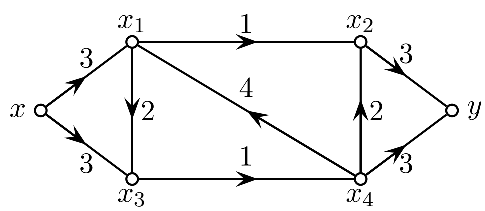
Exercises in Chapter 4

Exercises 4.1

4.1.1 Find all (x, y)-cuts and an (x, y)-flow $\mathbf{f}$ with value 2 in the following integral capacity network, and prove that $\mathbf{f}$ is maximum.



(Exercise 4.1.1)

4.1.2 (a) Prove that the equality (4.3).

(b) Prove that for any (x, y)-flow $\mathbf{f}$ in a capacity network $N = (G_{xy}, \mathbf{c})$ and $\emptyset \neq S \subset V(G)$,

$$\sum_{u \in S} (\mathbf{f}^+(u) - \mathbf{f}^-(u)) = \mathbf{f}^+(S) - \mathbf{f}^-(S).$$

(c) Construct an example to show

$$\sum_{u \in S} \mathbf{f}^+(u) \neq \mathbf{f}^+(S), \qquad \sum_{u \in S} \mathbf{f}^-(u) \neq \mathbf{f}^-(S).$$

4.1.3 Prove that (a) the assertion in the proof of Theorem 4.1: $\mathbf{f}'$ is an (x, y)-flow and val $\mathbf{f}' =$ val $\mathbf{f} + \sigma$; (b) the equality (4.5).

4.1.4 Prove that (a) the corollary 4.1;

(b) for any nonnegative real capacity network, there must be a maximum flow.

4.1.5 Let $\mathbf{f}$ be an (x, y)-flow in N and $B = (S, \overline{S})$ an (x, y)-cut. Prove that

$$\text{val } \mathbf{f} = \text{cap } B \Leftrightarrow \mathbf{f}(a) = \begin{cases} \mathbf{c}(a), & a \in (S, \overline{S}); \\ 0, & a \in (\overline{S}, S) \end{cases}$$

$$\Leftrightarrow \mathbf{f} \text{ is maximum and } B \text{ is minimum.}$$

4.1.6 Let $(S, \overline{S})$ and $(T, \overline{T})$ be two minimum (x, y)-cuts in N. Prove that both $(S \cup T, \overline{S \cup T})$ and $(S \cap T, \overline{S \cap T})$ are minimum (x, y)-cuts in N.

Exercises 4.2

4.2.1 (Menger's theorem) Let G be an undirected graph with two distinct vertices x and y. Prove that

(a) $\eta_G(x, y) = \lambda_G(x, y)$;

(b) $\zeta_G(x, y) = \kappa_G(x, y)$ if x and y are not adjacent in G.

4.2.2 Construct digraphs with two distinct x and y to show that the following statements are not true.

(a) If any two (x, y)-path and (y, x)-path have a common edge, then there is an edge which is contained in all of these paths.

(b) If $\eta_G(x, y) \geq k\,(\geq 1)$, then $\eta_G(y, x) \geq k$.

(c) If $\eta_G(x, y) \geq k\,(\geq 1)$ and $\eta_G(y, x) \geq k$, then there are k edge-disjoint (x, y)-paths $P_1, P_2, \cdots, P_k$ and k edge-disjoint (y, x)-paths $Q_1, Q_2, \cdots, Q_k$ which are pairwise-edge-disjoint.

4.2.3 Let G be a digraph with two distinct vertices x and y. Prove that

(a) if G is connected and each vertex of G other than x and y is balanced, and $d_G^+(x) - d_G^-(y) = k$, then $\eta_G(x, y) \geq k$;

(b) if G is balanced, then three statements in the exercise 4.2.2 are true.

4.2.4 Let G be undirected graph with two distinct vertices x and y. Prove that if G has diameter two, then there are $\min\{d_G(x), d_G(y)\}$ edge-disjoint xy-paths of length at most 4. (Peyrat [**?**])

4.2.5 Let G be a k-regular graph, $k \geq 2$. Prove that if there are k internally disjoint paths between any two vertices in G, then there are two vertices x and y such that at least one is of length at least $d(G)+1$ among any k internally disjoint (x, y)-paths.

