Exercises in Chapter 3

Exercises 3.1

3.1.1 Prove that

(a) a graph G is planar if and only if each of its components is planar;

(b) if G is a plane graph, then $v - \varepsilon + \phi = \omega + 1$.

3.1.2 Let G be a graph. Prove that

(a) if G is planar, $x \in V(G)$ and $e \in E(G)$ then G can be embedded in the plane in such a way that x (resp. e) is on the boundary of the exterior face of the embedding;

(b) G is planar if and only if each block of G is planar.

3.1.3 Prove that each simple planar graph of order $v \geq 3$ is a spanning subgraph of some plane triangulation.

3.1.4 Let G be a graph of order $v \geq 4$, and v_i be the number of i-degree vertices of G. Prove that

(a) if G is a plane triangulation, then

$$3v_3 + 2v_4 + v_5 = v_7 + 2v_8 + \cdots + (\Delta - 6)v_\Delta + 12;$$

(b) if $\delta(G) = 5$, then G contains at least 12 vertices of degree 5;

(c) if G is a tree, then

$$v_1 = v_3 + 2v_4 + 3v_5 + \cdots + (\Delta - 2)v_\Delta + 2.$$

3.1.5 Prove that, if G is a connected plane graph and each of its faces has degree four, then $\varepsilon = 2v - 4$.

3.1.6 Let G be a connected 3-regular plane graph, and ϕ_i be the number of the faces of i-degree of G. Prove that

(a) $12 = 5\phi_1 + 4\phi_2 + 3\phi_3 + 2\phi_4 + \phi_5 - \phi_7 - 2\phi_8 - \cdots$;

(b) G contains a face of degree fewer than 6.

3.1.7 Prove that

(a) if G is a connected planar graph with girth $g \geq 3$, then

$$\varepsilon \leq g(v - 2)/(g - 2);$$

(b) Petersen graph is non-planar.

3.1.8 (a) Prove that, if G is a simple planar graph of order $v \geq 11$, then G^c is non-planar. (W. T. Tutte (1973) showed that the assertion is true for $v \geq 9$.)

(b) Construct a simple planar graph G of order 8 such that G^c is also planar.

3.1.9 Let G be a simple planar graph. Prove that

(a) G contains at least 4 vertices of degree fewer than 6 if $v \geq 4$;

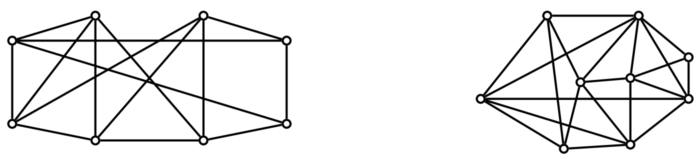
(b) there exists exactly one 4-regular plane triangulation.

3.1.10 Let $S = \{x_1, x_2, \cdots, x_n\}$ be a set of $n\,(\geq 3)$ points in the plane such that the distance between any two points is at least one. Prove that there are at most $3n-6$ pairs of points at distance exactly one.

3.1.11 Prove that if G is a simple planar graph with $v \geq 5$ and $\Delta = v-1$ then G contains two nonadjacent vertices of degree at most 3.

(N.Alon and Y.Caro, 1984)

3.1.12 Find a planar embedding of the following graphs in which each edge is a straight line segment.



(Exercise 3.1.12)

3.1.13 A graph G is called a *minimal non-planar graph* if G is non-planar and each subgraph of G is planar. Prove that

(a) both K_5 and $K_{3,3}$ are minimal non-planar;

(b) each minimal non-planar graph is a block.

3.1.14 Let G be a simple planar graph of order v. Prove that

(a) $\sum\limits_{x\in V} d_G(x)^2 \leq 2(v+3)^2 - 62$ if $v \geq 4$;

(b) $\sum\limits_{x\in V} d_G(x)^2 < 2(v+3)^2 - 62$ if $\delta \geq 4$.

3.1.15 If a graph G can be drawn in the 3-dimensional space $\mathscr{R}^3$, then G is said to be *embeddable in* $\mathscr{R}^3$ so that its edges intersect only at their end-vertices. Prove that

(a) all graphs are embeddable in $\mathscr{R}^3$;

(b) all simple graphs are embeddable in $\mathscr{R}^3$ so that each edge is a straight line segment.

