Chapter 6

Coloring Theory

Origin of Coloring Theory

1. Color of Maps: In the history of graph theory, the problems involving the coloring of graphs have received considerable attention – mainly because of one problem, the four-color problem proposed in 1852: whether four colors will be enough to color the countries of any map so that no two countries which have a common boundary are assigned the same color. Since more than 150 years, in the process of attempt at the four-color problem, one has greatly developed and enriched coloring theory of graphs.

2. Graph Theory Self: How many is the least number of independent subsets (resp. matchings) into which the vertex-set (resp. edge-set) of the graph can be partitioned. In view of this, the coloring theory provided in this chapter is a continuation and extensions of theory concerning independent sets and matchings.

3. Applications: Apart from its own theoretical interest, the study of coloring of graphs is also motivated by its increasing importance in applications of the real-world problems. Unfortunately, as far no efficient algorithm is known for solving these problems.

In this chapter, we will introduce basic concepts of vertex-coloring and edge-coloring of a graph and two graphic parameters, chromatic number and edge-chromatic number, closely related to the two types of colorings. We will present **two classical results on coloring theory of graphs, Brooks' theorem and Vizing's theorem. We will also present the equivalence of certain problems concerning vertex-coloring and edge-coloring with the four-color problem by Tait's theorem**.

6.1 Vertex Colorings

Definitions

Let G be a loopless graph. A **k-vertex coloring** of G is an assignment of k colors, $1, 2, \cdots, k$, to the vertices of G such that adjacent vertices are assigned different colors. In other words, a k-vertex coloring of G is a mapping

$$\pi: \ V(G) \to \{1, 2, \cdots, k\}$$

such that for each $i = 1, 2, \cdots, k$,

$$V_i = \{x \in V(G): \ \pi(x) = i\}$$

is an independent set of G or an empty set. The subset V_i is called a **color class** of π. We often write $\pi = (V_1, V_2, \cdots, V_k)$ for a k-vertex coloring.

The concept of coloring bears **no relation to orientations of edges, loops and parallel edges**. In discussing vertex colorings, therefore, we will restrict ourselves to simple undirected graphs. Figure 6.1 illustrates a 3-vertex coloring of C_5 and a 3-vertex coloring of Petersen graph, respectively.

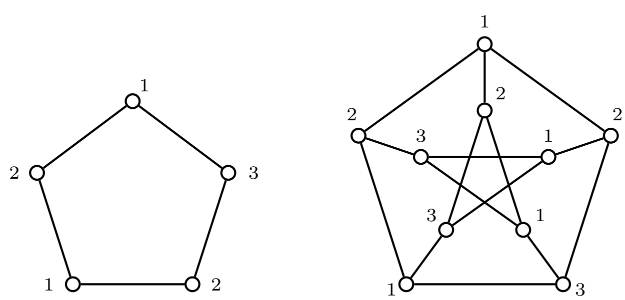

Figure 6.1: Two illustrations of 3-vertex colorings

G is said to be **k-vertex-colorable** if it has a k-vertex coloring. The (**vertex-**) **chromatic number**

$$\chi(G) = \min\{k: \ G \text{ is } k-\text{vertex}-\text{colorable}\}.$$

From definition, for a loopless graph G, if $\chi(G) = k$ then each color class V_i of a k-vertex coloring $\pi = (V_1, V_2, \cdots, V_k)$ is a nonempty independent set of G and $\{V_1, V_2, \cdots, V_k\}$ is a partition of $V(G)$. In other words, the chromatic number of G may be defined alternatively as the minimum number k of independent subsets into which $V(G)$ can be partitioned. Each such independent set is then a color class in the k-vertex coloring of G so defined.

Applications

A number of real-world problems show that determining chromatic number is of great importance. For example, suppose that a school assigns end-of-term examinations for several subjects for its students. It is imperative, of course, that two subjects should not be scheduled at the same time if some student is to attend both subjects. Furthermore, it is more efficient to minimize the number of periods used for examinations. This situation can be represented by a simple undirected graph G whose vertices represent the subjects and two vertices are linked by an undirected edge if and only if there is at least one student who is to attend both of the corresponding subjects. The least number of the periods required is then $\chi(G)$.

Simple Examples

For several special classes of graphs, the chromatic number is quite easy to determine. For example,

$\chi(G) = 1 \iff G \cong K_v^c$;

$\chi(G) = 2 \iff G$ is a nonempty bipartite graph;

$\chi(G) = v \iff G \cong K_v$, and

$\chi(C_{2n+1}) = 3$ for $n \geq 1$.

Critical k-chromatic Graphs

A graph G is called to be **k-chromatic** if $\chi(G) = k$. A graph G is said to be **critical k-chromatic** if $\chi(G) = k$ and $\chi(H) < k$ for every proper subgraph H of G.

