Exercises in Chapter 1

Exercises 1.1

1.1.1 Drawing graphical presentations of the following five graphs without parallel edges B, K, Q, D and G, respectively, where

(a) $V(B) = \{x_1x_2x_3 : \ x_i \in \{0,1\}\}$ and if $x, y \in V(B)$, $x = x_1x_2x_3$, then $(x,y) \in E(B)$ if and only if $y = x_2x_3\alpha,\ \alpha \in \{0,1\}$;

(b) $V(K) = \{x_1x_2x_3 : \ x_i \in \{0,1,2\}, x_2 \neq x_1, x_3 \neq x_2\}$ and if $x, y \in V(K), x = x_1x_2x_3$, then $(x,y) \in E(K)$ if and only if $y = x_2x_3\alpha,\ \alpha \in \{0,1,2\}$ and $\alpha \neq x_3$;

(c) $V(Q) = \{x_1x_2x_3 : \ x_i \in \{0,1\}\}$, if $x, y \in V(Q)$, $x = x_1x_2x_3$ and $y = y_1y_2y_3$, then $xy \in E(Q)$ if and only if $|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3| = 1$;

(d) $V(D) = \{0, 1, \cdots, 7\}$, and $E(D) = \{(i,j)$: there exists some $s \in \{1,2\}$ such that $j - i \equiv s \pmod 8$ $\}$;

(e) $V(G) = \{0, 1, \cdots, 7\}$, and $E(G) = \{ij$: there exists some $s \in \{1,4\}$ such that $|j - i| \equiv s \pmod 8$ $\}$.

1.1.2 Prove that for any simple graph G,

(a) $\varepsilon \leq v(v-1)$ if G is directed;

(b) $\varepsilon \leq \frac{1}{2}\ v(v-1)$ if G is undirected.

1.1.3 The symbols $\mathcal{D}_v$ and $\mathcal{G}_v$ denote the sets of all simple digraphs and all simple undirected graphs of order v, respectively. Prove that

(a) $|\mathcal{D}_v| = 2^{v(v-1)}$;

(b) $|\mathcal{G}_v| = 2^{v(v-1)/2}$.

1.1.4 Prove that there are 2^ε different oriented graphs for any undirected graph.

1.1.5 The symbols $\mathcal{D}(v,\varepsilon)$ and $\mathcal{G}(v,\varepsilon)$ denote the sets of all simple digraph and undirected graphs of order v and size ε, respectively. Prove that

(a) $|\mathcal{D}(v,\varepsilon)| = \binom{v(v-1)}{\varepsilon}$;

(b) $|\mathcal{G}(v,\varepsilon)| = \binom{v(v-1)/2}{\varepsilon}$.

Exercises 1.2

1.2.1 (a) Prove that if $G \cong H$, then $v(G) = v(H)$ and $\varepsilon(G) = \varepsilon(H)$.

(b) Construct a graph to show that the converse of (a) is not true.

1.2.2 Prove that if G is a bipartite simple graph G of order n, then

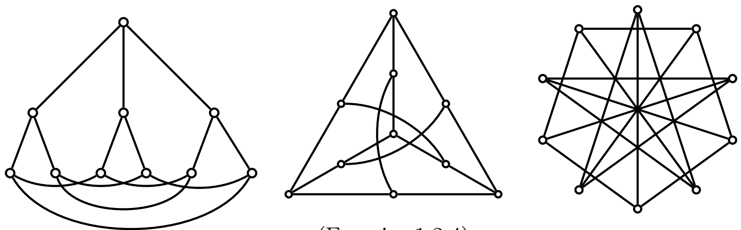
$$\varepsilon(G) \le \begin{cases} \frac{1}{4}\, n^2 & \text{if } n \text{ is even;} \\ \frac{1}{4}\, (n^2-1) & \text{if } n \text{ is odd.} \end{cases}$$

In Particular, $\varepsilon(K_{m,n}) = mn$.

1.2.3 Write out definition of k-partite graph and prove that

$$\varepsilon(K_n(k)) = \frac{1}{2}\, k(k-1)n^2.$$

1.2.4 Prove that the following three graphs are isomorphic to Petersen graph.



(Exercise 1.2.4)

(the exercise 1.2.4)

1.2.5 The *complement* G^c of a simple graph $G = (V, E)$ is the simple graph with the vertex-set V, and $(x, y) \in E(G^c) \Longleftrightarrow (x, y) \notin E(G)$. Prove that

(a) the complement of every tournament is a tournament;

(b) $G^c \cong H^c \Longleftrightarrow G \cong H$ if both G and H are simple.

1.2.6 A simple graph G is *self-complementary* if $G \cong G^c$. Prove that if G is self-complementary, then

(a) $\varepsilon(G) = \frac{1}{2}\, v(v-1)$ if G is directed;

(b) $\varepsilon(G) = \frac{1}{4}\, v(v-1)$ and $v \equiv 0$, or 1 (mod 4) if G is undirected.

1.2.7 Construct that

(a) two self-complementary tournaments of order four;

(a) a self-complementary undirected graph of order five.

Exercises 1.3

1.3.1 Prove that $\delta \le 2\varepsilon/v \le \Delta$ for any undirected graph.

1.3.2 Prove that there are always two vertices with exactly the same degree for any simple undirected graph of order at least two.