4.2.6 Let G be a k-regular bipartite graph with bipartition $\{X, Y\}$. Prove that

(a) if G is undirected, then for any $x \in X$ there is $y \in Y$ such that $\eta_G(x, y) \geq k$; (Y.O.Hamidoune and M.Las Vergnas (1988))

(b) if G is directed, then for any $x \in X$ there is $y \in Y$ and k edge-disjoint (x, y)-paths and k edge-disjoint (y, x)-paths such that all $2k$ of them are pairwise edge-disjoint. (Xu [**?**])

Exercises 4.3

4.3.1 Prove the conclusion (b) in Theorem 4.5.

4.3.2 Let G be a graph with $\lambda(G) = \lambda > 0$, and B be a λ-cut of G.

(a) Prove that there exists $\emptyset \neq S \subset V(G)$ such that $B = [S, \overline{S}]$ and two subgraph $G[S]$ and $G[\overline{S}]$ of G are connected if G is undirected.

(b) Construct a digraph to show that (a) is not always true.

4.3.3 Let k be an integer with $1 \leq k \leq v - 1$.

(a) Prove that, if G is a simple undirected graph of order v and $\delta(G) \geq \left\lceil \frac{1}{2}(v + k - 2) \right\rceil$, then $\kappa(G) \geq k$.

(b) Find a simple undirected graph G of order v such that $\delta(G) = \left\lceil \frac{1}{2}(v + k - 3) \right\rceil$ and $\kappa(G) < k$.

4.3.4 (a) prove that, if G is a simple undirected graph and $\delta(G) \geq v-2$, then $\kappa(G) = \delta(G)$.

(b) Find a simple undirected graph G of order $v \geq 4$ such that $\delta(G) = v-3$ and $\kappa(G) < \delta(G)$.

4.3.5 (a) Prove that, if G is a simple undirected graph with order v and $\delta(G) \geq \lfloor \frac{v}{2} \rfloor$, then $\lambda(G) = \delta(G)$.

(b) Find a simple undirected graph G of order v such that $\delta(G) = \lfloor \frac{v}{2} \rfloor - 1$ and $\lambda(G) < \delta(G)$.

4.3.6 Prove that (a) if L is the line graph $L(G)$ of G, then $\kappa(L) \geq \lambda(G)$;

(b) $\kappa(B(d,n)) = d-1$ and $\kappa(K(d,n)) = d$.

4.3.7 Prove that, if G is a 2-connected undirected graph with order at least 3, then there exist two adjacent vertices x and y in G such that $G - \{x, y\}$ is connected.

4.3.8 Prove that, if G is a plane triangulation of order at least 4, then the geometric dual G^* is simple, 3-regular and 2-edge-connected.

4.3.9 Prove that, if G is a 3-connected undirected graph with order at least 5, then there exists $e \in E(G)$ such that $\kappa(G \cdot e) \geq 3$. (W.T.Tutte, 1961)

4.3.10 Prove that, if G is a k-connected graph, $\{x, x_1, x_2, \cdots, x_k\}$ is any $k+1$ vertices of G, then there exist k internally disjoint (x, x_i)-paths $(i = 1, 2, \cdots, k)$. (Such a set of paths is called an *x-fan*.)

4.3.11 Prove that, if G is a $k\,(\geq 2)$-connected undirected graph, then any k vertices of G is contained in a cycle of G. (G.A.Dirac 1960)

4.3.12 Prove that the girth $g(G) \leq \lceil \frac{v}{k} \rceil$ for a $k\,(\geq 1)$-connected digraph G of order v. (This result is due to Hamidoune [?], who solved a special case of a conjecture of Behzad *et al.* [?] that $g(G) \leq \lceil \frac{v}{k} \rceil$ for any k-regular digraph G of order v.)

4.3.13 Prove that, if G is a strongly connected simple digraph with connectivity κ and diameter d, then $v \geq \kappa(d-3) + \delta^+ + \delta^- + 2$.

4.3.14 Let G be an undirected graph. Prove that

(a) if D is a k-connected oriented graph of G, then $\lambda(G) \geq 2k$;

(b) if $\lambda(G) \geq 2$, then G has a strongly connected orientation;

(c) if G eulerian and $\lambda(G) \geq 2k$ for some $k \geq 1$, then G has an orientation D such that $\lambda(D) \geq k$. (The conclusion (b) is due to Robbins [?]; and generalized by Nash-Williams [?] who showed that, if $\lambda(G) \geq 2k$, then G has an orientation D such that $\lambda(D) \geq k$.)

