Chapter 14

Menger-Type Problems in Parallel Systems

In the preceding chapter, we have studied two graph-theoretic parameters: $D_w(G)$ and $D'_w(G)$. By their definitions, if $D_w(G) = \ell$ (resp. $D'_w(G) = \ell$), then for any two vertices $x$ and $y$ and any $F \subseteq V(G) \setminus \{x, y\}$ (resp. $F \subseteq E(G)$) with $|F| \leq w - 1$, $G - F$ contains $(x, y)$-paths of length at most $\ell$.

In this section, we will discuss a dual problem: For a given connected graph $G$ and a positive integer $\ell$, determine the largest $w$ such that $D_w(G)$ (resp. $D'_w(G)$) is at most $\ell$. We will refer this problem to determining the bounded connectivity (resp. bounded edge-connectivity) of $G$ with respect to $\ell$. The problem is of great interest in parallel computing or processing systems since the parameter $w$ provides a measurement of fault tolerance when a transmission delay $\ell$ is given. In other words, the system can tolerate at most $w - 1$ faulty vertices and/or edges to ensure that the transmission delay of the remaining network does not exceed $\ell$.

The problem derived therefrom is for a given connected graph $G$ and a positive integer $\ell$, determining the largest number of internally (resp. edge-) disjoint paths of length at most $\ell$ between two vertices in $G$. We will refer this problem to determining the Menger number of $G$ with respect to $\ell$.

In addition, in this chapter, we will also discuss another related problem called the Rabin number.

Since these problems are closely related to Menger’s theorem, we call them Menger-type problems.

14.1 Menger-Type Problems

When a $w$-connected graph $G$ is used to model a parallel computing or processing system, one can use $w$ internally disjoint paths to transmit messages
simultaneously from a vertex \( x \) to another \( y \) in \( G \). If, however, the network is a real-time system, then the length of each path used must not exceed a preestablished bound \( \ell \) because any too long path is essentially useless in such a network. Such a consideration leads one to study the following two parameters.

Let \( x \) and \( y \) be any two vertices in \( G \), \( \ell \) be a given positive integer. The \((x, y)\)-**Menger number** with respect to \( \ell \), denote by \( \zeta_\ell(G; x, y) \), is the maximum number of internally disjoint \((x, y)\)-paths of length at most \( \ell \) in \( G \). If \((x, y) \notin E(G)\), then there are \((x, y)\)-separating sets in \( G \). An \((x, y)\)-separating set in \( G \) is called a **bounded** \((x, y)\)-separating set with respect to \( \ell \) if it destroys all \((x, y)\)-paths of length at most \( \ell \) in \( G \). The **bounded** \((x, y)\)-**connectivity** with respect to \( \ell \), denoted by \( \kappa_\ell(G; x, y) \), is defined as the minimum number of vertices in a bounded \((x, y)\)-separating set with respect to \( \ell \) in \( G \).

![Figure 14.1: An example explaining the Menger number and the bounded connectivity](image)

For example, for the graph \( G \) shown in Figure 14.1,

\[
\zeta_4(G; x, y) = 1 = \kappa_4(G; x, y); \quad \zeta_5(G; x, y) = 1, \quad \kappa_5(G; x, y) = 2.
\]

To avoid the relatively non-significant cases in which \( \ell < d(G; x, y) \) or \( \ell = 1 \), we suppose \( 2 \leq d(G; x, y) \leq \ell \). Since the length of any path in \( G \) with order \( n \) does not exceed \( n - 1 \), we suppose \( \ell \leq n - 1 \). By the definitions, for any two integers \( \ell' \) and \( \ell \) with \( d(G; x, y) \leq \ell' \leq \ell \leq n - 1 \), it is clear that

\[
\zeta_{\ell'}(G; x, y) \leq \zeta_{\ell}(G; x, y), \quad \kappa_{\ell'}(G; x, y) \leq \kappa_{\ell}(G; x, y).
\]

For \( \ell = n - 1 \), no restriction is imposed on the length of any path in \( G \). Thus,

\[
\zeta_{n-1}(G; x, y) = \zeta(G; x, y), \quad \kappa_{n-1}(G; x, y) = \kappa(G; x, y).
\]

By Menger’s theorem, we have the equality

\[
\zeta_{n-1}(G; x, y) = \kappa_{n-1}(G; x, y).
\]

However, the graph shown in Figure 14.1 for \( \ell = 5 \) shows that, in general, the equality does not hold, but it is trivial for any positive integer \( \ell \) that

\[
\zeta_\ell(G; x, y) \leq \kappa_\ell(G; x, y),
\]

(14.1.1)
as to destroy all \((x, y)\)-paths of length at most \(\ell\) we need delete at least one vertex from each path of length at most \(\ell\).