3.1.16 The *thickness* of G, $\vartheta(G)$, is the minimum number of planar graphs into which the edges of G can be partitioned. It is clear that $\vartheta(G) = 0$ if and only if G is planar. Prove that

(a) if G is a simple graph of order $v \geq 3$ then $\vartheta(G) \geq \left\lceil \dfrac{\varepsilon}{3v-6} \right\rceil$, and the equality holds for all K_v's with $3 \leq v \leq 8$;

(b) $\vartheta(K_9) = \vartheta(K_{10}) = 3$;

(c) $\vartheta(K_v) \geq \left\lfloor \dfrac{v+7}{6} \right\rfloor$.

(It has been shown that the equality holds for all K_v's with $v \geq 3$ and $v \neq 9, 10$ by L. W. Beineke and F. Harary (1965), V. B. Alekeev and V. S. Gonchakov (1976).)

3.1.17 The *crossing number* of G, $r(G)$, is the minimum number of pairwise intersections of its edges when G is drawn in the plane. Obviously, $r(G) = 0$ if and only if G is planar. Prove that $r(K_5) = r(K_{3,3}) = 1$ and $r(K_6) = 3$.

(It has been prove that

$$r(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$$

for $\min\{m, n\} \leq 6$ by Kleitman [**?**] and

$$r(K_v) = \frac{1}{4} \left\lfloor \frac{v}{2} \right\rfloor \left\lfloor \frac{v-1}{2} \right\rfloor \left\lfloor \frac{v-2}{2} \right\rfloor \left\lfloor \frac{v-3}{2} \right\rfloor$$

for $v \leq 10$ by Guy [**?**]. Moreover, it has been conjectured that the two equalities hold for any positive integers m, n and v.)

Exercises 3.2

3.2.1 Prove that (a) any subdivision of a non-planar graph is also non-planar;

(b) any subgraph of a planar graph is also planar.

3.2.2 Prove that G is planar if either $\varepsilon < 9$ or $v < 5$.

3.2.3 Prove that for any three vertices x, y, z of a simple planar graph G,

$$d_G(x) + d_G(y) + d_G(z) \leq 2v + 2.$$

3.2.4 Prove that if G is a minimal planar graph, then G has a basic cycles together with one additional cycle such that this collection of cycles contains each edge of G exactly twice.

3.2.5 Prove that if C is a cycle of a planar graph G, then G has a planar embedding $\widetilde{G}$ such that C partitions all faces of $\widetilde{G}$ into two parts, $\operatorname{Int} C$ and Ext C, one part is contained in Int C and the other in Ext C.

3.2.6 Prove that if G is a plane graph of odd order and contains a Hamilton cycle, then G has even (≥ 2) faces of odd-degree.

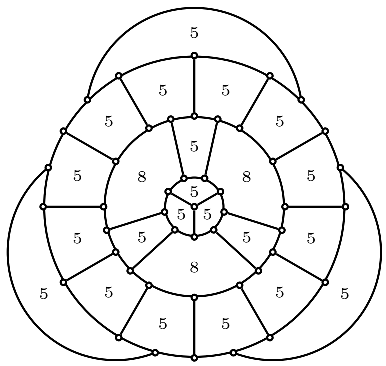
3.2.7 Prove that

(a) if G is a loopless plane graph with a Hamilton cycle C, then

$$\sum_{i=1}^{v} (i-2)(\phi_i' - \phi_i'') = 0$$

where ϕ_i' and ϕ_i'' are the numbers of faces of degree i contained in $\operatorname{Int} C$ and $\operatorname{Ext} C$, respectively; (È. Ja. Grinberg [?])