4.3.15 Let v, δ, κ and λ be given nonnegative integers. Prove that there exists a simple graph G of order v such that $\delta(G) = \delta, \kappa(G) = \kappa, \lambda(G) = \lambda$ if and only if one of the following condition holds:

(a) $0 \leq \kappa \leq \lambda \leq \delta \leq \lfloor \frac{v}{2} \rfloor$;

(b) $1 \le 2\delta + 2 - \upsilon \le \kappa \le \lambda \le \delta < \upsilon - 1$;

(c) $\kappa = \lambda = \delta = \upsilon - 1$.

4.3.16 The *k-diameter* of a k-connected graph G, denoted by $d_k(G)$, is the maximum integer d for which for any two vertices x and y in G, there exist k internally disjoint (x, y)-paths of length at most d. Prove that

(a) $d_k(G) \le \upsilon - k + 1$ for any k-connected undirected graph G;

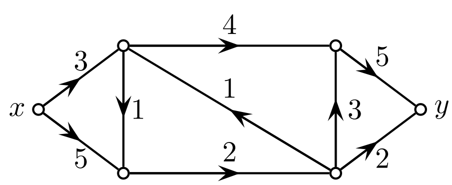
(b) $d_k(G) \ge d(G) + 1$ for any $k\,(\ge 2)$-regular k-connected graph G;

(c) $d_{d-1}(B(d, n)) = n + 1$ and $d_d(K(d, n)) = n + 2$ by making use of Example 1.8.2 and the exercises 1.8.9.

Exercises 4.4

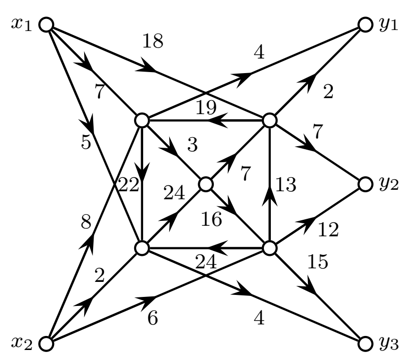
4.4.1 Prove that the $\widetilde{\mathbf{f}} \in \mathscr{E}(G)$ given in the expression (4.10) is an (x, y)-flow in N with val $\widetilde{\mathbf{f}} =$ val $\mathbf{f} + \sigma_P(y)$.

4.4.2 Find a maximum (x, y)-flow and a minimum (x, y)-cut in the following network.



(Exercise 4.4.2)

4.4.3 Some commodities will be shipped from their producing areas x_1 and x_2 to their markets y_1, y_2, y_3 by the following transport system. Design transport scheme by which freight volume is as large as possible.



(Exercise 4.4.3)

4.4.4 Prove that the labelling method is avail for a nonnegative rational capacity.

4.4.5 Let N be an integral capacity network. Prove that

(a) if the revised flow $\widetilde{\mathbf{f}}$ is obtained from a shortest $\mathbf{f}$-incrementing path in the labelling method, then a maximum flow can be obtained by execution of the algorithm at most $\frac{1}{2}\,\upsilon\,\varepsilon$ times;

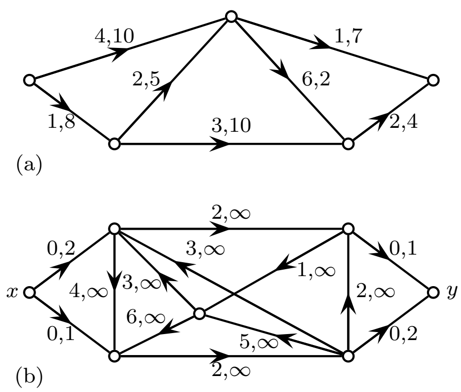
(b) the labelling method is an efficient algorithm and its complexity is $O(\upsilon\,\varepsilon^2)$.

Exercises 4.5

4.5.1 Prove that the function $\widetilde{\mathbf{f}} \in \mathscr{E}(G)$ defined by (4.11) is an (x, y)-flow in N, and val $\widetilde{\mathbf{f}} =$ val $\mathbf{f}$.

4.5.2 Prove Theorem 4.10 and prove that Klein's algorithm is efficient.

4.5.3 Use Klein's algorithm to find minimum-cost maximum (x, y)-flows in the following networks, respectively, where the ordered pair (b, c) of digits nearby the edge a denotes the values of the cost function $\mathbf{b}$ and the capacity function $\mathbf{c}$ on a, respectively, that is, $b = \mathbf{b}(a)$ and $c = \mathbf{c}(a)$.