In fact, there exist some special values of \(\ell\) for which the equality in (14.1.1) holds. Lovász, Neumann-Lara and Plummer [326] studied this problem first and found some such values of \(\ell\)’s.

**Theorem 14.1.1**  
For any undirected graph \(G\) with two non-adjacent vertices \(x\) and \(y\),
\[
\zeta_4(G; x, y) = \kappa_4(G; x, y).
\]

**Proof.**  
Partition \(V(G)\) into disjoint subsets \(V_{ij}\) as follows:
\[
z \in V_{ij} \iff d(G - y; x, z) = i \text{ and } d(G - x; z, y) = j.
\]
Since \(V_{11}\) must be contained in any \((x, y)\)-separating set with respect to 4, and any \((x, y)\)-path passing through \(V_{ij}\) for \(i + j > 4\) must have length more than 4, we may ignore \(V_{11}\) and all \(V_{ij}\)’s for \(i + j > 4\). Thus, the remaining graph \(H\) has the structure as shown in Figure 14.2.

![Figure 14.2: H and D constructed in the proof of Theorem 14.1.1](image)

We construct a digraph \(D\) from \(H\) as follows. \(V(D) = V(H)\) and
\[
(u, v) \in E(D) \iff uv \in E(H) \text{ and } d(H; x, u) < d(H; x, v).
\]
Hence \(D\) has the appearance of Figure 14.2.

It is easy to see that each \((x, y)\)-path in \(D\) has length at most 4 and corresponds to an \(xy\)-path in \(H\) of length at most 4. This fact means that
\[
\zeta_4(G; x, y) \geq \zeta_4(H; x, y) \geq \zeta(D; x, y).
\]
By Menger’s theorem applied to \(D\), we have that
\[
\zeta_4(G; x, y) \geq \zeta(D; x, y) = \kappa(D; x, y).
\]  (14.1.2)
In order to complete the proof, by the inequality (14.1.1) we need only prove \( \zeta_4(G; x, y) \geq \kappa_4(G; x, y) \). By (14.1.2), we only need to prove \( \kappa(D; x, y) \geq \kappa_4(G; x, y) \).

To the end, let \( S \) be an \((x, y)\)-separating set in \( D \) with \(|S| = \kappa(D; x, y)\). If there is an \((x, y)\)-path \( P \) in \( H - S \) with length at most 4, then from \( P \) we can construct an \((x, y)\)-path in \( D \) whose length is at most 4 and avoids \( S \), which contradicts the choice of \( S \). Thus, there are no \((x, y)\)-paths in \( H - S \) with length at most 4, that is, \( S \) is an \((x, y)\)-separating set with respect to 4 in \( G \), which means \( \kappa(D; x, y) = |S| \geq \kappa_4(G; x, y) \).

The theorem follows.

**Theorem 14.1.2** For any undirected graph \( G \) with two non-adjacent vertices \( x \) and \( y \), if \( d = d(G; x, y) \), then

\[
\zeta_d(G; x, y) = \kappa_d(G; x, y).
\]

**Proof.** We construct a digraph \( D \) by orienting some edges of \( G \) according to the following rule:

\[
(u, v) \in E(D) \iff uv \in E(G) \text{ and } d(G; u, y) > d(G; v, y).
\]

Then, clearly, a shortest \((x, y)\)-path in \( G \) yields an \((x, y)\)-path in \( D \).

On the other hand, we claim that an \((x, y)\)-path in \( D \) must correspond a shortest \((x, y)\)-path in \( G \). By our rule of orientation.

In fact, let \( P = (x_0, x_1, \ldots, x_h) \) be an \((x, y)\)-path in \( D \), where \( x_0 = x \) and \( x_h = y \). By the definition of \( D \), for each edge \((x_{i-1}, x_i)\) in \( P \), we have \( d(G; x_{i-1}, y) > d(G; x_i, y) \) for each \( i = 1, 2, \ldots, h \). This implies that the distance decreases by 1 as we traverse each directed edge along \( P \) toward \( y \). Thus \( P \) has at most \( d \) edges, namely, \( h \leq d \), and so \( P \) corresponds a shortest \((x, y)\)-path in \( G \).

