Exercises in Chapter 5

Exercises 5.1

5.1.1 Prove that $\left\lceil \dfrac{v}{1+\Delta} \right\rceil \le \alpha'(G) \le \left\lfloor \dfrac{v}{2} \right\rfloor$ for any graph G without isolated vertices. (Weinstein, 1963)

5.1.2 Prove that

(a) n-cube Q_n has n edge-disjoint perfect matchings;

(b) complete graph K_{2n} has $(2n-1)$ edge-disjoint perfect matchings;

(c) complete graph K_{2n} has $(2n-1)!!$ distinct perfect matchings;

(d) $K_{n,n}$ has $n\,!$ different perfect matchings;

(e) for any $k\,(\ge 2)$, there exists a k-regular simple graph that contains no perfect matching.

5.1.3 Prove that every plane triangulation of order $v\,(\ge 4)$ contains a bipartite subgraph with $\frac{2}{3}\,\varepsilon$ edges.

5.1.4 A *line* of a matrix is a row or a column of the matrix. Prove that the minimum number of lines containing all the 1's of a $(0,1)$-matrix is equal to the maximum number of 1's, no two of which are in the same line.

5.1.5 Let $A_1, A_2, \cdots, A_m$ be subsets of a set S. A *system of distinct representatives* for the family $\{A_1, A_2, \cdots, A_m\}$ is a subset $\{a_1, a_2, \cdots, a_m\}$ of S such that $a_i \in A_i$ for each $i = 1, 2, \cdots, m$, and $a_i \ne a_j$ for $i \ne j$. Prove that $\{A_1, A_2, \cdots, A_m\}$ has a system of distinct representatives if and only if $|\bigcup_{i\in J} A_i| \ge |J|$ for all subsets J of $\{1, 2, \cdots, m\}$. (P.Hall, 1935)

5.1.6 Prove that a tree T has a perfect matching if and only if $o(T-x) = 1$ for any $x \in V(T)$.

5.1.7 Prove that, if G is a bipartite graph with bipartition $\{X, Y\}$, then

$$\alpha'(G) = |X| - \max\{|S| - |N_G(S)| : \ \forall\ S \subseteq X\}. \qquad \text{(O.Ore, 1955)}$$

5.1.8 Prove that, if G is a graph and $r = \max\{o(G-S) - |S| : \ \forall\ S \subset V(G)\}$, then $\alpha'(G) = \frac{1}{2}\,(v - r)$. (C.Berge, 1958)

5.1.9 Let Γ be a finite group and H be a subgroup of Γ. Prove that there exist elements $a_1, a_2, \cdots, a_n \in \Gamma$ such that $a_1H, a_2H, \cdots, a_nH$ are the left cosets of H and $Ha_1, Ha_2, \cdots, Ha_n$ are the right cosets of H.

(P.Hall, 1935)

5.1.10 Let $\mathbf{A}$ be an $m \times n\,(m \le n)$ matrix. The *permanent* of $\mathbf{A}$, denoted by $\mathrm{Per}\,(\mathbf{A})$, is defined as the sum of products of m entries from different rows and columns of $\mathbf{A}$. Prove that, if $\mathbf{A}$ is the adjacency matrix of a bipartite graph G with bipartition $\{X, Y\}$ and $|X| \le |Y|$, then

(a) the number of matchings of G saturating X is equal to $\mathrm{Per}\,(\mathbf{A})$;

(b) $K_{n,n}$ has $n\,!$ different perfect matchings.

5.1.11 A k-regular spanning subgraph of G is called a *k-factor* of G. A graph G is called to be *k-factorable* if it can be expressed as the union of edge-disjoint k-factors. Prove that

(a) G has 1-factor if and only if G has a perfect matching;

(b) Petersen graph is not 1-factorable, but the union of one 1-factor and one 2-factor;

(c) K_{2n} and $K_{n,n}$ is 1-factorable;

(d) K_{2n+1} is 2-factorable;

(e) a simple graph G is 2-factorable if and only if G is $2k$-regular;

(f) every $2k$-regular graph is 2-factorable for $k \geq 1$.

5.1.12 Deduce the following theorems

(a) Hall's theorem (5.1) from Tutte's theorem (5.2);

(b) Hall's theorem (5.1) from König's theorem (5.3);

(c) König's theorem (5.3) from Menger's theorem (4.3);

(d) König's theorem (5.3) from the max-flow min-cut theorem (4.1);

(e) Menger's theorem (4.3) from Hall's theorem (5.1).

