Exercises in Chapter 6

Exercises 6.1

6.1.1 Let G be a critical k-chromatic graph with $k \geq 3$. Prove that

(a) for any separating set S of G, the induced subgraph $G[S]$ is not a complete graph;

(b) G is 2-connected.

6.1.2 Prove that if G is a critical k-chromatic graph then $v(G) \neq k+1$.

6.1.3 Prove that

(a) any k-chromatic graph contains a critical k-chromatic subgraph;

(b) any k-chromatic simple graph with minimum order must be a critical k-chromatic graph;

(c) $\chi(G) \leq 1 + \max\{\delta(H) : \ H \subseteq G\}$ for any graph G;

(d) $\chi(G) \leq 1 + l(G)$ for any graph G, where $l(G)$ is the length of longest path in G.

6.1.4 Prove that

(a) $\left\lceil \dfrac{v}{\alpha} \right\rceil \leq \chi(G) \leq v + 1 - \alpha$ for any graph G;

(b) $\dfrac{v^2}{v^2 - 2\varepsilon} \leq \chi(G) \leq \dfrac{1}{2} + \sqrt{2\,\varepsilon + \dfrac{1}{4}}$ for any simple graph G.

6.1.5 Prove that if any two odd cycles in a graph G have a vertex in common then $\chi(G) \leq 5$.

6.1.6 Prove that Brooks' theorem is equivalent to the following statement: if G is k-critical ($k \geq 4$) and not complete, then $2\,\varepsilon \geq v(k-1)+1$.

6.1.7 (a) Prove that a generalized Brooks' theorem: if a graph G has either $\Delta(G) = 2$ and no odd component or $\Delta(G) \geq 3$ and no component that contains $K_{\Delta+1}$, then $\chi(G) \leq \Delta(G)$.

(b) Prove that for any integers k and m with $2 \leq k \leq m$, there exists a graph G such that $\Delta(G) = m$ and $\chi(G) = k$.

6.1.8 Prove that an undirected graph G is k-vertex colorable if and only if G has an orientation D in which each directed path is of length at most $k-1$.

Exercises 6.2

6.2.1 Prove that

(a) each of bipartite graphs, 3-regular hamiltonian graphs and even complete graphs belongs to class one;

(b) each of odd cycles and odd complete graphs belongs to class two.

6.2.2 Prove that a simple graph G belongs to class two if it satisfies one of the following conditions:

(a) $\varepsilon > \Delta\alpha'$;

(b) $\varepsilon > \Delta \left\lfloor \frac{v}{2} \right\rfloor$; (L.W.Beineke and R.J.Wilson, 1973)

(c) G has odd order and is nonempty, regular;

(d) G is regular and contains a cut-vertex. (V.G.Vizing, 1965)

6.2.3 Prove that $\chi'(G) = \chi(L(G))$ if G is nonempty.

6.2.4 Prove that a complete k-partite graph $K_n(k)$ belongs to class one if nk is even; to class two otherwise. (R. Laskar and W.Hare, 1971)

6.2.5 Prove that $\chi'(G) \leq \left\lfloor \frac{3}{2}\,\Delta \right\rfloor$ if G is loopless. (C.E.Shannon, 1949)

6.2.6 Prove that $\chi'(G) \leq 3\Delta - 2$ if G is simple and $\Delta \geq 3$ by using Brooks' theorem (6.2) and the exercise 6.2.3.

Exercises 6.3

6.3.1 Prove that a 2-edge-connected plane graph G is 2-face-colorable if and only if and G contains no vertex of odd degree.

6.3.2 Prove that a plane triangulation G is 3-face-colorable if and only if and G contains no vertex of odd degree.

6.3.3 Prove that

(a) every Hamiltonian plane graph is 4-face-colorable;

(b) every Hamiltonian 3-regular graph is 3-edge-colorable.

6.3.4 Prove that every plane graph is 4-face-colorable if and only if every simple 2-edge-connected 3-regular plane graph is 4-face-colorable.

6.3.5 Suppose that the plane is divided into several regions by $n\,(\geq 1)$ lines. Prove that these regions can be colored with two colors so that no two regions that share a length of common border are assigned the same color.

6.3.6 Prove that every 3-regular plane graph G is 3-face-colorable if and only if G contains no face of odd degree.

6.3.7 Prove that if every 3-regular plane graph is 4-face-colorable, then the four-color conjecture holds.

6.3.8 Prove that the four-color conjecture is equivalent to *Tait's conjecture*: every 3-connected 3-regular simple planar graph is 3-edge-colorable.