1.3.3 (a) Prove that if a digraph D is both δ^+-regular and δ^--regular, then $\delta = \delta^+ = \delta^-$, and hence D is δ-regular.

(b) Construct a digraph that is δ^+-regular but not δ^--regular.

1.3.4 Let $v \geq 2$. Prove that

(a) there exists a simple digraph D of order v such that for any two distinct vertices x and y

$$d_D^+(x) \neq d_D^+(y) \quad \text{and } d_D^-(x) \neq d_D^-(y);$$

(b) there exists a simple digraph D of order v such that the number of vertices of odd out-degree and the number of vertices of odd in-degree both are odd;

(c) there exists a r-regular simple digraph for any integer r with $r < v$.

1.3.5 Prove that for any tournament D,

$$\sum_{x \in V} d_D^+(x)^2 = \sum_{x \in V} d_D^-(x)^2 = \sum_{x \in V} (v - d_D^-(x))^2 - v^2.$$

1.3.6 Prove that

(a) any k (> 0)-regular bipartite graph is equally bipartite;

(b) any k-regular tournament has order $v = 2k + 1$.

1.3.7 Let X and Y be two subsets of $V(G)$. Prove that

(a) $d_G^+(X \cap Y) + d_G^+(X \cup Y) \leq d_G^+(X) + d_G^+(Y)$ if G is a digraph;

(b) $d_G^-(X \cap Y) + d_G^-(X \cup Y) \leq d_G^-(X) + d_G^-(Y)$ if G is a digraph;

(c) $d_G(X \cap Y) + d_G(X \cup Y) \leq d_G(X) + d_G(Y)$ if G is an undirected graph.

1.3.8 The symbol $\varepsilon_{\min}$ denotes the minimum number of edges in a simple undirected graph of order v that there is at least one edge among any three vertices. Prove that

$$\varepsilon_{\min} = \begin{cases} k^2 - k, & \text{if } v = 2k; \\ k^2, & \text{if } v = 2k + 1. \end{cases}$$

Exercises 1.4

1.4.1 Prove that for any bipartite undirected graph G with $\Delta(G) = \Delta$,

(a) there exists a Δ-regular bipartite graph H such that $G \subseteq H$;

(b) there exists a Δ-regular bipartite simple graph F such that $G \subseteq F$ if G is simple.

1.4.2 Prove that any loopless undirected graph G contains a k-partite spanning subgraph H such that $(1 - \frac{1}{k})d_G(x) \leq d_H(x)$ for any $x \in V(G)$.

1.4.3 (a) Let G be an undirected graph of order v, and n be an integer with $2 \leq n < v - 1$. Prove that if $v \geq 4$ and all induced subgraphs by n vertices in G have the same numbers of edges, then G is either complete or empty.

(b) Give an example to show that the conclusion in (a) is false for digraphs.

(c) Let G be a digraph of order v, and n be an integer with $2 \leq n < v - 1$. Prove or disprove that if all induced subgraphs by n vertices in G are regular, then G is either complete or empty.

1.4.4 Let $L = L(G)$ be the line graph of G. Prove that

(a) L contains no parallel edges and contains a loop at vertex a if and only if a is a loop in G;

(b) $d_L^+(a) = d_G^+(y)$ and $d_L^-(a) = d_G^-(x)$ for any $a \in E(G)$ with $\psi_G(a) = (x, y)$, in particular, if G is d-regular, then so is L;

(c) if G is undirected then $d_L(e) = d_G(x) + d_G(y) - 2$ for any $e \in E(G)$ with $\psi_G(e) = xy$, particularly, L is $(2d-2)$-regular if G is d-regular;

(d) $\varepsilon(L) = \sum\limits_{x \in V(G)} d_G^+(x) d_G^-(x)$ if G is directed, and

(e) $\varepsilon(L) = \frac{1}{2} \sum\limits_{x \in V(G)} (d_G(x))^2 - \varepsilon(G)$ if G is undirected.

1.4.5 The *join* $G_1 \vee G_2$ of disjoint undirected graphs G_1 and G_2 is the undirected graph obtained from $G_1 + G_2$ by joining each vertex of G_1 to each vertex of G_2. Prove that

(a) $K_{m,n} \cong K_m^c \vee K_n^c$;

(b) $\varepsilon(G_1 \vee G_2) = \varepsilon(G_1) + \varepsilon(G_2) + v(G_1)\ v(G_2)$.

1.4.6 Prove that the cartesian product $G_1 \times G_2$ of two simple graphs G_1 and G_2 satisfies the following properties.

(a) $v(G_1 \times G_2) = \nu(G_1) v(G_2)$.

(b) For any $xy \in V(G_1 \times G_2)$, where $x \in V(G_1)$ and $y \in V(G_2)$,

$$d^+_{G_1 \times G_2}(xy) = d^+_{G_1}(x) + d^+_{G_2}(y),\ \ d^-_{G_1 \times G_2}(xy) = d^-_{G_1}(x) + d^-_{G_2}(y)$$

if G is directed, and

$$d_{G_1 \times G_2}(xy) = d_{G_1}(x) + d_{G_2}(y)$$

if G is undirected. In particular, $G_1 \times G_2$ is $r_1 + r_2$ regular if G_1 and G_2 are r_1- and r_2-regular, respectively.

