Exercises in Chapter 2

Exercises 2.1

2.1.1 Let G be a digraph and x be a vertex of G. Prove that

(a) G is an out-tree rooted at x if and only if G contains no directed cycle, $d_G^-(x) = 0$ and $d_G^-(y) = 1$ for any y other than x in G;

(b) G is an in-tree rooted at x if and only if G contains no directed cycle, $d_G^+(x) = 0$ and $d_G^+(y) = 1$ for any y other than x in G.

2.1.2 Let G be a nontrivial tree. Prove that

(a) two end-vertices of any longest path in G are of degree one;

(b) all longest paths have at least one vertex in common;

(c) G contains at least $(v-k)$ paths of length at least k if $d(G) \geq 2k-3$ and $k \geq 2$.

2.1.3 Let G be a nontrivial tree, v_i denote the number of i-degree vertices in G. Prove that

(a) $v_1 \geq \Delta(G)$ and G is a path if $v_1 = 2$;

(b) $v_1 \geq v_i + 2$ for each $i = 3, 4, \cdots, \Delta$ if $\Delta \geq 3$;

(c) $v_1 = 2 + \sum\limits_{x \in U} (d_G(x) - 2)$, where $U = \{x \in V(G) : \ d_G(x) \geq 3\}$.

2.1.4 Let T be a tree of order at least two. Prove that if $\{X, Y\}$ is a bipartition of T with $|X| = |Y| + k$, then there are at least $(k+1)$ vertices of degree one in X.

2.1.5 Let G_1 and G_2 be two distinct trees with the same vertex-set V. Prove that $d(G_1) = d(G_2)$ if $G_1 - x \cong G_2 - x$ for any $x \in V$. (P.J.Kelly, 1957)

2.1.6 Let G be a forest with exactly $2k\,(k \geq 1)$ vertices of odd degree. Prove that there are k edge-disjoint paths $P_1, P_2, \cdots, P_k$ in G such that $E(G) = E(P_1) \cup E(P_2) \cup \cdots \cup E(P_k)$.

2.1.7 Let G be a nontrivial tree of order k. Prove that any simple undirected graph H with $\delta(H) \geq k-1$ contains a subgraph isomorphic to G.

2.1.8 Let G be a tree and $G_i = (V_i, E_i)\,(i = 1, 2, \cdots, k)$ be sub-trees of G, $B = V_1 \cap V_2 \cap \cdots \cap V_k$. Prove that

(a) $B \neq \emptyset$ if $V_i \cap V_j \neq \emptyset$ for $1 \leq i \neq j \leq k$;

(b) $G[B]$ is a tree if $B \neq \emptyset$.

2.1.9 Prove that any graph contains at least $\varepsilon - v + \omega$ distinct cycles and $v - \omega$ distinct bonds.

2.1.10 Prove that if G is a strongly connected digraph and x is any vertex of G then there are a spanning out-tree and a spanning in-tree rooted at x in G.

2.1.11 Prove that if G is a connected graph and S is a nonempty proper subset of $V(G)$ then the cut $[S, \overline{S}]$ is minimal if and only if both $G[S]$ and $G[\overline{S}]$ are connected.

2.1.12 Prove that every cut can be expressed as a union of several edge-disjoint bonds.

2.1.13 Prove that the symmetric difference $B \Delta B'$ of two distinct cuts B and B' is still a cut, and so contains a bond.

2.1.14 Prove that any complete graph of order at least four contains at least two edge-disjoint spanning trees.

2.1.15 The *tree graph* of a connected graph G, denoted by $T(G)$, is a simple undirected graph whose vertex-set consists of all spanning trees of G, and two spanning trees T_i and T_j are linked by an undirected edge if and only if T_i and T_j have exactly $v - 2$ edges in common. Prove that $T(G)$ is connected for any graph G.