(Exercise 4.5.3)

4.5.4 Suppose that $\mathbf{f}$ is an (x, y)-flow in a network $N = (G_{xy}, \mathbf{b}, \mathbf{c})$, P an $\mathbf{f}$-incrementing path with cost as little as possible, $\widetilde{\mathbf{f}}$ the revised flow based on P.

(a) Prove that, if $\mathbf{f}$ is a minimum-cost maximum-flow, then so is $\widetilde{\mathbf{f}}$.

(b) Prove that finding an $\mathbf{f}$-incrementing path with minimum cost in N is equivalent to finding a shortest xy-path in $G(\mathbf{f})$.

(c) Using (a) and (b), design an algorithm for finding a minimum-cost maximum-flow in N, and by which shows that the flow obtained by the exercise 4.5.3 is a minimum-cost maximum-flow.

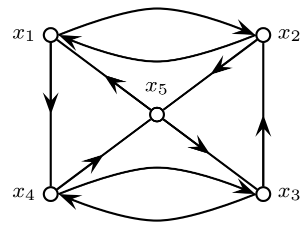
(R.G.Busacker and P.J.Gowen, 1961)

Exercises 4.6

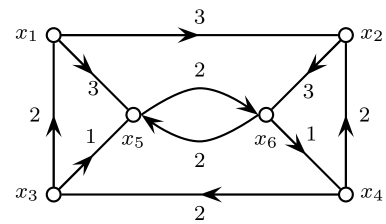
4.6.1 Find an Euler directed circuit in the following digraph, applying the first Edmonds-Johnson's algorithm.

4.6.2 Find an optimal tour of the following weighted digraph and its weight-sum.

4.6.3 Prove that Edmonds-Johnson's algorithm for finding an optimal postman tour in a non-eulerian weighted digraph is efficient.



(Exercise 4.6.1)



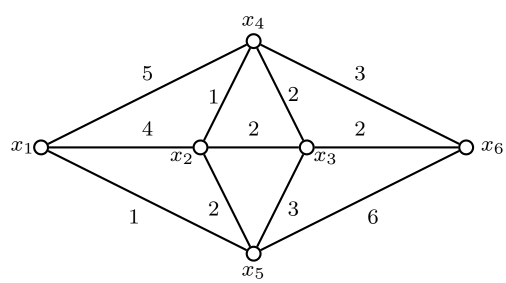
(Exercise 4.6.2)

4.6.4 Prove that a postman tour P of a weighted undirected graph $(G, \mathbf{w})$ is optimal if and only if it satisfies the following conditions:

(i) no edge in G appears in P than two times;

(ii) for any cycle C of G, the weight-sum of the edges in C that belong to multiple-edges is at most $\frac{1}{2}\,\mathbf{w}(C)$. (M.G.Guan, 1960)

4.6.5 Find an optimal tour of the following weighted undirected graph and its weight-sum.



(Exercise 4.6.5)

Exercises 4.6

4.7.1 Prove that the ratio of two neighboring sides of any squared rectangle is rational. (This result is due to M. Dehn (1930), and generalized by Sprague (1940) who showed that a rectangle has a perfect squaring if and only if the ratio of two neighboring sides is rational.)

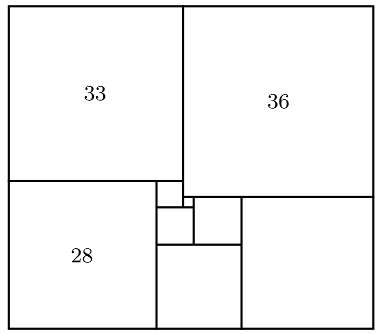
4.7.2 Prove that $\mathbf{g} \in \mathscr{B}(D)$ defined by (4.14) is an (x, y)-flow.

4.7.3 Using the way described this section, find an (x, y)-flow $\mathbf{g} \in \mathscr{B}(D)$, where the digraph D is shown in Figure **??** (b), from which construct a squared rectangle.

4.7.4 It has been shown that there are two essentially different squared rectangles of order 9; the one shown in Figure **??**, the other indicated as follows, which is a 69×61 rectangle. Find the side length of each square.

4.7.5 Prove that there is no perfect squared rectangle of order less than 9.

4.7.6 A *perfect equilateral triangle* is an equilateral triangle dissected into a finitely many (but at least two) smaller equilateral triangles, no two of the same size. Prove that there exists no perfect equilateral triangle.



(the exercise 4.7.5)

4.7.7 A *perfect cube* is a cube dissected into a finitely many (but at least two) smaller cubes, no two of the same size. Prove that there exists no perfect cube.