Recall

Vertex Colorings:

A **vertex k-coloring** of G is an assignment of k colors, $1, 2, \cdots, k$, to the vertices of G such that adjacent vertices are assigned different colors. In other words, a k-vertex coloring of G is a mapping

$$\pi : V(G) \to \{1, 2, \cdots, k\}$$

such that for each $i = 1, 2, \cdots, k$,

$$V_i = \{x \in V(G) : \pi(x) = i\}$$

is an independent set of G or an empty set. The subset V_i is called a **color class** of π. We often write $\pi = (V_1, V_2, \cdots, V_k)$ for a k-vertex coloring.

G is said to be **vertex k-colorable** if it has a vertex k-coloring. The (**vertex-**) **chromatic number**

$$\chi(G) = \min\{k : G \text{ is vertex k} - \text{colorable}\}.$$

A graph G is called to be **k-chromatic** if $\chi(G) = k$, and to be **critical k-chromatic** if $\chi(G) = k$ and $\chi(H) < k$ for every proper subgraph H of G.

Basic Results

Theorem 6.1 (Dirac, 1952) $\lambda(G) \geq k - 1$ **for any critical** $k(\geq 2)$**-chromatic graph** G**.**

Corollary $\chi(G) \leq \Delta(G) + 1$ **for any simple graph** G**.**

Theorem 6.2 (Brooks, 1941) **If** G **is a connected simple graph and is neither an odd cycle nor a complete graph, then** $\chi(G) \leq \Delta(G)$**.**

6.2 Edge Colorings

Definitions

Let G be a loopless graph. An **edge k-coloring** of G is an assignment of k colors, $1, 2, \cdots, k$, to the edges of G such that adjacent edges are assigned different colors. In other words, an edge k-coloring of G is a mapping

$$\pi' : E(G) \to \{1, 2, \cdots, k\}$$

such that for each $i = 1, 2, \cdots, k$,

$$E_i = \{e \in E(G) : \pi'(e) = i\}$$

is a matching of G or empty set. We often write $\pi' = (E_1, E_2, \cdots, E_k)$ for an edge k-coloring, where E_i is called an **edge-color class** of π'.

The concept of edge-coloring bears no relation to orientations of edges. Therefore, in discussing edge colorings, we will restrict ourselves to undirected graphs. Figure 6.3 illustrate a 3-edge coloring of odd cycle C_5 and a 4-edge coloring of Petersen graph, respectively.

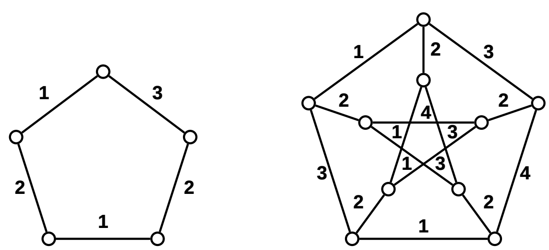


Figure 6.3: **Two illustrations of edge colorings**

G is said to be **edge k-colorable** if it has an edge k-coloring. The **edge-chromatic number**

$$\chi'(G) = \min\{k : G \text{ is edge} k - \text{colorable}\}.$$

Clearly,

$$\chi'(G) \geq \Delta(G) \quad \text{for any loopless graph } G. \tag{6.1}$$

From definition, if $\chi'(G) = k$ then each edge-color class E_i of any edge k-coloring $\pi = (E_1, E_2, \cdots, E_k)$ is a nonempty matching of G. In other words, the edge-chromatic number k of G may be defined alternatively as the minimum number of matchings into which $E(G)$ may be partitioned. Each such matching is then an edge-color class in the k-edge coloring of G so defined.

Applications

A number of real-world problems can be described by edge coloring of a graph. For example, in a school, there are m teachers $x_1, x_2, \cdots, x_m$ and n classes $y_1, y_2, \cdots, y_n$. Given that the teacher x_i is required to teach the class y_j for p_{ij} periods, schedule a complete timetable in the minimum possible number of periods.

Construct a bipartite graph G with bipartition $\{X, Y\}$, where $X = \{x_1, x_2, \cdots, x_m\}$ and $Y = \{y_1, y_2, \cdots, y_n\}$, and vertices x_i and y_j are jointed by p_{ij} undirected edges.

