_2, x_3, x_4, x_6\}$ is the set of vertices of odd degree in T, $M = \{x_2x_3, x_4x_6\}$ is a minimum weight perfect matching of $K_6[V']$. $G^* = T \oplus M$, shown in (d), has an Euler circuit $C_0 = (x_1, x_5, x_2, x_3, x_2, x_4, x_6, x_1)$. Deleting a repeated vertex x_2 from C_0 results in a Hamilton cycle $C = (x_1, x_5, x_2, x_3, x_4, x_6, x_1)$ in $(K_6, \mathbf{w}')$ with $\mathbf{w}'(C) = 12$. Because the edge x_3x_4 of C is not in G, C corresponds a salesman route $P = (x_1, x_5, x_2, x_3, x_2, x_4, x_6, x_1)$ with $\mathbf{w}(P) = 12$ which visits each vertex of G at least once.

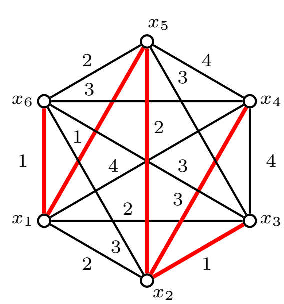
Notice that G contains only one Hamilton cycle C shown in (a), and so is optimal. However, $\mathbf{w}(C) = 13 > 12 = \mathbf{w}(P)$. ∎



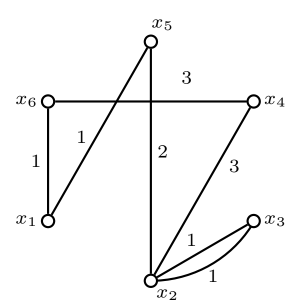
(a) $(G, \mathbf{w})$ and an optimal cycle C

$$\mathbf{W}' = \begin{pmatrix} 0 & 2 & 2 & 4 & 1 & 1 \\ 2 & 0 & 1 & 4 & 2 & 3 \\ 2 & 1 & 0 & 4 & 3 & 3 \\ 4 & 3 & 4 & 0 & 4 & 3 \\ 1 & 2 & 3 & 4 & 0 & 2 \\ 1 & 3 & 3 & 3 & 2 & 0 \end{pmatrix}$$

(b) the weighted distance matrix of $(G, \mathbf{w})$



(c) $(K_6, \mathbf{w}')$ and a minimum tree



(d) $G^* = T \oplus M$

Figure 5.16: An application of Christofides' algorithm

Performance Ratio of Approximation Algorithms

A quality measurement of an approximation algorithm is the **performance ratio**. Let L be a value obtained by an approximation algorithm and L_0 be an exact value. We require a quality guarantee for the approximation algorithm which could, for a minimization (resp. maximization) problem, be stated in the form:

$$\boxed{1 \leq \frac{L}{L_0} \leq \alpha \quad (\text{resp. } 1 \leq \frac{L_0}{L} \leq \alpha).} \tag{5.10}$$

We would, of course, like α to be as close to one as possible. For an approximation algorithm, if there exists a constant α such that (5.10) holds, then the approximation algorithm is called an **α-approximation algorithm**.

Theorem 5.13 **Christofides' algorithm is a $\frac{3}{2}$-approximation algorithm.**

Proof: Suppose that the triangle inequality is satisfied in a weighted complete graph $(K_v, \mathbf{w})$, C_0 and C are an Euler circuit and a Hamilton cycle, respectively, obtained by Christofides' approximation algorithm. Then

$$\mathbf{w}(C) \leq \mathbf{w}(C_0) = \mathbf{w}(T) + \mathbf{w}(M), \tag{5.11}$$

where T is a minimum spanning tree of $(K_v, \mathbf{w})$, M is a minimum weight perfect matching in $G' = K_v[V']$, and V' is the set of vertices of odd degree in T.

Let C^* be an optimal cycle in $(K_v, \mathbf{w})$ and T' be a spanning tree of K_v obtained by deleting any one edge from C^*. Then

$$\mathbf{w}(T) \leq \mathbf{w}(T') < \mathbf{w}(C^*) \tag{5.12}$$

Let C' be a Hamilton cycle in $G' = K_v[V']$ obtained by following C^*. Because the triangle inequality is satisfied in $(K_v, \mathbf{w})$, $\mathbf{w}(C') \leq \mathbf{w}(C^*)$. Since C' is an even cycle, the set of edges of C' can be divided into two edge-disjoint perfect matchings M_1 and M_2. Without loss of generality, suppose $\mathbf{w}(M_1) \leq \mathbf{w}(M_2)$. Thus, M_1 is a perfect matching in G' and

$$\mathbf{w}(M) \leq \mathbf{w}(M_1) \leq \frac{1}{2}\mathbf{w}(C') \leq \frac{1}{2}\mathbf{w}(C^*). \tag{5.13}$$

By (5.11), (5.12) and (5.13), we have

$$\mathbf{w}(C) < \mathbf{w}(C^*) + \frac{1}{2}\mathbf{w}(C^*) = \frac{3}{2}\mathbf{w}(C^*),$$

that is,

$$\frac{L_0}{L} = \frac{\mathbf{w}(C)}{\mathbf{w}(C^*)} < \frac{3}{2}$$

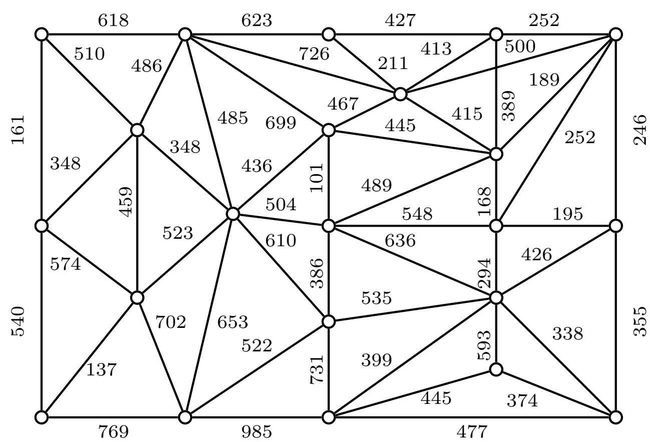
as desired. ∎

Remarks

No such an approximation algorithm for the travelling salesman problem is as far found whose performance ratio is smaller than one of Christofides' algorithm. Although this algorithm can efficiently solve one class of travelling salesman problem, Sahni and Gonzalez (1976) have proved that unless the NP-complete problems have solutions in polynomial-time, there is no algorithm with a constant of performance ratio for the optimal cycle problem in $(K_v, \mathbf{w})$ within which the triangle inequality is not satisfied.

There is a great volume of literature associated with the travelling salesman problem. See, for example, Bellmore and Nemhauser (1968) for a survey of earlier works and Lawler *et al.* (1985) for more.

Exercise 5.5.4: Solve the travelling salesman problem in the following traffic system (the minimum weight is 8117)(cited from Mathematics Today, 1978).



(Exercise 5.5.5)