Simple Examples

K_v^c is 1-chromatic;

a nonempty bipartite graph is 2-chromatic and

an odd cycle is 3-chromatic and

K_v is v-chromatic.

G is critical 1-chromatic $\iff G \cong K_1$;

G is critical 2-chromatic $\iff G \cong K_2$;

G is critical 3-chromatic $\iff G \cong K_3$.

Basic Properties of Critical k-chromatic graphs

Clearly, from definition, any critical k-chromatic graph is **simple** certainly.

Generally, no necessary and sufficient condition for a graph to be k-chromatic or critical k-chromatic has been known so far. However, it is clear that **any k-chromatic graph contains a critical k-chromatic subgraph**.

Theorem 6.1 $\lambda(G) \geq k-1$ **for any critical** $k\,(\geq 2)$**-chromatic graph** G**.**

Proof: Suppose that G is a critical k-chromatic graph with $k \geq 2$. If $k = 2$ then theorem holds clearly since, in this case, $G \cong K_2$, and so $\lambda(G) = 1$. Suppose below $k \geq 3$ and suppose to the contrary that $\lambda(G) < k-1$. There exists a subset $S \subset V(G)$ such that $|[S, \overline{S}]| = \lambda(G) < k-1$. Since G is critical k-chromatic, both $G_1 = G[S]$ and $G_2 = G[\overline{S}]$ are $(k-1)$-vertex-colorable. Suppose that

$$\pi_1 = (U_1, U_2, \cdots, U_{k-1}) \quad \text{and} \quad \pi_2 = (W_1, W_2, \cdots, W_{k-1})$$

are $(k-1)$-vertex colorings of G_1 and G_2, respectively. Construct a bipartite simple graph H with bipartition $\{X, Y\}$ as follows. $X = \{x_1, x_2, \cdots, x_{k-1}\}$ and $Y = \{y_1, y_2, \cdots, y_{k-1}\}$, $x_i y_j \in E(H) \Longleftrightarrow E_G(U_i, W_j) = \emptyset$. It follows from $|E_G(S, \overline{S})| = \lambda(G) < k-1$ that

$$\varepsilon(H) > (k-1)^2 - (k-1) = (k-1)(k-2).$$

By Corollary 5.3 of König's theorem, H contains a perfect matching, say $M = \{x_i y_{j_i} : \ i = 1, 2, \cdots, k-1\}$. Thus, $V_i = U_i \cup W_{j_i}$ is an independent set of G for each $i = 1, 2, \cdots, k-1$. Therefore, $\pi = (V_1, V_2, \cdots, V_{k-1})$ is a $(k-1)$-vertex coloring of G, which contradicts the hypothesis of $\chi(G) = k$. Thus, $\lambda(G) \geq k-1$, and theorem follows. ∎

Corollary 6.1.1 $\delta(G) \geq k-1$ **for any critical** k**-chromatic graph** G**.**

Proof: It is immediate from Theorem 4.4 and Theorem 6.1. ∎

Corollary 6.1.2 $\chi(G) \leq \Delta(G) + 1$ **for any simple graph** G**.**

Proof: Suppose that $\chi(G) = k$ and H is a critical k-chromatic subgraph of G. By Corollary 6.1.1, $\delta(H) \geq k-1$. Thus

$$\Delta(G) \geq \Delta(H) \geq \delta(H) \geq k-1 = \chi(G) - 1,$$

and the corollary follows. ∎

Sequential Coloring Algorithm

A heuristic algorithm for Δ-vertex coloring of a graph is called **sequential coloring**.

Example 6.1.1 Consider the Grótzsch graph G, shown in Figure 6.2 We can give a 4-vertex coloring by the sequential coloring.