2.1.16 Prove that if a graph G contains k edge-disjoint spanning trees then, for any partition $\{V_1, V_2, \cdots, V_n\}$ of $V(G)$, the number of edges whose end-vertices are in different parts of the partition is at least $k(n-1)$.

(W.T.Tutte (1961) and C.St.J.A.Nash-Williams (1961) proved that this necessary condition for G to contain k edge-disjoint spanning trees is also sufficient.)

2.1.17 A spanning subgraph T of a connected graph G is called to be *distance-preserving* from a vertex x in G if $d_T(x, y) = d_G(x, y)$ for every vertex y. Prove that for every vertex x of a connected graph G, there exists a spanning tree T that is distance-preserving from x. (O.Ore, 1962)

2.1.18 Let G be a connected undirected simple graph and x be a vertex of G. Prove that there is an oriented graph D that contains a spanning out-tree T rooted at x satisfying the following conditions:

(a) $T + a$ contains a directed cycle for any directed edge a of the cotree $\overline{T}$;

(b) there is a directed edge a of $\overline{T}$ such that $C \subseteq T + a$ for any directed cycle C of D.

Exercises 2.2

2.2.1 Give a proof of Theorem 2.6 (b).

2.2.2 Let C be a cycle in a loopless digraph G. Prove that

(a) $\mathbf{f}_c \in \mathscr{E}(G)$ is a cycle-vector in G;

(b) all cycle-vectors in G form a subspace of $\mathscr{E}(G)$.

2.2.3 Let B be a bond in a digraph G. Prove that

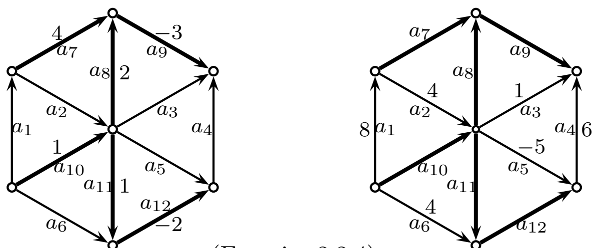
(a) $\mathbf{g}_B \in \mathscr{E}(G)$ is a bond-vector in G;

(b) all bond-vectors in G form a subspace of $\mathscr{E}(G)$.

2.2.4 In the following digraph G, the heavy edges indicate a spanning tree T of G.

(a) Extend the weights on T to a bond-vector $\mathbf{g}$ in G, write out the basis matrix $\mathbf{B}_T$ of $\mathscr{B}(G)$ corresponding to T and the expression of $\mathbf{g}$ by using this basis.

(b) Extend the weights on $\overline{T}$ to a cycle-vector $\mathbf{f}$ in G, write out the basis matrix $\mathbf{C}_T$ of $\mathscr{C}(G)$ corresponding to T and the expression of $\mathbf{f}$ by using this basis.



(Exercise 2.2.4)

2.2.5 Let F be a spanning forest in a digraph G, $\mathbf{B}_F = (\mathbf{B}_1\ \mathbf{I}_{v-\omega})$ and $\mathbf{C}_F = (\mathbf{I}_{\varepsilon-v+\omega}\ \mathbf{C}_2)$ be the basis matrices of $\mathscr{B}(G)$ and, respectively, $\mathscr{C}(G)$ corresponding to F. Prove that

(a) $\mathbf{C}_F(\mathbf{B}_F)^T = \mathbf{O}$;

(b) $\mathbf{C}_F = \left(\mathbf{I}_{\varepsilon-v+\omega}\ \ -\mathbf{B}_1^T\right)$;

(c) $\mathbf{B}_F = \left(-\mathbf{C}_2{}^T\ \mathbf{I}_{v-\omega}\right)$.