Now, in any one period, each teacher can teach at most one class, and each class can be taught by at most one teacher. Thus a teaching schedule for one period corresponds to a matching of G and, conversely, each matching of G corresponds to a possible assignment of teachers to classes for one period. Our problem, therefore, is to partition the edges of G into as few matchings as possible or, equivalently, to color the edges of G with as few colors as possible. This the minimum number is $\chi'(G)$.

For several special classes of graphs, the edge-chromatic number is quite easy to determine. For example,

$\chi'(C_n)$ is equal to 2 if n is even, and 3 if n is odd;

$\chi'(K_n)$ is equal to $n - 1$ if n is even, and n if n is odd; and

for a bipartite graph H, $\chi'(H) = \Delta(H)$ by Corollary 5.9.1.

Relations between Two Colorings

From definition, the problem of determining the edge-chromatic number of a graph G can be immediately transformed into that of dealing with chromatic number by considering its line graph $L(G)$, namely, if G is nonempty,

$$\chi'(G) = \chi(L(G)).$$

This observation appears to be of little value in computing edge-chromatic number, however, since chromatic numbers are extremely difficult to evaluate in general.

Vizing's Theorem

In what must be introduced the fundamental result on edge colorings is the following theorem, known as Vizing's theorem in the literature, first found by Vizing (1964).

Theorem 6.3 (Vizing's theorem) For any loopless nonempty simple graph,

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1. \tag{6.2}$$

Proof: Deleted for details.

Classification Problem

Vizing's theorem, or the inequality (6.2), gives us a simple way of classifying simple graphs into two classes.

A simple graph G is said to belong to **class one** if $\chi'(G) = \Delta(G)$, and to **class two** if $\chi'(G) = \Delta(G) + 1$. The problem of deciding which graphs belong to which class is the so-called **classification problem**.

For example: a complete graph K_{2n} and a bipartite graph belong to class one, and an odd cycle C_{2n+1} and a complete graph K_{2n+1} belong to class two, but the general classification problem has been proved to be NP-hard by Holyer (1981).

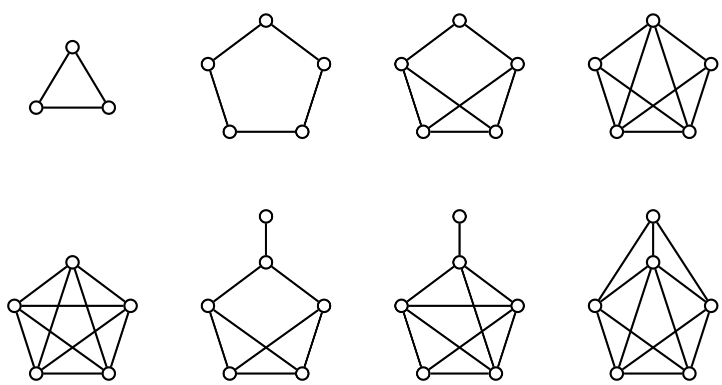


Figure 6.4: **All connected simple graphs of order at most six of class two**

It seems that graphs of class two are relatively scarce. For example, of the 143 connected simple graphs of order at most six, only eight belong to class two (see Figure 6.4). A more general result of this kind is due to Erdös and Wilson (1977), who proved that almost all nonempty simple graphs belong to class one, that is,

$$\lim_{v \to \infty} \frac{|C^1(v)|}{|C^1(v) \cup C^2(v)|} = 1,$$

where $C^1(v)$ and $C^2(v)$ are the set of all nonempty simple graphs of order v belonging to class one and two, respectively.

No progress has been made on the more difficult problem of deciding which class contains almost all graphs with a given maximum degree Δ, even for $\Delta = 3$ this is unknown.

Classification of Planar Graphs

There exist planar graphs of class one with the maximum degree Δ for any $\Delta \geq 2$. For example, the star graph $K_{1,\Delta}$ is such a graph. There exist also planar graphs of class two with the maximum degree Δ for $\Delta = 2, 3, 4, 5$. For example, $\chi'(K_3) = 3 = \Delta(K_3) + 1$; other three planar graphs of class two are shown in Figure 6.5.