(c) $\varepsilon(G_1 \times G_2) = v(G_1)\varepsilon(G_2) + v(G_2)\varepsilon(G_1)$.

(d) The cartesian product satisfies commutative and associative laws if we identify isomorphic graphs, that is, $G_1 \times G_2 = G_2 \times G_1$ and $(G_1 \times G_2) \times G_3 = G_1 \times (G_2 \times G_3)$.

(e) $Q_n = K_2 \times K_2 \times \cdots \times K_2$ of n identical complete graph K_2.

1.4.7 F.R.Ramsey [149] in 1930 proved the well-known *Ramsey's Theorem*: For given positive integers k and l, there exists a smallest integer $r = r(k, l)$ such that every simple undirected graph of order r contains either K_k or K_l^c as its subgraph. The number $r(k, l)$ is known as the *Ramsey number*. Prove that

(a) $r(k, l) = r(l, k)$, $\quad r(1, k) = 1$, $\quad r(2, k) = k$ and $r(3, 3) = 6$;

(b) $r(k, l) \leq r(k, l-1) + r(k-1, l)$, and the strict inequality holds if $r(k, l-1)$ and $r(k-1, l)$ are both even for $k \geq 3$ and $l \geq 3$;

(c) $r(3, 4) = 9$, $R(3, 5) = 14$, $r(4, 4) = 18$.

(Other Ramsey numbers known to date are $r(3, 6) = 18$ [102], $r(3, 7) = 23$ [77], $r(3, 8) = 28$ [126] and $r(3, 9) = 36$ [79].)

1.4.8 Prove that if an undirected graph G with vertex set V contains no K_{k+1} as its subgraph, then there exists a complete k-partite graph H with vertex-set V such that $d_G(x) \leq d_H(x)$ for every $x \in V(G)$. Moreover, the equality holds if and only if $G \cong H$. (P.Erdös [54])

1.4.8 Prove that (*Turán's theorem*) if an undirected graph G contains no K_{k+1} as its subgraph, then $\varepsilon(G) \leq \varepsilon(T_{k,v})$. Moreover, the equality holds if and only if $G \cong T_{k,v}$. (P.Turán [161])

Exercises 1.5

1.5.1 (a) Prove that any xy-walk (resp. (x,y)-walk) necessarily contain an xy-trail (resp. (x,y)-trail).

(b) Prove that any xy-trail (resp. (x,y)-trail) necessarily contain an xy-path (resp. (x,y)-path).

(c) Prove that any directed closed walk can be expressed as the union of several edge-disjoint closed trails, and construct an example to show that the term "directed" can not be deleted.

(d) Prove that any (directed) circuit can be expressed as the union of several edge-disjoint (directed) cycles.

1.5.2 Prove that any simple digraph contains a directed path of length at least $\max\{\delta^+, \delta^-\}$.

1.5.3 Prove that if G is a strongly connected digraph and $x, y \in V(G)$, then there exists an (x,y)-walk going through every vertex in G.

1.5.4 Prove that

(a) a graph is connected if and only if $[S, \overline{S}] \neq \emptyset$ for any nonempty proper subset S of V;

(b) a digraph is strongly connected if and only if both $(S, \overline{S}) \neq \emptyset$ and $(\overline{S}, S) \neq \emptyset$ for any nonempty proper subset S of V;

(c) a balanced digraph is strongly connected if and only if it is connected;

(d) a digraph contains a directed path from a vertex x_0 to any other vertex if and only if $(S, \overline{S}) \neq \emptyset$ for any nonempty proper subset S of V containing x_0.

1.5.5 Prove that a graph G of order at least three is connected if and only if there exist two vertices x and y in G such that $G-x$ and $G-y$ both are connected.

1.5.6 Let G be a simple undirected graph and $\omega = \omega(G)$. Prove that

(a) $\varepsilon(G) \leq \frac{1}{2}\,(v-\omega)(v-\omega+1)$;

(b) G is connected if $\varepsilon(G) > \frac{1}{2}(v-1)(v-2)$;

(c) G is connected if $d_G(x) + d_G(y) \geq v-1$ for any two nonadjacent vertices x and y.

1.5.7 Let G be a simple digraph with ω strongly connected components. Prove that

(a) $\varepsilon(G) \leq (v-\omega)(v-\omega+1)+\frac{1}{2}\,(\omega-1)(2v-\omega)$;

(b) $\omega = 1$, that is, G is strongly connected if $\varepsilon(G) > (v-1)^2$.

1.5.8 Let G be a simple digraph of order $v > 1$. Prove that

(a) G is strongly connected if $d_G^+(x) + d_G^-(y) \geq v-1$ for any two vertices x and y satisfying $(x,y) \notin E(G)$;

(b) G is strongly connected if $\varepsilon > v(v-1)-(k+1)(v-k-1)$ and $\delta \geq k$.