2.2.6 Let $\mathbf{f}$ be a cycle-vector and $\mathbf{g}$ be a bond-vector in $\mathscr{E}(G)$, and let F be a spanning forest of G, $\mathbf{B}_F$ and $\mathbf{C}_F$ be the basis matrices of $\mathscr{B}(G)$ and $\mathscr{C}(G)$ corresponding to F, respectively. Prove that

(a) $\mathbf{f}$ is uniquely determined by $\mathbf{f}|\overline{F}$ and $\mathbf{f} = (\mathbf{f}|\overline{F}\,)\mathbf{C}_F$;

(b) $\mathbf{g}$ is uniquely determined by $\mathbf{g}|F$ and $\mathbf{g} = (\mathbf{g}|F)\mathbf{B}_F$.

2.2.7 Let $\mathbf{C}$ be a basis matrix of the cycle space $\mathscr{C}(G)$, F be spanning forest of G and $\mathbf{C}_F$ be the basis matrix of $\mathscr{C}(G)$ corresponding to F. Prove that $\mathbf{C}$ is uniquely determined by $\mathbf{C}\,|\overline{F}$, and $\mathbf{C} = (\mathbf{C}\,|\overline{F})\mathbf{C}_F$.

2.2.8 Let G be a loopless digraph. Prove that

(a) any cycle of G can be represented into symmetric differences of several fundamental cycles;

(b) any bond of G can be represented into symmetric differences of several fundamental bonds.

Exercises 2.3

2.3.1 Let $\mathbf{M}$ be the incidence matrix of a connected loopless digraph G, $\mathbf{K}$ be a matrix obtained from $\mathbf{M}$ by deleting any one of its rows, $\mathbf{C}$ be unimodular basis matrix of the cycle-space $\mathscr{C}(G)$. Prove that

(a) $\varsigma(G) = \pm\det\begin{bmatrix}\mathbf{K}\\ \mathbf{C}\end{bmatrix}$.

(b) the algebraic cofactor of any entry in $\mathbf{M}\mathbf{M}^T$ is equal to $\varsigma(G)$.

2.3.2 Let G be a connected, labelled and loopless undirected graph with the vertex-set $\{x_1, x_2, \cdots, x_v\}$ and $\mathbf{A}$ be the adjacency matrix of G. Let $\mathbf{B}$ be the $v \times v$ diagonal matrix with the main diagonal elements $b_{ii} = d_G(x_i)$ for each $i = 1, 2, \cdots, v$. Prove that the algebraic cofactor of any entry in $\mathbf{B} - \mathbf{A}$ is equal to $\varsigma(G)$.

2.3.3 A matrix is *totally unimodular* if all square submatrices have determinants $0, +1$ or -1. Prove that

(a) any basis matrix of $\mathscr{B}$ and $\mathscr{C}$ corresponding to a spanning forest is totally unimodular;

(b) the incidence matrix of a loopless digraph is totally unimodular;

(c) the incidence matrix of a simple undirected graph G is totally unimodular if and only if G is bipartite. (J.Egerváry, 1931)

2.3.4 Prove that the following labelled undirected graphs

(a) $\varsigma(C_n) = n$, where C_n is a cycle of order n;

(b) $\varsigma(K_n - e) = (n-2)n^{n-3}$, where $K_n - e$ denotes a subgraph of a complete graph K_n by deleting an edge e;

(c) $\varsigma(K_{m,n}) = m^{n-1}n^{m-1}$, where $K_{m,n}$ is a complete bipartite graph.

2.3.5 Let G be a nontrivial connected loopless graph, and e an edge of G. Prove that $\varsigma(G) = \varsigma(G \cdot e) + \varsigma(G - e)$.

2.3.6 Let G be a digraph without directed cycles, x a vertex of G, and let ς_x denote the number of spanning out-trees rooted at x of G.

(a) Prove that $\varsigma_x(G) = \prod\limits_{y \in V \setminus \{x\}} d_G^-(y)$.

(b) Verify the conclusion in (a) for the tournament T_4 in Figure **??**.