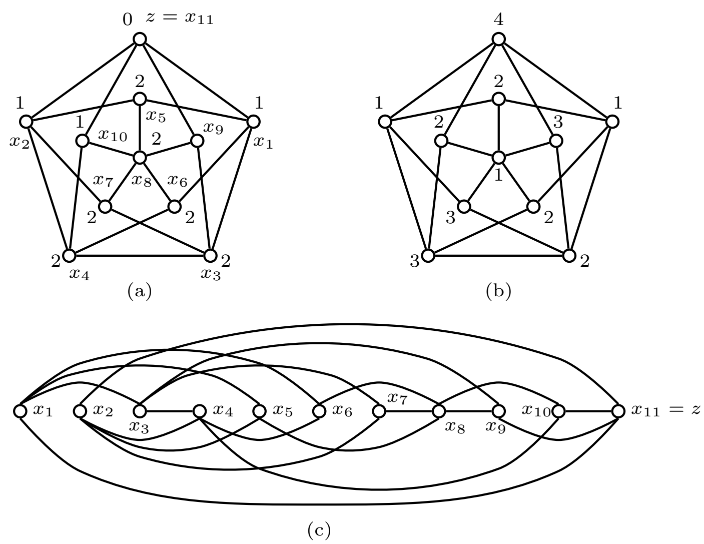

Figure 6.2: The sequential coloring of Grótzsch graph

Arbitrarily choose a vertex z of the Grótzsch graph G and then count the distance $d_G(z,x)$ for every $x \in V(G)$, indicated by the digits nearby vertices in Figure 6.2 (a). Label each vertex of G by $x_1, x_2, \cdots, x_{11}$ such that $x_1, x_2 \in N_G(z)$ and $x_1x_2 \notin E(G)$ (such two vertices must exist since $d_G(z) \geq 3$ and G contains no triangle) and $d_G(z,x_i) \geq d_G(z,x_{i+1})$ for each $i = 3, 4, \cdots, 11$, and so $z = x_{11}$, see Figure 6.2 (a) or (c).

First assign the color 1 to both x_1 and x_2; then successively color $x_3, x_4, \cdots, x_{11}$, each with the first available color in the list $1, 2, 3, 4$. The resulting coloring, a 4-vertex coloring, is shown in (b). ∎

Brooks' Theorem

We have seen $\chi = \Delta + 1$ for any odd cycle and any complete graph. In fact, it has been prove that odd cycles and complete graphs are only two types of graphs for which $\chi = \Delta + 1$. This is the following classical theorem, known as Brooks' theorem, the proof given here is due to Lovász (1975).

Theorem 6.2 (Brooks, 1941) **If G is a connected simple graph and is neither an odd cycle nor a complete graph, then $\chi(G) \leq \Delta(G)$.**

Proof: Suppose that $\chi(G) = k$ and H is a critical k-chromatic subgraph of G. If $H \cong K_k$, then $\Delta(H) = \Delta(K_k) = k - 1$. Thus $\chi(G) = k = \Delta(H) + 1 \leq \Delta(G)$. If H is an odd cycle, then $\chi(G) = \chi(H) = 3 \leq \Delta(G)$ since G is not an odd cycle.

Suppose now that H is neither an odd cycle nor a complete graph. Then $k \geq 4$ and $v(H) = p \geq 5$. Moreover, $\delta(H) \geq 3$ by Corollary 6.1.1. Subject to these hypotheses, we will prove that H is $\Delta(H)$-vertex-colorable.

Since H is not a complete graph, there are $x, y, z \in V(H)$ such that $xy \notin E(H)$, but $xz, yz \in E(H)$. Let $x = x_1, y = x_2$ and let $x_3, x_4, \cdots, x_p$ be an ordering of the vertices in $H - \{x, y\}$ such that $d_H(x_i, z) \geq d_H(x_{i+1}, z)$ for each $i = 3, 4, \cdots, p$. Then $z = x_p$. Let $h = \Delta(H)$. Then $h \leq \Delta(G)$.

We can now describe an h-vertex coloring of G: assign color 1 to both x_1 and x_2; then successively color $x_3, x_4, \cdots, x_p$, each with the first available color in the list $1, 2, \cdots, h$. By the construction of the sequence $x_1, x_2, \cdots, x_p$, each vertex x_i ($i = 1, 2, \cdots, p - 1$) is adjacent to some vertex x_j with $j > i$, and therefore to at most $h - 1$ vertices x_ℓ with $\ell < i$. It follows, when its turn comes to be colored, x_i is adjacent to at most $h - 1$ colors, and thus that one of the colors $1, 2, 3, \cdots, h$ will be available. Finally, since x_p is adjacent to both x_1 and x_2 that have been assigned the color 1, it is adjacent to at most $h - 2$ other vertices and can be assigned one of the colors $2, 3, \cdots, h$, and so theorem follows. ∎

Example 6.1.2 Consider Petersen graph G. Since it contains odd cycles, G is not bipartite, and so $\chi(G) \geq 3$. On the other hand, G is neither an odd cycle nor a complete graph, then, by Theorem 6.2, $\chi(G) \leq \Delta(G) = 3$. Thus, Petersen graph is a 3-chromatic graph. ∎

Example 6.1.3 (Roy ,1967; Gallai, 1968) Let G be a digraph with $\chi(G) = \chi$. Then G contains a directed path of length at least $\chi - 1$.

Proof: Let E' be a minimal subset of $E(G)$ such that $G' = G - E'$ contains no directed cycle, which implies that $G' + e$ contains a directed cycle for any $e \in E'$. Suppose that the length of a longest directed path in G' is k. It is sufficient to prove $\chi \leq k + 1$. We can do this by constructing a $(k + 1)$-vertex coloring of G.