1.5.9 Let G be an undirected graph. Prove that

(a) G contains no cut-edge if G contains no vertex of degree odd;

(b) G contains no cut-edge if G is k (≥ 2)-regular and bipartite;

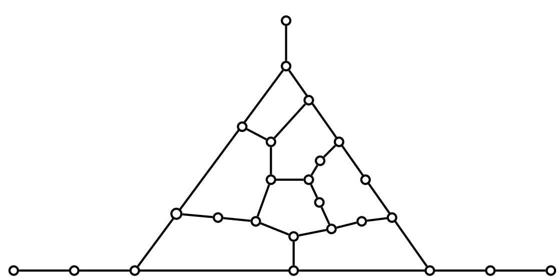
(c) if b_x denotes the number of blocks containing the vertex x in G, then the number of all blocks in G

$$b(G) = \omega(G) + \sum_{x \in V(G)} (b_x - 1).$$

1.5.10 Prove that

(a) any two longest paths in any connected graph must have a vertex in common;

(b) all longest paths in the following graph have no vertex in common.



(Exercise 1.5.10)

(the exercise 1.5.10)

1.5.11 Let G be a simple undirected digraph. Prove that

(a) if G is disconnected, then G^c is connected;

(b) G and G^c both are connected if and only if G and G^c both contains no complete bipartite graph as their spanning subgraph.

Exercises 1.6

1.6.1 Prove that $d_G(x,z) \leq d_G(x,y) + d_G(y,z)$ for any three vertices x, y and z of a strongly connected digraph G.

1.6.2 Prove that if G is a (Δ, k)-Moore digraph, then G is simple and Δ-regular, contains no cycle of length at most k and there is only unique (x, y)-path of length at most k for any pair (x, y) of vertices in G.

1.6.3 Let $L(G)$ be the line graph of a graph G of order $v \geq 2$. Prove that

(a) $L(G)$ is strongly connected if and only if G is strongly connected;

(b) if G is strongly connected then $L(G) \cong G$ if and only if G is a directed cycle;

(c) the above conclusions are true if G is undirected.

1.6.4 Let G be a simple undirected graph. Prove that

(a) if G is disconnected, then $d(G^c) \leq 2$;

(b) if $d(G) > 3$, then $d(G^c) < 3$;

(c) if $d(G) = 2$ and $\Delta(G) = v - 2$, then $\varepsilon \geq 2v - 4$.

1.6.5 Prove that if G is a connected undirected graph of the maximum degree Δ and diameter d, then

$$v \leq \begin{cases} 2d+1, & \text{for } \Delta = 2; \\ \dfrac{\Delta(\Delta-1)^d - 2}{\Delta - 2}, & \text{for } \Delta \geq 3, \end{cases}$$

and, hence,

$$d \geq \begin{cases} \left\lfloor \dfrac{1}{2} v \right\rfloor, & \text{for } \Delta = 2; \\ \left\lceil \log_{(\Delta-1)} \dfrac{v(\Delta-2)+2}{\Delta} \right\rceil, & \text{for } \Delta \geq 3. \end{cases}$$

1.6.6 Prove that

(a) $\mathrm{rad}\,(G) \leq d(G) \leq 2\,\mathrm{rad}\,(G)$ for any undirected graph G;

(b) if C_n is an undirected cycle, then the mean distance of C_n

$$m(C_n) = \begin{cases} \dfrac{n+1}{4}, & \text{if } n \text{ is odd;} \\ \dfrac{n^2}{4(n-1)}, & \text{if } n \text{ is even.} \end{cases}$$

Exercises 1.7

1.7.1 Prove Corollary 1.6.2 and Corollary 1.6.3.

1.7.2 Prove that

(a) any graph with $\varepsilon \geq v$ contains a cycle;

(b) any connected 2-regular undirected graph is a cycle;

(c) any strongly connected 1-regular digraph is a directed cycle.

1.7.3 Suppose that G is a simple digraph.

(a) Prove that if $k = \max\{\delta^+, \delta^-\} > 0$, then G contains a directed cycle of length at least $k+1$.

(b) Prove that if $\varepsilon > \frac{1}{2}\, v(v-1)$, then G contains a directed cycle.

(c) Construct a simple digraph with $\varepsilon = \frac{1}{2}\, v(v-1)$ such that it contains no directed cycle.

1.7.4 Suppose that T is a tournament. Prove that

(a) if $k = \max\{\delta^+, \delta^-\} > 0$, then T contains a directed cycle of length at least $2k+1$;

(b) if T is strongly connected and $v \geq 4$, then there exists $S \subseteq V(T)$ such that $|S| \geq 2$ and $T - x$ is strongly connected for any $x \in S$;

(c) if T contains a directed k-cycle, then T contains a directed l-cycle for each $l = 3, 4, \cdots, k$.

1.7.5 Prove that if G is a simple undirected graph with $\delta \geq 3$, then G contains even cycle and the greatest common factor of all lengths of cycles in G is either 1 or 2.

1.7.6 Prove that if G is a connected simple undirected graph with $v > 2\delta$, then G contains a path of length at least 2δ.

1.7.7 Let G be a non-bipartite simple undirected graph and k be a given integer. Prove that if $k \geq 2$ and $\delta > \left\lfloor \dfrac{2v}{2k+1} \right\rfloor$, then G contains an odd cycle of length at most $(2k-1)$.

1.7.8 Prove that a simple graph G of order $v \geq 4$ contains two different cycles with exactly one edge in common if it satisfies one of the following conditions:

(a) $\delta(G) \geq 3$; (b) $\varepsilon(G) = 2v - 3$.