2.3.7 Let G be a connected and loopless digraph with the vertex-set $\{x_1, x_2, \cdots, x_v\}$, $\mathbf{A}$ the adjacency matrix of G and $\mathbf{M}$ the incidence matrix of G. Let $\mathbf{S}$ be the $v \times v$ diagonal matrix with main diagonal elements $s_{ii} = d_G^-(x_i)$, $\mathbf{N} = \mathbf{S} - \mathbf{A}$, N_{ij} the algebraic cofactor of the entry n_{ij} in $\mathbf{N}$; $\mathbf{M}_i$ a matrix obtained from $\mathbf{M}$ by deleting the row x_i; $\overline{\mathbf{M}}_i$ a matrix obtained from $\mathbf{M}_i$ by replacing 0 by 1; let ς_i denote the number of spanning out-trees rooted at x_i of G. Prove that

(a) if $\varepsilon = v - 1$, then G is an out-tree rooted at x_i if and only if $N_{ii} = 1$ for every $i = 1, 2, \cdots, v$;

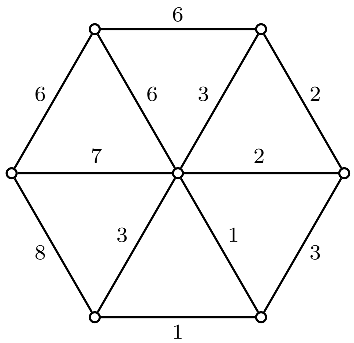
(b) $\varsigma_i(G) = N_{ii}$; (W.T.Tutte, 1964)

(c) $\varsigma_i(G) = \det(\overline{\mathbf{M}}_i \mathbf{M}_i^T)$; (W.T.Tutte, 1948)

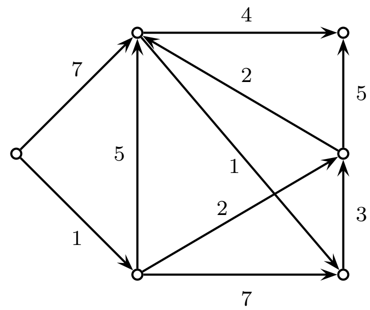
(d) the total number of spanning out-trees of a complete digraph of order n is equal to n^{n-1}.

Exercises 2.4

2.4.1 Constructing a minimum tree in the following weighted graph by using Prim's algorithm.



(Exercise 2.4.1 and 2.4.3)



(Exercise 2.4.5)

2.4.2 Prove that if every edge of a connected simple graph G has different weight then a minimum tree constructed by Prim's algorithm is unique.

2.4.3 **Kruskal's algorithm** for finding a minimum tree in a weighted connected graph $(G, \mathbf{w})$ is stated as follows.

1. Choose $e_1 \in E(G)$ such that $\mathbf{w}(e_1)$ is as small as possible.
2. If $e_1, e_2, \cdots, e_i$ have been chosen, then choose $e_{i+1} \in E(G)\backslash\{e_1, e_2, \cdots, e_i\}$ such that $\mathbf{w}(e_{i+1})$ is as small as possible and $G[\{e_1,\ e_2, \cdots, e_{i+1}\}]$ contains no cycle.
3. Stop when Step 2 can not be implemented further.

(a) Prove that a subgraph of G constructed by Kruskal's algorithm is a minimum tree in G.

(b) Constructing a minimum tree in the above weighted graph by using Kruskal's algorithm.

2.4.4 Can Prim's and Kruskal's algorithms be adapted to find a *maximum-weight spanning tree* in a weighted connected graph? If so, how?

2.4.5 By making a modification to Prim's algorithm, we can obtain the following algorithm for finding a *minimum weight spanning out-tree* rooted at x_0 in a weighted strongly connected digraph $(G, \mathbf{w})$ for a given vertex x_0 in G.