It follows that

\[
\zeta_d(G; x, y) = \zeta(D; x, y), \quad \kappa_d(G; x, y) = \kappa(D; x, y).
\]

Menger’s theorem applied to \( D \), we have \( \zeta_d(G; x, y) = \kappa_d(G; x, y) \).

The result stated in Theorem 14.1.2 is also found by Entringer, Jackson and Slater [138]. In general, we have the following two results on the relations between \( \zeta_\ell(G; x, y) \) and \( \kappa_\ell(G; x, y) \), due to Lovász, Neumann-Lara and Plummer [326].

**Theorem 14.1.3** For any undirected graph \( G \) with two non-adjacent vertices \( x \) and \( y \), if \( m = \ell - d(G; x, y) \geq 0 \) for some integer \( \ell \), then

\[
\kappa_\ell(G; x, y) \leq (m + 1)\zeta_\ell(G; x, y).
\]
14.1. MENGER-TYPE PROBLEMS

**Proof.** The proof proceeds by induction on \( m \geq 0 \). For \( m = 0 \), the theorem is proved by Theorem 14.1.2. Now by the induction hypothesis, assume that the theorem holds for some smaller integer \( m_0 \geq 0 \), and suppose \( d = d(G; x, y) \) and \( m = \ell - d = m_0 + 1 \).

Let \( X \) be a minimum set of vertices separating all shortest \( xy \)-paths in \( G \). By Theorem 14.1.2, we have

\[
|X| = \kappa_d(G; x, y) = \zeta_d(G; x, y) \leq \zeta_\ell(G; x, y). \tag{14.1.3}
\]

Consider the subgraph \( H = G - X \). If \( d(H; x, y) > \ell \), then \( X \) separates all \( xy \)-paths of length at most \( \ell \) in \( G \). It follows from (14.1.3) that

\[
\kappa_\ell(G; x, y) \leq |X| \leq \zeta_\ell(G; x, y) \leq m \zeta_\ell(G; x, y),
\]

and we are done. So suppose \( d(H; x, y) \leq \ell \), say \( d(H; x, y) = \ell - r \) for some non-negative integer \( r \). Note that \( X \) separates all shortest \( xy \)-paths in \( G \), there are no \( xy \)-paths of length \( d \) in \( H \), which implies \( d(H; x, y) > d \). It follows that

\[
0 \leq r = \ell - d(H; x, y) < \ell - d = m. \tag{14.1.4}
\]

So the induction hypothesis applied to vertices \( x \) and \( y \) in \( H \), we have

\[
\kappa_\ell(H; x, y) \leq (r + 1)\zeta_\ell(H; x, y). \tag{14.1.5}
\]

Let \( Y \) be the minimum \((x, y)\)-separating set with respect to \( \ell \) in \( G \). Then, by (14.1.5),

\[
|Y| \leq |X| + (r + 1)\zeta_\ell(H; x, y) \leq |X| + (r + 1)\zeta_\ell(G; x, y). \tag{14.1.6}
\]

It follows from (14.1.3), (14.1.6) and (14.1.4) that

\[
\kappa_\ell(G; x, y) = |Y| \leq |X| + (r + 1)\zeta_\ell(G; x, y) \\
\leq (r + 2)\zeta_\ell(G; x, y) \leq (m + 1)\zeta_\ell(G; x, y)
\]

as required. \( \blacksquare \)

**Theorem 14.1.4** For any undirected graph \( G \) with two non-adjacent vertices \( x \) and \( y \),

\[
\kappa_\ell(G; x, y) \leq \left\lceil \frac{\ell}{2} \right\rceil \zeta_\ell(G; x, y).
\]

**Proof.** We first consider the case that \( d(G; x, y) \geq 1 + \left\lceil \frac{\ell}{2} \right\rceil \). Let \( m = \ell - d(G; x, y) \). If \( \ell = 2k \) for some \( k \) \((\geq 1)\), then \( d(G; x, y) \geq 1 + k \) and

\[
m + 1 = \ell - d(G; x, y) + 1 \leq 2k - (1 + k) + 1 = k = \left\lceil \frac{\ell}{2} \right\rceil.
\]
If \( \ell = 2k + 1 \) for some \( k \geq 1 \), then \( d(G; x, y) \geq 2 + k \) and

\[
m + 1 = \ell - d(G; x, y) + 1 \leq (2k + 1) - (2 + k) + 1 = k = \left\lfloor \frac{\ell}{2} \right\rfloor.
\]

Thus, by Theorem 14.1.3 the theorem holds if \( d(G; x, y) \geq 1 + \left\lfloor \frac{\ell}{2} \right\rfloor \).