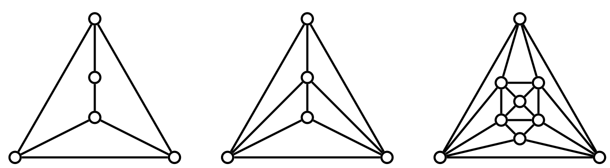


Figure 6.5: **Three planar graphs of class two**

Vizing (1965) has proved that there are no planar graphs of class two with the maximum degree $\Delta \geq 8$. In his another paper, Vizing (1968) conjectured that there is no planar graph of class two with the maximum degree $\Delta = 6$ or 7. Zhang (2000) has proved that Vizing's conjecture is true for $\Delta = 7$. We have not, however, known whether or not Vizing's conjecture is true for $\Delta = 6$.

6.3 The Four-Color Problem

Four-color Problem

This problem can be traced back to 1852. While coloring a map of the counties of England, Francis Guthrie, a London student, noticed that four colors are sometimes needed, and then proposed a conjecture to his brother Frederick that

four colors are always sufficient to color the countries of any map so that no two countries which have a common boundary are assigned the same color.

But Frederick was unable to prove this and brought the problem to the attention of Augustus De Morgan, Professor of Mathematics at University College, London, who mentioned it on a number of occasion, giving credit to Francis Guthrie for proposing it. However, it was not until 1878 (after De Morgan's death) that the problem became widely known. At a meeting of the London Mathematical Society in that year, Arthur Cayley asked whether the problem had been solved, and shortly afterwards wrote a note (1879) in which he attempted to explain where the difficulty lies.

Face-colorings of Plane Graphs

Note that a map can be regarded as a plane graph with its countries as the faces of the graph. This leads to a concept of face-colorings of a plane graph.

A **k-face coloring** π^* of a plane graph G is an assignment of k colors, $1, 2, \cdots, k$, to the faces of G such that no two faces which have a common boundary are assigned the same color. We write $\pi^* = (F_1, F_2, \cdots, F_k)$ for a k-face coloring, where $F_i = \{f \in F(G) : \pi^*(f) = i\}$. A plane graph G is said to be **k-face-colorable** if it has a k-face coloring. The **face-chromatic number** of G

$$\chi^*(G) = \min\{k : G \text{ is } k\text{-face-colorable}\}.$$

For example $\chi^*(K_4) = 4$.

From definition, by making use of the geometric dual G^* of a plane graph G, we immediately obverse the following relationship between the face-chromatic number of G and the vertex-chromatic number of G^*:

$$\chi^*(G) = \chi(G^*). \tag{6.3}$$

Four-Color Conjecture

By Corollary 3.3.2, all planar embeddings of a given connected planar graph have the same number of faces. By (6.4), therefore, the four-color conjecture can be equivalently stated as the following formula.

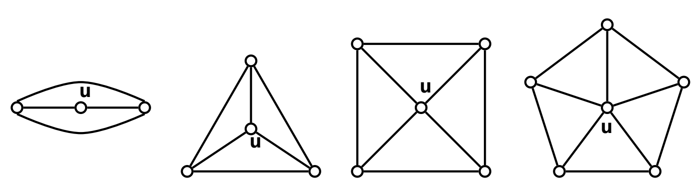
Four-Color Conjecture Every plane graph is 4-face-colorable or every planar graph is 4-vertex-colorable.

The four-color conjecture is one of the best-known conjecture in the whole of mathematics. The problem of deciding whether the four-color conjecture is true or not is called the **four-color problem**.

Kempe's Ideas

The first serious attempt at a proof of the four-color conjecture seems to have been made by Kempe (1879), a barrister and keen amateur mathematician who was Treasurer, and later President, of the London Mathematical Society. In 1879, he published a "proof" of the four-color conjecture. In order to describe Kempe's ideas in his proof in modern terminology, we need a few definitions.