Exercises 5.2

5.2.1 Prove that G is a bipartite graph if and only if

(a) $\alpha(H) \geq \frac{1}{2}\,v(H)$ for any subgraph H of G;

(b) $\alpha(H) = \beta'(H)$ for any subgraph H of G without isolated vertices.

5.2.2 Let $\{V_1, V_2, \cdots, V_n\}$ be a partition of $V(G)$ and V_i is a maximal independent set of G for each $i = 1, 2, \cdots, n$. Let H be a simple graph with vertex-set $\{u_1, u_2, \cdots, u_n\}$ and $u_iu_j \in E(H) \iff E_G(V_i, V_j) \neq \emptyset$. Prove that H is a complete graph.

5.2.3 Prove that $\alpha'(G) = \alpha(L(G))$, where $L(G)$ is the line graph of a non-empty graph G.

5.2.4 Construct graphs to show that

(a) the condition"$d_G(x) + d_G(y) \geq v$" in Theorem 5.6 can not be improved as "$d_G(x) + d_G(y) \geq v - 1$";

(b) the condition"$\delta(G) \geq \frac{1}{2}\,v$" in Corollary 5.6 can not be improved as "$\delta(G) \geq \lfloor \frac{1}{2}\,v \rfloor$".

5.2.5 Prove that a simple graph G is hamiltonian if it satisfies one of the following conditions.

(a) G is k-regular k-connected and $v = 2k + 1$;

(b) $T = \{x \in V(G) :\ d_G(x) = v - 1\}$ and $|T| \geq \alpha(G)$.

5.2.6 Let G be a simple graph. Prove that, if $\delta(G) \geq \frac{1}{3}[v(G)+\kappa(G)]$, then $\alpha(G) \leq \delta(G)$.

5.2.7 Let G be a loopless digraph. Prove that G contains an independent set I such that $d_G(I, y) \leq 2$ for any $y \in V(G) \setminus I$, where $d_G(I, y) = \min\{d_G(x, y) : x \in I\}$. (V.Chvàtal, and L.Lovász, 1974)

Exercises 5.3

5.3.1 (a) Let M and N be two disjoint matchings of G, and $|M| > |N|$. Prove that G has two disjoint matchings M' and N' such that $M' \cup N' = M \cup N$, $|M'| = |M| - 1$ and $|N'| = |N| + 1$.

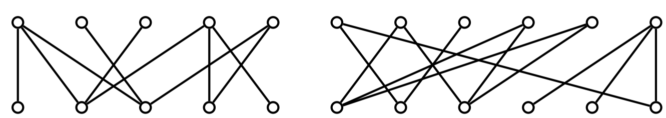
(b) Let G be a bipartite graph. Prove that if $p \geq \Delta$, then there exist p disjoint matchings $M_1, M_2, \cdots, M_p$ such that $E(G) = M_1 \cup M_2 \cup \cdots \cup M_p$ and $\lfloor \frac{\varepsilon}{p} \rfloor \leq |M_i| \leq \lceil \frac{\varepsilon}{p} \rceil$ for any $1 \leq i \leq p$.

5.3.2 Give another proof of Hall's theorem (Theorem 5.1) by making use of Theorem 5.8.

5.3.3 (a) Prove that the Hungarian method is an $O(v\varepsilon^2)$ algorithm.

(b) Describe an efficient algorithm for finding a maximum matching in a bipartite graph.

5.3.4 Test whether or not the following two graphs have perfect matchings. If no perfect matching exists, then finding a maximum matching such that it contains all maximum degree vertices.



(a) (b)

(Exercise 5.3.4)

5.3.5 A *Latin rectangle* is an $m \times n$ matrix in which the entries are integers in the range from 1 to n. No entry appears more than once in any row or any column. A Latin rectangle is called a *Latin square* if $m = n$.

(a) Add two other two rows to the matrix such which is a Latin square.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 2 & 5 & 1 \\ 5 & 1 & 4 & 2 & 3 \end{pmatrix}$$

(b) Prove that for any $m \times n \, (m < n)$ Latin rectangle $\mathbf{A}$, other $n - m$ rows can be added to $\mathbf{A}$ such which is a Latin square.

Exercises 5.4

5.4.1 Prove that (a) $\alpha_l > 0$ and $T \subset N_{G_l}(S)$;

(b) Kuhn-Munkres' algorithm is efficient;

(c) Theorem 5.11.

