Spanning trees, Dense subgraphs, Clustering, Flows and parity subgraphs

Cun-Quan Zhang
West Virginia University

An optimization problem. *For a multi-graph G with n vertices, m edges, and a given integer k, determine wether G contains k edge-disjoint spanning trees.*

This optimization problem has been extensively studied and a long list of algorithms have been introduced with the following complexity: Cunningham (1984) $O(nm^8)$; Cunningham (1985) $O(nm)\phi_M$ or $O(n^4m)$; Gabow (1991) $O(n^4m^2\log^2 W)$; Gusfield (1991) $O(n^3m)$; Barahona (1995) $O(n^2)\phi_M$; Gabow (1998) $O(n^2m\log\frac{n^2}{m})$; Barahona (2004) $O(n)\phi_M$; *etc.*, where $\phi_M = O(nm\log\frac{n^2}{m})$ is the complexity of the min-cut max-flow problem (by Goldberg and Tarjan) and W is the maximum edge capacity. Here, we are to present a new algorithm with the complexity $O(n^2h^2)$.

The theoretical part of the algorithm also provided some very useful graph theory lemmas, which has been applied in the proof of some new results related to cycle covers, integer flows and circular flows.

By applying a theorem of Tutte and Nash-Williams, and a theorem by Itai and Rodeh, it is known that *Every $2k$-edge-connected graph G contains at least k edge-disjoint parity subgraphs of G.* This is an extensively used lemma in the studies of flows and cycle covers. And it is improved recently that *if the odd-edge-connectivity of a graph G is $2k+1$, then G contains k edge-disjoint parity subgraphs.*

Integer flow was originally introduced by Tutte as a generalization of map coloring problems. Circular flow was introduced as a real line extension of integer flow problem. The flow index of a graph is defined as the smallest rational number r that G admits a nowhere-zero r-flow.

One of major open problems in the area of integer flow is that *every graph with odd-edge-connectivity at least 5 admits a nowhere-zero 3-flow* (by Tutte). Two of the major partial results to this conjecture are (1) *every 4-edge-connected graph admits a nowhere-zero 4-flow* (by Jaeger); (2) *the flow index of a 6-edge-connected graph is < 4* (by Galluccio and Goddyn). The above result is further improved as follows: *the flow index of every odd-7-edge-connected graph is < 4.*

A weak version of 3-flow conjecture was proposed by Jaeger that *there is an integer h such that every h-edge-connected multigraph admits a nowhere-zero 3-flow.* The following is a partial results to this conjecture. *Every $4\lceil\log_2 n\rceil$-edge-connected multigraph with n vertices admits a nowhere-zero 3-flow* (by Lai and Zhang). It is further improved that *if the odd-edge-connectivity of a multigraph G with n vertices is at least $4\lceil\log_2 n\rceil$, then G admits a nowhere-zero 3-flow.*