Clearly, it is sufficient to consider plane triangulations for the four-color conjecture. A plane graph is called a **configuration** if each of its bounded faces is a triangle. The four graphs shown in Figure 6.6, for example, are configurations, denoted by $\mathbf{O}, \mathbf{P}, \mathbf{Q}, \mathbf{R}$, respectively. A set $\mathscr{F}$ consisting of finite configurations is called **unavoidable complete** if every plane triangulation must contain at least one element of $\mathscr{F}$. By Corollary 3.4.3, it is clear that the set $\mathscr{F} = \{\mathbf{O}, \mathbf{P}, \mathbf{Q}, \mathbf{R}\}$ is an unavoidable complete set.



O P Q R

Figure 6.6: **An unavoidable complete set** $\mathscr{F}$

Suppose that there exists a counterexample to the four-color conjecture. We may choose a counterexample with order as small as possible, such a plane triangulation $\mathbf{G}$ is called a **minimal graph**. So $\chi(\mathbf{G}) = 5$ and $\chi(\mathbf{H}) \leq 4$ for any plane graph $\mathbf{H}$ with fewer vertices than $\mathbf{G}$.

Kempe attempted to prove there exists no minimal graph.

Suppose that G is a minimal graph. So G must contain at least one of configurations in $\mathscr{F} = \{O, P, Q, R\}$. If G contains either O or P, then $\chi(G - u) \le 4$. Since $d_G(u) \le 3$, there is always a spare color that can be used to color u for any 4-vertex coloring of $G - u$. This implies $\chi(G) \le 4$, a contradiction. Therefore, G contains neither O nor P.

Suppose that G contains the configuration Q. Let $N_G(u) = \{u_1, u_2, u_3, u_4\}$, and let $\pi = \{V_1, V_2, V_3, V_4\}$ be a 4-vertex coloring of $G - u$. Without loss of generality, we may suppose $u_i \in V_i$ for each $i = 1, 2, 3, 4$. Then either u_1 and u_4 are not connected in $G_{14} = G[V_1 \cup V_4]$ or u_2 and u_3 are not connected in $G_{23} = G[V_2 \cup V_3]$; for otherwise, a u_1u_4-path in G_{14} and a u_2u_3-path in G_{23} have a common vertex with two different colors, see Figure 6.7, where the digit i nearby a vertex indicates the color used in the vertex. Without loss of generality, suppose that u_1 and u_4 are not connected in G_{14}. By interchanging the colors 1 and 4 in the component containing u_1 of G_{14}, we can obtain a spare color 1 that can be used to color u, which results is a 4-vertex coloring of G, a contradiction. Therefore, G can not contain the configuration Q.

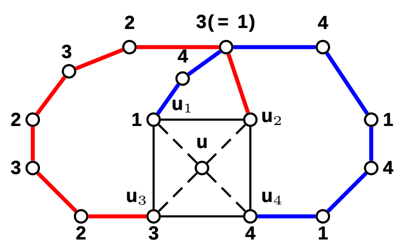


Figure 6.7: **An illustration of Kempe's argument**

Using the same way, Kempe "proved" that G can not contain the configuration R. Thus, G contains none of elements of $\mathscr{F}$, which contradicts to the fact that $\mathscr{F} = \{O, P, Q, R\}$ is an unavoidable complete set. So Kempe regarded he had proved the four conjecture.

Heawood's Counterexample

Eleven years later, in 1890, Heawood gave a counterexample, shown in Figure 6.8, showing that Kempe's discussion of the configuration **R** is incorrect.

