Exercises in Chapter 6

Exercises 6.1

6.1.1 Let $G$ be a critical $k$-chromatic graph with $k \geq 3$. Prove that
(a) for any separating set $S$ of $G$, the induced subgraph $G[S]$ is not a complete
graph;
(b) $G$ is 2-connected.

6.1.2 Prove that if $G$ is a critical $k$-chromatic graph then $v(G) \neq k + 1$.

6.1.3 Prove that
(a) any $k$-chromatic graph contains a critical $k$-chromatic subgraph;
(b) any $k$-chromatic simple graph with minimum order must be a critical
$k$-chromatic graph;
(c) $\chi(G) \leq 1 + \max\{\delta(H) : H \subseteq G\}$ for any graph $G$;
(d) $\chi(G) \leq 1 + l(G)$ for any graph $G$, where $l(G)$ is the length of longest path
in $G$.

6.1.4 Prove that
(a) $\left\lceil \frac{v}{\alpha} \right\rceil \leq \chi(G) \leq v + 1 - \alpha$ for any graph $G$;
(b) $\frac{v^2}{v^2 - 2\varepsilon} \leq \chi(G) \leq \frac{1}{2} + \sqrt{2\varepsilon + \frac{1}{4}}$ for any simple graph $G$.

6.1.5 Prove that if any two odd cycles in a graph $G$ have a vertex in common then
$\chi(G) \leq 5$.

6.1.6 Prove that Brooks’ theorem is equivalent to the following statement: if $G$ is
$k$-critical ($k \geq 4$) and not complete, then $2\varepsilon \geq v(k - 1) + 1$.

6.1.7 (a) Prove that a generalized Brooks’ theorem: if a graph $G$ has either $\Delta(G) =
2$ and no odd component or $\Delta(G) \geq 3$ and no component that contains $K_{\Delta+1}$,
then $\chi(G) \leq \Delta(G)$.
(b) Prove that for any integers $k$ and $m$ with $2 \leq k \leq m$, there exists a graph
$G$ such that $\Delta(G) = m$ and $\chi(G) = k$.

6.1.8 Prove that an undirected graph $G$ is $k$-vertex colorable if and only if $G$ has
an orientation $D$ in which each directed path is of length at most $k - 1$.

Exercises 6.2

6.2.1 Prove that
(a) each of bipartite graphs, 3-regular hamiltonian graphs and even complete
graphs belongs to class one;
(b) each of odd cycles and odd complete graphs belongs to class two.
6.2.2 Prove that a simple graph $G$ belongs to class two if it satisfies one of the following conditions:
(a) $\varepsilon > \Delta \alpha'$;
(b) $\varepsilon > \Delta \left\lfloor \frac{\Delta}{2} \right\rfloor$; (L.W. Beineke and R.J. Wilson, 1973)
(c) $G$ has odd order and is nonempty, regular;
(d) $G$ is regular and contains a cut-vertex. (V.G. Vizing, 1965)

6.2.3 Prove that $\chi'(G) = \chi(L(G))$ if $G$ is nonempty.

6.2.4 Prove that a complete $k$-partite graph $K_n(k)$ belongs to class one if $nk$ is even; to class two otherwise. (R. Laskar and W. Hare, 1971)

6.2.5 Prove that $\chi'(G) \leq \left\lceil \frac{3}{2} \Delta \right\rceil$ if $G$ is loopless. (C.E. Shannon, 1949)

6.2.6 Prove that $\chi'(G) \leq 3\Delta - 2$ if $G$ is simple and $\Delta \geq 3$ by using Brooks’ theorem (6.2) and the exercise 6.2.3.

**Exercises 6.3**

6.3.1 Prove that a 2-edge-connected plane graph $G$ is 2-face-colorable if and only if $G$ contains no vertex of odd degree.

6.3.2 Prove that a plane triangulation $G$ is 3-face-colorable if and only if $G$ contains no vertex of odd degree.

6.3.3 Prove that
(a) every Hamiltonian plane graph is 4-face-colorable;
(b) every Hamiltonian 3-regular graph is 3-edge-colorable.

6.3.4 Prove that every plane graph is 4-face-colorable if and only if every simple 2-edge-connected 3-regular plane graph is 4-face-colorable.

6.3.5 Suppose that the plane is divided into several regions by $n$ ($\geq 1$) lines. Prove that these regions can be colored with two colors so that no two regions that share a length of common border are assigned the same color.

6.3.6 Prove that every 3-regular plane graph $G$ is 3-face-colorable if and only if $G$ contains no face of odd degree.

6.3.7 Prove that if every 3-regular plane graph is 4-face-colorable, then the four-color conjecture holds.

6.3.8 Prove that the four-color conjecture is equivalent to Tait’s conjecture: every 3-connected 3-regular simple planar graph is 3-edge-colorable.