1.7.9 A Δ-regular undirected graph of diameter k with the largest order is called a *maximum* (Δ, k)*-graph*. Used $n(\Delta, k)$ to denote the order of a maximum (Δ, k)-graph. Exercise 1.6.5 gives an upper bound of $n(\Delta, k)$ for $\Delta \geq 2$. Prove that

(a) $n(\Delta, 1) = \Delta + 1$ and a complete graph $K_{\Delta+1}$ is, to up isomorphism, the unique maximum $(\Delta, 1)$-graph;

(b) $n(2, k) = 2k + 1$ and a $(2k+1)$-cycle is, to up isomorphism, the unique maximum $(2, k)$-graph;

(c) $n(3, 2) = 10$ and the Petersen graph is a maximum $(3, 2)$-graph.

(In addition, Hoffman and Singleton [97] showed $n(7, 2) = 50$, and Elspas [53] showed $n(3, 3) = 20$, $n(4, 2) = 15$ and $n(5, 2) = 24$. These are the only known exact values of $n(d, k)$ so far.)

1.7.10 A Δ-regular undirected graph with girth at least g having the least order is called a (Δ, g)-*cage*. Used $f(\Delta, g)$ to denote order in a (Δ, g)-cage. When $g \geq 3$, Example 1.7.2 gives a lower bound of $f(\Delta, g)$.

(a) Prove that the diameter $d(G) \le g$ and the girth $g(G) = g$.

(b) Complete the proof of Example 1.7.2 for the case that g is even.

(c) Prove that $v(G) \le \frac{\Delta}{\Delta-2}(\Delta-1)^g$.

(d) Prove that $f(2,g) = g$ and a g-cycle is, to up isomorphism, the unique $(2,g)$-cage.

(e) Verify that, to up isomorphism, $K_{\Delta+1}$ is the unique $(\Delta,3)$-cage and $K_{\Delta,\Delta}$ is the unique $(\Delta,4)$-cage.

(f) The Petersen graph is a $(3,5)$-cage.

(g) Can you depict $(3,6)$-, $(3,7)$-, $(4,5)$-cages and more?

1.7.11 Suppose that G is a digraph without a directed cycle. Prove that

(a) $\delta^- = 0$;

(b) there is an ordering $x_1, x_2, \cdots, x_v$ of V such that every directed edge of G with head x_i has its tail in $\{x_1, x_2, \cdots, x_{i-1}\}$ for each $i = 1, 2, \cdots, v$.

1.7.12 The *converse* $\overleftarrow{G}$ of a digraph G is a digraph obtained from G by reversing the orientation of each edge.

(a) Prove that (i) $d^+_{\overleftarrow{G}}(x) = d^-_G(x)$; (ii) $d_{\overleftarrow{G}}(x,y) = d_G(y,x)$.

(b) By using part (ii) of (a), deduce from the exercise 1.7.11 (a) that if G is a digraph without a directed cycle, then $\delta^+ = 0$.

1.7.13 Let $G_1, G_2, \cdots, G_\omega$ be all strongly connected components of a digraph G. The *condensation* $\hat{G}$ of G is a simple digraph with ω vertices $u_1, u_2, \cdots, u_\omega$ and $(u_i, u_j) \in E(\hat{G})$ if and only if $E_G(V(G_i), V(G_j)) \ne \emptyset$. Prove that

(a) $\hat{G}$ contains no directed cycle;

(b) a simple digraph G contains no directed cycle if and only if $G \cong \hat{G}$.

1.7.14 A tournament T is called to be *transmissible* if, whenever (x,y) and (y,z) are edges of T, then (x,z) is also an edge of T. A sequence $(s_1, s_2, \cdots, s_n)$ of nonnegative integers is called a *score sequence* of a tournament if there exists a tournament T of order n whose vertices can be labelled as $x_1, x_2, \cdots, x_n$ such that $d^+_T(x_i) = s_i$ for $i = 1, 2, \cdots, n$. Prove that

(a) a tournament T is transmissible if and only if T contains no directed cycle;

(b) if a tournament T is transmissible, then $d^+_T(x) \ne d^+_T(y)$ and $d^-_T(x) \ne d^-_T(y)$ for any two vertices x and y of T;

(c) A non-decreasing sequence S of $n\,(\ge 1)$ nonnegative integers is a score sequence of a transmissible tournament of order n if and only if $S = (0, 1, 2, \cdots, n-1)$;

(d) there exists exactly one transmissible tournament of order n;

(e) there exists exactly one tournament of order n without a directed cycle;

(f) the condensation of a tournament is transmissible;

(g) every transmissible tournament contains exactly one Hamilton directed path;

(h) if T is not a transmissible tournament, then T contains at least three Hamilton directed paths;

(i) any tournament of order 2^{n-1} contains a transmissible tournament as its subgraph.

Exercises 1.8

1.8.1 Prove that a digraph G is eulerian if and only if G is connected and there are edge-disjoint directed cycles $C_1, C_2, \cdots, C_m$ such that $G = C_1 \oplus C_2 \oplus \cdots \oplus C_m$.

1.8.2 Suppose that G is a connected digraph and satisfies the condition

$$\sum_{x \in V} |d^+_G(x) - d^-_G(x)| = 2l, \quad l \geq 1.$$

Prove that there are l edge-disjoint directed trails $T_1, T_2, \cdots, T_l$ such that $G = T_1 \oplus T_2 \oplus \cdots \oplus T_l$.