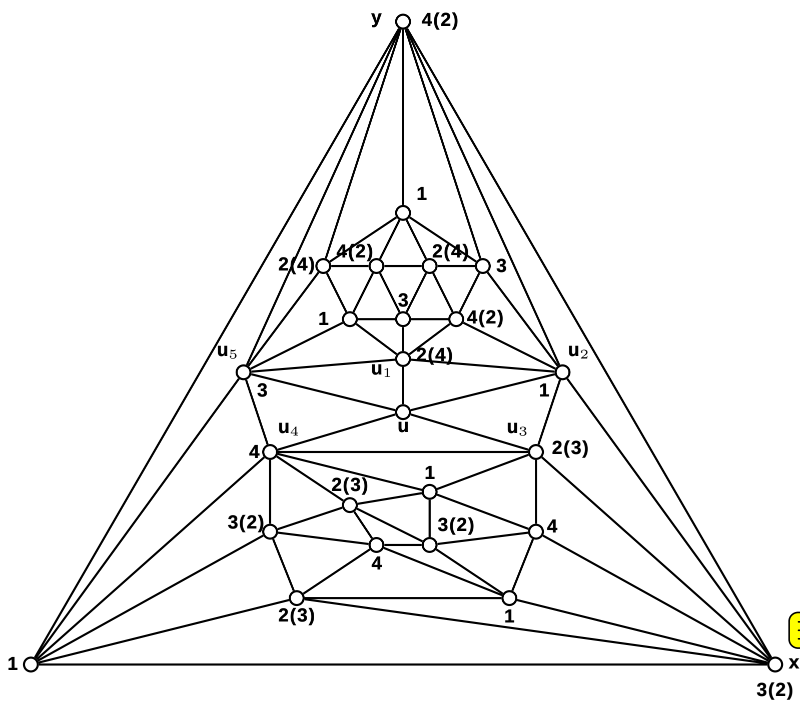


Figure 6.8: **A counterexample to Kempe's proof**

Therefore, Kempe's proof is invalid. Making use of Kempe's argument, Heawood proved the following theorem, known as the **five-color theorem** on planar graphs.

Theorem 6.4 $\chi(G) \leq 5$ **for any planar graph G.**

Development of Kempe's Method

For some years afterwards, the flaw in Kempe's proof seems not to have been recognized as serious, but as the years went by and nobody found a satisfactory way around the difficulty, it gradually became realized that the problem was much harder than originally supposed. Since then, many mathematicians have intended to prove the conjecture. Although Kempe's proof was fallacious, his several important ideas contributed in the proof has became the foundation for almost all subsequent attempts on the problem.

A configuration is called to be **reducible** if it can not be contained in any minimal graph. Kempe only proved that the configurations **O, P, Q** all are reducible, but he was unable to prove that the configuration **R** is reducible. Kempe's ideas showed that in order to prove the four-color conjecture, it suffices to find an unavoidable complete set of reducible configurations. Since the last of the configurations in $\mathscr{F}$ = **{O, P, Q, R}** has not been shown to be reducible, it is natural to ask whether it can be replaced by any other configurations to form another unavoidable complete set. In 1904, Wernicke found an unavoidable complete set, and, in 1913, Birkhoff, shown in Figure 4.9 (a) and (b), respectively. Unfortunately, they were unable to show that the last two configurations are reducible. Since then, many mathematicians have joined the search for reducible configuration, and thousand of such configurations are found.

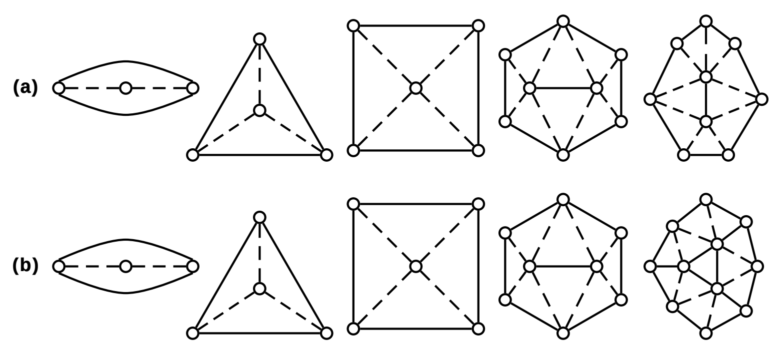


Figure 6.9: **Two unavoidable complete sets**

In 1969, Heesch developed the two main ingredients needed for the ultimate proof - **reducibility and discharging**. It was he who conjectured that a suitable development of this method would solve the four-color problem.

Proofs by Computers

Using Heesch's idea, in 1976, Appel and Haken announced a proof of the four-color conjecture by using a computer to exhibited an unavoidable set of 1936 reducible configurations.

Another similar proof of the conjecture, but simpler than Appel and Haken's in several respects, in 1997, was given by Robertson **et al.** who exhibited an unavoidable set of 633 reducible configurations.

Problem ???

However, their proof has not been fully accepted. There has remained a certain amount of doubt about it validity, basically for a main reason: part of the proof uses a computer and can not be verified by hand. Thus, the proof of the four-color conjecture by hand is still necessary.