5.4.2 Find a maximum-weight and a minimum-weight perfect matching in $K_{5,5}$ with weight matrices, respectively,

$$\mathbf{W} = \begin{pmatrix} 9 & 8 & 5 & 3 & 2 \\ 6 & 7 & 8 & 6 & 9 \\ 5 & 8 & 1 & 4 & 7 \\ 7 & 7 & 0 & 3 & 6 \\ 9 & 8 & 6 & 4 & 5 \end{pmatrix} \qquad \mathbf{W}' = \begin{pmatrix} 3 & 2 & 1 & 2 & 3 \\ 1 & 4 & 2 & 1 & 2 \\ 5 & 1 & 2 & 3 & 1 \\ 3 & 2 & 6 & 4 & 1 \\ 1 & 2 & 3 & 1 & 2 \end{pmatrix}$$

5.4.3 Let $\mathbf{A}$ be an n-square matrix. A *diagonal line* of $\mathbf{A}$ is a set of n entries from different rows and columns of $\mathbf{A}$, and its weight is the sum of these n entries.

(a) Find a maximum-weight and a minimum-weight diagonal line of the following matrices and their weights, respectively,

$$\mathbf{A} = \begin{pmatrix} 4 & 5 & 8 & 10 & 11 \\ 7 & 6 & 5 & 7 & 4 \\ 8 & 5 & 12 & 9 & 6 \\ 6 & 6 & 13 & 10 & 7 \\ 4 & 5 & 7 & 9 & 8 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 9 & 8 & 0 & 0 & 0 \\ 0 & 0 & 8 & 6 & 9 \\ 0 & 8 & 0 & 4 & 7 \\ 0 & 7 & 0 & 3 & 6 \\ 9 & 8 & 6 & 4 & 0 \end{pmatrix}$$

(b) Prove that all diagonal lines of $\mathbf{B}$ admit the same weight.

5.4.4 Six jobs $J_1, J_2, \cdots, J_6$ need to be processed, the time t_{ij} of adaptation from job J_i to job J_j is as follows:

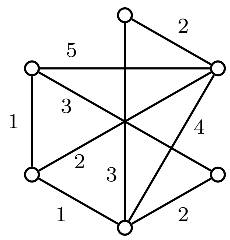
$$\mathbf{T} = (t_{ij}) = \begin{pmatrix} 0 & 3 & 2 & 5 & 1 & 3 \\ 2 & 0 & 4 & 5 & 4 & 1 \\ 1 & 3 & 0 & 1 & 2 & 2 \\ 4 & 2 & 2 & 0 & 1 & 3 \\ 3 & 1 & 4 & 5 & 0 & 2 \\ 2 & 5 & 3 & 1 & 2 & 0 \end{pmatrix}$$

Find a sequencing of jobs as optimal as possible.

Exercises 5.5

5.5.1 Find an approximately optimal route in the following weighted graph by Christofides' algorithm.

5.5.2 Suppose that the distance matrix of six cities is as follows. Find an approximation solution for the travelling salesman problem by the Christofides' algorithm.



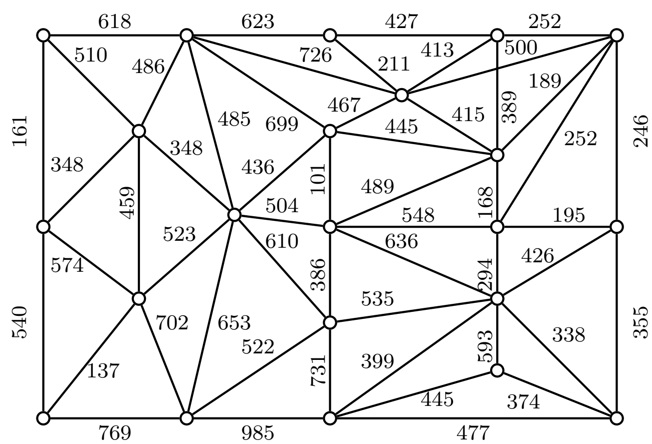
(Exercise 5.5.1)

$$\begin{pmatrix} 0 & 56 & 35 & 2 & 51 & 60 \\ 56 & 0 & 21 & 57 & 78 & 70 \\ 35 & 21 & 0 & 36 & 68 & 68 \\ 2 & 57 & 36 & 0 & 51 & 61 \\ 51 & 78 & 68 & 51 & 0 & 13 \\ 60 & 70 & 68 & 61 & 13 & 0 \end{pmatrix}$$

(Exercise 5.5.2)

5.5.3 Suppose that the triangle inequality is satisfied in a weighted complete graph $(K_v, \mathbf{w})$. Prove that if C is an optimal cycle and T is a minimum tree in $(K_v, \mathbf{w})$ then $\mathbf{w}(C) \leq 2\mathbf{w}(T)$.

5.5.4 Solve the travelling salesman problem in the following traffic system (the minimum weight is 8117).



(Exercise 5.5.5)