1.8.3 Suppose that G is a connected digraph, x and y two distinct vertices of G. Prove that if

$$\begin{aligned} &d^+_G(x) - d^-_G(x) = l = d^-_G(y) - d^+_G(y); \text{ and} \\ &d^+_G(u) = d^-_G(u), \ \forall\ u \in V \setminus \{x, y\}, \end{aligned}$$

then there are at least l edge-disjoint (x, y)-paths in G.

1.8.4 Suppose that G is an undirected graph. Prove that there is an oriented graph D of G such that $|d^+_D(x) - d^-_D(x)| \leq 1$ for any $x \in V$.

1.8.5 Prove that the cartesian product of two eulerian graphs is an eulerian graph, hence $2n$-cube Q_{2n} is an eulerian graph.

1.8.6 (a) Prove that if G is a connected digraph, $|d^+_G(x) - d^-_G(x)| \leq 1$ for any $x \in V$ and any edge of G is not contained in odd number of directed cycles, then G is eulerian.

(b) Give an example to show that the converse of (a) is not true.

1.8.7 Prove that a connected undirected G is eulerian if and only if every edge of G lies on an odd number of cycles. (The necessity is due to Toida [160] and the sufficiency to McKee [127])

1.8.8 Let G be an eulerian graph and x be a vertex of G. Prove that every trail of G with origin x (maybe closed trail) can be extended to an Euler circuit of G if and only if $G - x$ contains no cycle.

1.8.9 By making a little modification of the way in Example 1.8.2, prove that there are d internally disjoint (x, y)-paths for any two distinct vertices x and y of $K(d, n)$, one of length at most n, $d - 2$ of length at most $n + 1$, and one of length at most $n + 2$. (Du, Hsu and Lyuu [43])

Exercises 1.9

1.9.1 (a) Prove that every hamiltonian graph contains no cut-vertex.

(b) Prove that every hamiltonian digraph is strongly connected.

(c) Prove that every hamiltonian bipartite graph is an equally bipartite graph.

(d) Give examples to show that the converse propositions of (a), (b) and (c) are not true.

1.9.2 (a) Prove the corollaries 1.10.1 $\sim$ 1.10.6 making use of Theorem 1.10.

(b) Give examples to show that the converse propositions of the corollaries 1.10.1 $\sim$ 1.10.6 are not true.

(c) Give examples to show that the degree-conditions in the corollaries 1.10.1 $\sim$ 1.10.5 can be improved.

1.9.3 (a) Prove that every strongly connected simple digraph with $v \geq 3$ and $\varepsilon > (v-1)(v-2)+2$ is hamiltonian.

(b) Construct a strongly connected and non-hamiltonian digraph with $v \geq 3$ and $\varepsilon = (v-1)(v-2)+2$.

1.9.4 (a) Prove that every connected simple undirected graph with $v \geq 3$ and $\varepsilon > \frac{1}{2}\,(v-1)(v-2)+1$ is hamiltonian.

(b) Construct a connected and non-hamiltonian undirected graph with $v \geq 3$ and $\varepsilon = \frac{1}{2}\,(v-1)(v-2)+1$.

1.9.5 The *closure* $c(G)$ of an undirected graph G is the graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least v until no such pair remains. Prove that $c(G)$ is well defined and a simple graph G is hamiltonian if and only if $c(G)$ is hamiltonian.

(Bondy and Chvatal [17])

1.9.6 Let G be a simple undirected graph with $v \geq 3$ and without a cut-vertex. Prove, using the exercise 1.9.5, that if $\max\{d_G(x), d_G(y)\} \geq \frac{1}{2}\,v$ for any two vertices x and y with distance two in G, then G is hamiltonian. (Genghua Fan [59])

1.9.7 Let $G = (X \cup Y, E)$ be a simple bipartite undirected graph with $|X| = |Y| = n\,(\geq 2)$. Prove that G is hamiltonian if it satisfies one of the following conditions:

(a) $d_G(x) + d_G(y) > n$ for any $x \in X$ and $y \in Y$ with $xy \notin E$;

(b) $\delta > \frac{1}{2}\,n$;

(c) $\varepsilon > n^2 - n + 1$.

1.9.8 (a) Prove that a digraph G is Eulerian if and only if its line digraph $L(G)$ is hamiltonian.

(b) Prove that the de Bruijn digraph $B(d,n)$ and the Kautz digraph $K(d,n)$ are hamiltonian.

(c) Construct a non-eulerian undirected graph G whose line graph $L(G)$ is hamiltonian.

1.9.9 Let n be an integer, G be a simple undirected graph, and $F \subseteq E(G)$ with $|F| = n$. Prove that if $d_G(x) + d_G(y) \geq \frac{1}{2}(v+n)$ for any two nonadjacent vertices x and y of G, and $G[F]$ is the union of several disjoint paths, then G contains a Hamilton cycle C with $F \subseteq E(C)$.