For each $i = 1, 2, \cdots, k + 1$, let V_i be a subset of $V(G)$: $x \in V_i$ if and only if the length of a longest directed path in G' with origin x is $i-1$. Then $\{V_1, V_2, \cdots, V_{k+1}\}$ is a partition of $V(G)$. We first prove that V_i satisfies the following properties.

(i) G' contains no directed path whose origin and terminus both are in V_i for any $i\,(1 \leq i \leq k + 1)$. For otherwise, consider a directed (x, y)-path P in G' with $x, y \in V_i$. Then G' contains a directed path Q of length $i - 1$ with origin y. Since G' contains no directed cycle, $P \cup Q$ is a directed path in G' whose length is at least i. This contradicts the choice of $x \in V_i$.

(ii) V_i is an empty set or an independent set of G for each $i = 1, 2, \cdots, k + 1$. By contradiction. Suppose that x and y are two distinct vertices in some V_i and are adjacent in G. Thus there exists $e \in E(G)$ such that $\psi(e) = (x, y)$. Then $e \in E'$ since G' contains no (x, y)-path by (i). Thus $G' + e$ contains a directed cycle, say C. However, $C - e$ is a (y, x)-path, which contradicts (i).

By (ii), $\pi = (V_1, V_2, \cdots, V_{k+1})$ is a $(k + 1)$-vertex coloring of G, that is, $\chi(G) \leq k + 1$. ∎

Example 6.1.4 (V. Chvátal and J. Komlós, 1971) Let G be a simple digraph with $\chi(G) > mn$ and $\mathbf{f} \in \mathscr{V}(G)$. Then G contains either a directed path $(x_0, x_1, \cdots, x_m)$ such that $\mathbf{f}(x_0) \leq \mathbf{f}(x_1) \leq \cdots \leq \mathbf{f}(x_m)$ or a directed path $(y_0, y_1, \cdots, y_n)$ such that $\mathbf{f}(y_0) > \mathbf{f}(y_1) > \cdots > \mathbf{f}(y_n)$.

Proof: Construct two spanning subgraphs G_1 and G_2 of G as follows. For $(x, y) \in E(G)$,

$$(x, y) \in E(G_1) \Longleftrightarrow \mathbf{f}(x) \leq \mathbf{f}(y) \text{ or}$$

$$(x, y) \in E(G_2) \Longleftrightarrow \mathbf{f}(x) > \mathbf{f}(y).$$

Clearly, $G = G_1 \oplus G_2$. Suppose that $\chi(G_1) \leq m$ and $\chi(G_2) \leq n$, and let $\pi_1 = (V_1, V_2, \cdots, V_m)$ and $\pi_2 = (V_1', V_2', \cdots, V_n')$ be an m-vertex coloring of G_1 and an n-vertex coloring of G_2, respectively. Set

$$V_{ij} = \{x \in V(G) : \ x \in V_i \cap V'_j\}, \quad 1 \le i \le m, 1 \le j \le n.$$

V_i and V'_j are either an independent set or an empty set of G, so is V_{ij}. Thus,

$$\pi = \{V_{ij} : \ 1 \le i \le m, \ 1 \le j \le n\}$$

is an mn-vertex coloring of G. This implies $\chi(G) \le mn$, which contradicts the hypothesis. Therefore, we have either $\chi(G_1) > m$ or $\chi(G_2) > n$.

If $\chi(G_1) > m$, then, by Example 6.1.3, G_1 contains a directed path P of length at least $\chi(G) - 1 \ge m$. Let $(x_0, x_1, \cdots, x_m)$ be a section of P of length m. By construction of G_1, we have $\mathbf{f}(x_0) \le \mathbf{f}(x_1) \le \cdots \le \mathbf{f}(x_m)$.

Similarly, if $\chi(G_2) > n$, then G_2 contains a directed path $(y_0, y_1, \cdots, y_n)$ such that $\mathbf{f}(y_0) > \mathbf{f}(y_1) > \cdots > \mathbf{f}(y_n)$. ∎

Example 6.1.5 (P. Erdös and G. Szekeres, 1935) Any sequence of $mn + 1$ distinct integers contains either an increasing subsequence of m terms or a decreasing subsequence of n terms.

Proof: Let $(a_1, a_2, \cdots, a_{mn+1})$ be any sequence of $mn + 1$ distinct integers. Construct a simple digraph $G = (V, E)$ as follows. $V(G) = \{a_1, a_2, \cdots, a_{mn+1}\}$, and $(a_i, a_j) \in E(G) \Longleftrightarrow a_i < a_j$. It is easy to see that G is a tournament and $\chi(G) = mn + 1$. Let $\mathbf{f} \in \mathscr{V}(G)$ such that $\mathbf{f}(a_i) = a_i$ for each $i = 1, 2, \cdots, mn + 1$. The conclusion follows immediately from Example 6.1.4. ∎

Exercises: 6.1.4; 6.1.5

Thank You !