1. Set $\mathbf{l}(x_0) = 0,\ \mathbf{l}(x) = \infty\ (x \neq x_0)$, $V_0 = \{x_0\}$ and $T_0 = x_0, k = 0$.
2. For any $x \in N_G^+(x_{k-1}) \cap \overline{V}_{k-1}$, if $\mathbf{w}(x_{k-1}, x) < \mathbf{l}(x)$, then replace $\mathbf{l}(x)$ by $\mathbf{w}(x_{k-1}, x)$. So choose $x_k \in \overline{V}_{k-1}$ that $\mathbf{l}(x_k) = \min\{\mathbf{l}(x) :\ x \in \overline{V}_{k-1}\}$. Let $e_k = (u, x_k),\ u \in V_{k-1}$ such that $\mathbf{w}(e_k) = \mathbf{l}(x_k)$. Let $V_k = V_{k-1} \cup \{x_k\}$, and $T_k = T_{k-1} + e_k$.
3. Stop when step 2 can not be implemented further.

(a) Prove that a subdigraph constructed by this algorithm is a minimum weight spanning out-tree rooted at x_0 (if such an out-tree exists).

(b) Using the algorithm find a minimum weight spanning out-tree rooted at x_0 in the above digraph.

(c) Prove that this algorithm can be used to verify whether or not a connected digraph with a given vertex x_0 contains a spanning out-tree rooted at x_0.

Exercises 2.5

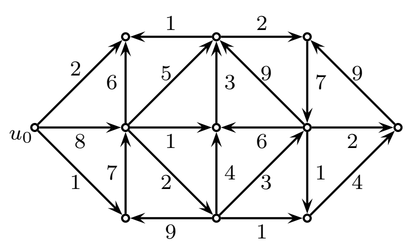
2.5.1 Prove that the complexity of Dijsktra's algorithm is $O(v^2)$.

2.5.2 A company has branches in each of six cities $x_1, \cdots, x_6$. The fare for a direct fight from x_i to x_j is given by the (i, j)th entry in the following matrix:

$$\begin{pmatrix} 0 & 50 & \infty & 40 & 25 & 10 \\ 50 & 0 & 15 & 20 & \infty & 25 \\ \infty & 15 & 0 & 10 & 20 & \infty \\ 40 & 20 & 10 & 0 & 10 & 25 \\ 25 & \infty & 20 & 10 & 0 & 55 \\ 10 & 25 & \infty & 25 & 55 & 0 \end{pmatrix}$$

Design a scheme of cheapest routes between pair of cities.

2.5.3 Find shortest directed paths from x_0 to all other vertices in the following weighted digraph.



(Exercise 2.5.3)

2.5.4 Suppose that there are a full twelve-gallon jug of eight, and two empty jugs of five and three gallons capacity, respectively. Question:

(a) whether or not it is possible to divide the wine equally. If so, what is the simplest way?

(b) whether or not it is possible to divide the wine equally if replace the jug of three gallons by seven.

2.5.5 A wolf, a goat and a cabbage are on one bank of a river. A ferryman wants to take them across, but, since his boat is small, he can take only one of them at a time. For obvious reason, neither the wolf and the goat nor the goat and the cabbage can be left unguarded. How is the ferryman going to get them across the river?

Exercises 2.6

2.6.1 Let $T = \{e_1, e_2, e_4\}$ be a spanning tree in the digraph shown in Figure **??**. Write out

(a) a system of independent equations by the KCL;

(b) a system of independent equations by the KVL;

(c) the expression of $\mathbf{w}$ in $\mathbf{w}_c$;

(d) the expression of $\mathbf{u}$ in $\mathbf{u}_t$.

2.6.2 Prove that Tellegen's theorem: if two electrical networks N and $\overline{N}$ have the same associated digraph G, then

(a) $\mathbf{u}^{\mathbf{T}}\overline{\mathbf{w}} = \mathbf{O}$ and

(b) $\mathbf{w}^{\mathbf{T}}\overline{\mathbf{u}} = \mathbf{O}$,

where $\mathbf{u}$, $\overline{\mathbf{u}}$ and $\mathbf{w}$, $\overline{\mathbf{w}}$ are the column vectors of the voltages and the currents in N and $\overline{N}$, respectively.