1.9.10 Prove that the following five problems are equivalent: To determine whether or not (Nash-Williams [139])

(a) an undirected graph contains a Hamilton cycle;

(b) an undirected graph contains a Hamilton path;

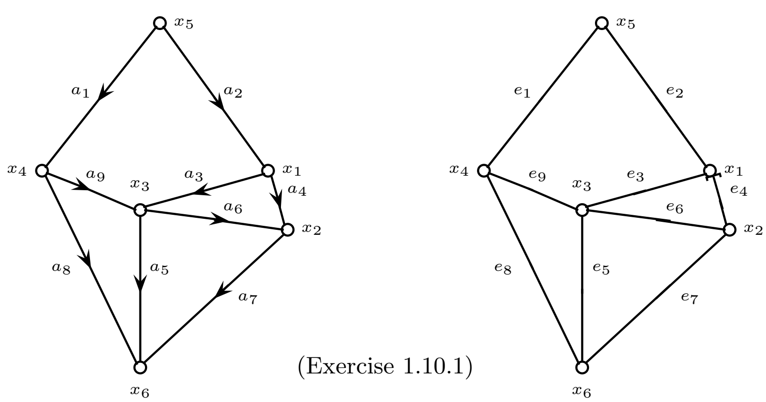
(c) a digraph contains a Hamilton directed cycle;

(d) a digraph contains a Hamilton directed path;

(e) a bipartite graph contains a Hamilton cycle.

Exercises 1.10

1.10.1 Write out the adjacency and incidence matrices of the following graphs



(Exercise 1.10.1)

1.10.2 Let $\mathbf{A}$ be the adjacency matrix of a graph. Question that

(a) what do the row sum and column sum of $\mathbf{A}$ represent, respectively?

(b) what does the sum of all elements in $\mathbf{A}$ represent?

1.10.3 Let $\mathbf{M}$ be the incidence matrix of a digraph D or an undirected graph G with the vertex-set $\{x_1, x_2, \cdots, x_v\}$. Prove that

(a) the sum of all positive (resp. negative) entries on ith row of $\mathbf{M}(D)$ is $d^+_D(x_i)$ (resp. $d^-_D(x_i)$); the sum of all entries on ith row of $\mathbf{M}(G)$ is $d_G(x_i)$;

(b) the jth column sum of $\mathbf{M}(D)$ is 0, while the jth column sum of $\mathbf{M}(G)$ is 2;

(c) rank $(\mathbf{M}) \leq v - \omega$;

(d) $\mathbf{M}$ is permutation equivalent to $\begin{pmatrix} \mathbf{M}_{11} & \mathbf{O} \\ \mathbf{O} & \mathbf{M}_{22} \end{pmatrix} \Longleftrightarrow D$ (or G) is disconnected.

1.10.4 Let $\mathbf{A}$ be the adjacency matrix of a digraph D or an undirected graph G. Prove that

(a) $\mathbf{A}$ is permutation similar to

(i) $\begin{pmatrix} \mathbf{O} & \mathbf{A_{12}} \\ \mathbf{A}_{21} & \mathbf{O} \end{pmatrix} \Longleftrightarrow D$ (or G) is bipartite,

(ii) $\begin{pmatrix} \mathbf{A}_{11} & \mathbf{O} \\ \mathbf{O} & \mathbf{A}_{22} \end{pmatrix} \Longleftrightarrow D$ (or G) is disconnected,

(iii) $\begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{O} & \mathbf{A}_{22} \end{pmatrix} \Longleftrightarrow D$ is not strongly connected,

(iv) an upper triangular matrix $\Longleftrightarrow D$ contains no directed cycle of length at least 2;

(b) D is strongly connected $\Longleftrightarrow \mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^{v-1} > 0$;

(c) if D is a strongly connected tournament with $v \geq 5$, then $\mathbf{A}^{d+3} > 0$, where $d = d(D)$ is diameter of D.

1.10.5 Let $\mathbf{A}$ be the adjacency matrix of an undirected graph G with the vertex-set $\{x_1, x_2, \cdots, x_v\}$, $\mathbf{M}$ the incidence matrix of any oriented graph D of G, and let $\mathbf{B}$ be the $v \times v$ diagonal matrix with main diagonal elements $b_{ii} = d_G(x_i)$.

(a) Prove $\mathbf{M}\mathbf{M}^T = \mathbf{B} - \mathbf{A}$.

(b) Prove that algebraic cofactors of all entries in $\mathbf{M}\mathbf{M}^T$ are identical.

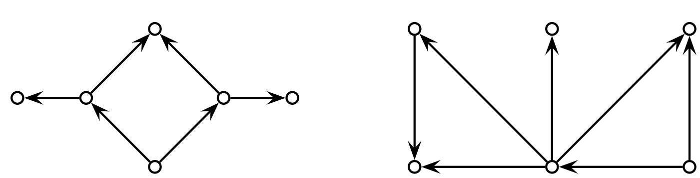
(c) Verify (a) for the graph shown in Exercise 1.10.1 and count the algebraic cofactor of the entry $(1, 1)$ in $\mathbf{M}\mathbf{M}^T$ (the value is 66).