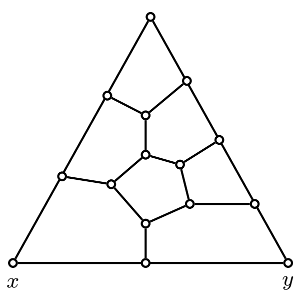
(b) Grinberg graph is non-planar;



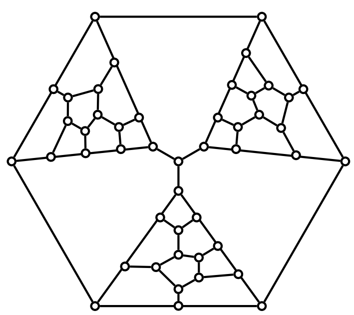
(Exercise 3.2.7 (b) Grinberg graph)

(c) the following graph contains no xy-Hamilton path;

(d) Tutte graph is non-planar.



(Exercise 3.2.7 (c))



(Exercise 3.2.7 (d) Tutte graph)

Exercises 3.3

3.3.1 Let G be a plane graph and G^* be the geometric dual of G.

(a) Prove that G^* is a connected plane graph.

(b) Prove that G is isomorphic to the geometric dual G^{*^*} of G^* if and only if G is connected.

(c) Construct a plane graph G such that G is not isomorphic to the geometric dual G^{**} of G^*.

(d) Construct an example to show that isomorphic plane graphs may have non-isomorphic geometric duals.

3.3.2 Prove that

(a) G has the geometric (combinatorial) dual if and only if every connected component of G has the geometric (combinatorial) dual;

(b) G has the geometric (combinatorial) dual if and only if every subdivision of G has the geometric (combinatorial) dual.

3.3.3 Prove that

(a) Theorem 3.2 using (3.1) and Theorem 3.7 (b);

(b) the geometric dual of any plane graph without odd-vertices is bipartite;

(c) any plane graph has even faces of odd degree.

3.3.4 Prove that if G is a connected plane graph then $\varsigma(G) = \varsigma(G^*)$, where G^* is the geometric dual of G.

3.3.5 A plane graph is *self-dual* if it is isomorphic to its geometric dual.

(a) Prove that if G is self-dual plane graph then $\varepsilon = 2v - 2$.

(b) Find a self-dual plane graph of order v for each $v \geq 4$.

(c) Prove that a *wheel* W_n $(= K_1 \vee C_{n-1})$ is self-dual.

3.3.6 A bond B of a connected graph G is called a *Hamilton cut* if two connected components of $G - B$ both are trees. Prove that

(a) a connected plane graph G contains a Hamilton cycle if and only if the geometric dual G^* contains a Hamilton cut;

(b) if G contains a Hamilton cut B, then

$$\sum_{i=1}^{\Delta}(i-2)(v_i' - v_i'') = 0$$

where v_i' and v_i'' are the numbers of i-degree vertices of G in two connected components G_1 and G_2 of $G - B$, respectively.

Exercises 3.4

3.4.1 Prove that there is no such convex polyhedron that has odd faces of odd degree.

3.4.2 Prove that any convex polyhedron contains at least 6 edges.

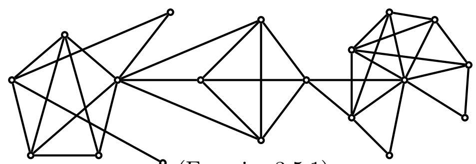
3.4.3 Prove that there is no such convex polyhedron that has 7 edges.

3.4.4 Prove that excepting the tetrahedron there is no such convex polyhedron that each of its vertices is adjacent to all of others.

3.4.5 Prove that any convex polyhedron contains either a face of degree three or a vertex of degree three.

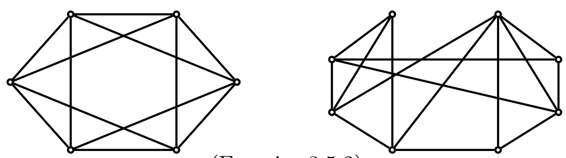
Exercises 3.5

3.5.1 Preprocess the following graph by applying the way described in the beginning of this section.



(Exercise 3.5.1)

3.5.2 Test planarity of the following graph by applying the DMP algorithm.



(Exercise 3.5.2)

3.5.3 Prove that the Petersen graph, K_5 and $K_{3,3}$ are non-planar by applying the DMP algorithm.

3.5.4 (a) Describe the main operations involved in the DMP algorithm.

(b) Show that the DMP algorithm is efficient.