(**d**) Prove that $\mathbf{B} - \mathbf{A}$ is semi-positive definite, and that G is connected if and only if rank $(\mathbf{B} - \mathbf{A}) = v - 1$. (D.Raghavarao, 1977)

1.10.6 Let $\mathbf{A}$ be the adjacency matrix of a graph G (undirected or directed). The eigenvalues of $\mathbf{A}$ is referred to as the *eigenvalues* of G; the characteristic polynomial det $(\lambda\mathbf{I} - \mathbf{A})$ is referred to as the *characteristic polynomial* of G. Suppose that characteristic polynomial of G is

$$P_G(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A}) = \lambda^v + c_1\lambda^{v-1} + \cdots + c_{v-1}\lambda + c_v.$$

(a) Count the characteristic polynomials of the following two graphs.



(Exercise 1.10.6)

(b) Prove
$$c_k = \sum_{H \in \mathcal{H}_k} (-1)^{\omega(H)}, \qquad k = 1, 2, \cdots, v,$$
where $\mathcal{H}_k$ is the set of (1-) 2-regular subgraphs with order k of (di)graphs of G. (M.Milic (1964), H. Sachs (1964), L.Spialter (1964))

(c) Prove that $c_1 = 0$; $-c_2 = \varepsilon$; and $-c_3$ is equal to twice the number of triangles in G.

(d) Prove that if $\lambda_1, \lambda_2, \cdots, \lambda_v$ are all eigenvalues of G, then

(i) $\lambda_1 + \lambda_2 + \cdots + \lambda_v = -c_1$;

(ii) the number of different directed closed walks of length k in G is $(\lambda_1^k + \lambda_2^k + \cdots + \lambda_v^k)$.

(e) Let λ be the maximum eigenvalue of G. Prove that

(i) $\delta^+ \le \lambda \le \Delta^+$, and $\delta^- \le \lambda \le \Delta^-$ (or $\delta \le \lambda \le \Delta$),

and the equalities hold if and only if G is regular;

(ii) if G is strongly connected and regular, then λ has the multiplicity 1.

(f) Prove that a strongly connected digraph of diameter d has at leat $d + 1$ distinct eigenvalues.

1.10.7 (a) Let $\mathbf{A}$ be the adjacency matrix of a digraph G. Prove that there is a polynomial $p(x)$ such that $\mathbf{J} = p(\mathbf{A})$ if and only if G is strongly connected and regular.

(b) Let C_n be a directed cycle of order n, $\mathbf{A}$ be the adjacency matrix of C_n. Find a polynomial $p(x)$ such that $\mathbf{J} = p(\mathbf{A})$.

1.10.8 (a) Let G be an undirected graph of diameter 2. Prove that if $\Delta \neq 2, 3, 7$ or 57, then $v \le \Delta^2$.

(This result is due to Hoffman and Singleton [97]. In fact, Erdös, Fajtlowicz and Hoffman [55] have shown $v \le \Delta^2 - 1$.)

(b) Construct two undirected graphs with diameter 2 and the maximum degree $\Delta = 2$ and 3, respectively, such that $v = \Delta^2 + 1$.

(Hoffman and Singleton [97] have constructed such a graph for $\Delta = 7$. However, whether there exists an undirected graph of order $v = \Delta^2 + 1$ and maximum degree $\Delta = 57$ is unknown.)

Exercises 1.11

1.11.1 Let matrices
$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

(a) Drawing associated digraphs with $\mathbf{A}$ and $\mathbf{B}$, respectively.

(b) Prove that $\mathbf{A}$ is not primitive, but $\mathbf{B}$ is primitive and $e(\mathbf{B}) = 9$.

1.11.2 Prove that

(a) every strongly connected digraph with a loop must be primitive;

(b) $e(G) \leq d(G)$, and hence $e(G) \leq n-1$, for any strongly connected digraph G with a loop at every vertex, where $d(G)$ is diameter of G of order n.

1.11.3 Prove $e(\mathbf{A}) = n+2$, where $\mathbf{A}$ is an $n(\geq 5)$-square matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & \cdots & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

1.11.4 Let T_n be a strongly connected tournament of order n (≥ 4). Prove that

(a) $e(T_n) \geq 3$;

(b) $e(T_n) \neq 3$ for $n \leq 6$;

(c) there is T_5 such that $e(T_5) = k$ for any k with $4 \leq k \leq 7$;

(d) there is T_6 such that $e(T_6) = k$ for any k with $4 \leq k \leq 8$;

(e) if there is T_n such that $e(T_n) = k$, then there are T_{n+1} and T'_{n+1} such that $e(T_{n+1}) = k$ and $e(T'_{n+1}) = k+1$ for any n and k with $n \geq 5$ and $3 \leq k \leq n+2$;

(f) there is T_n such that $e(T_n) = k$ for any n and k with $n \geq 7$ and $3 \leq k \leq n+2$.

1.11.5 An n-square matrix $\mathbf{A}$ is called to be *reducible* if there is a permutation matrix $\mathbf{P}$ such that

$$\mathbf{P}\mathbf{A}\mathbf{P}^T = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{O} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix},$$

where $\mathbf{A}_{11}$ is an l-square matrix, $1 \leq l \leq n-1$; and to be *irreducible* otherwise. Prove that

(a) a nonnegative $n(>1)$-square matrix $\mathbf{A}$ is irreducible if and only if $G(\mathbf{A})$ is strongly connected;

(b) if $\mathbf{A}$ is irreducible nonnegative n-square matrix with at least k (≥ 1) non-zero diagonal elements, then $\mathbf{A}$ is primitive and $e(\mathbf{A}) \leq 2n-k-1$.