Now, we suppose that \( d(G; x, y) \leq \left\lfloor \frac{\ell}{2} \right\rfloor \). Choose such an integer \( h \), to be defined, that \( d(G; x, y) \leq h \leq \ell \). Let \( P_0 \) be a shortest \( xy \)-path in \( G \). Construct a new graph \( G_1 \) from \( G \) by removing all internal vertices of \( P_0 \). Clearly, \( d(G_1; x, y) \geq d(G; x, y) \). Let \( P_1 \) be a shortest \( xy \)-path in \( G_1 \). Remove from \( G_1 \) all internal vertices of \( P_1 \) to obtain \( G_2 \). Continue in this manner until we obtain a graph \( G_r \) in which all \( xy \)-paths have length larger than \( h \), but \( G_{r-1} \) contains a shortest \( xy \)-path \( P_{r-1} \) of length at most \( h \). For convenience, let us denote \( G_r \) by \( H \). Then \( d(H; x, y) \geq h + 1 \).

Since \( r \) internally disjoint \( xy \)-paths have been removed from \( G \) to get \( H \) and all discarded paths have length less than \( d(H; x, y) \), we have

\[
\zeta_\ell(G; x, y) \geq \zeta_\ell(H; x, y) + r.
\]

Since the number of all discarded vertices from \( G \) to obtain \( H \) are at most \( r(h - 1) \), we have

\[
\kappa_\ell(G; x, y) \leq \kappa_\ell(H; x, y) + r(h - 1).
\]

If \( x \) and \( y \) are not connected in \( H \), we have \( \kappa_\ell(H; x, y) = \zeta_\ell(H; x, y) = 0 \). By (14.1.8) and (14.1.7), we have

\[
\kappa_\ell(G; x, y) \leq r(h - 1) \leq (h - 1) \zeta_\ell(G; x, y).
\]

Choose \( h = \left\lfloor \frac{\ell}{2} \right\rfloor + 1 \), which satisfies \( d(G; x, y) \leq h = \left\lfloor \frac{\ell}{2} \right\rfloor + 1 \leq \ell \). Substituting \( h = \left\lfloor \frac{\ell}{2} \right\rfloor + 1 \) into (14.1.9) yields

\[
\kappa_\ell(G; x, y) \leq \left\lfloor \frac{\ell}{2} \right\rfloor \zeta_\ell(G; x, y)
\]

as required.

If \( x \) and \( y \) are connected in \( H \), by Theorem 14.1.3 we have that

\[
\kappa_\ell(H; x, y) \leq (\ell - d(H; x, y) + 1) \zeta_\ell(H; x, y) \leq (\ell - h) \zeta_\ell(H; x, y).
\]

Combining (14.1.8) with (14.1.10), we obtain by (14.1.7)

\[
\kappa_\ell(G; x, y) \leq (\ell - h) \zeta_\ell(H; x, y) + r(h - 1)
\]

\[
\leq (\ell - h) (\zeta_\ell(G; x, y) - r) + r(h - 1)
\]

\[
= (\ell - h) \zeta_\ell(G; x, y) + r(2h - \ell - 1).
\]
If we choose $h$ to be the greatest integer so that $2h - \ell - 1 \leq 0$, then $h \leq \left\lfloor \frac{\ell}{2} \right\rfloor$, that is, $h = \left\lfloor \frac{\ell}{2} \right\rfloor$, which satisfies $d(G; x, y) \leq h = \left\lfloor \frac{\ell}{2} \right\rfloor < \ell$. Thus

$$\ell - h = \ell - \left\lfloor \frac{\ell}{2} \right\rfloor = \left\lceil \frac{\ell}{2} \right\rceil.$$  \hfill (14.1.12)

Since $r$ is non-negative and $2h - \ell - 1 \leq 0$, by (14.1.11) and (14.1.12), we have that

$$\kappa_\ell(G; x, y) \leq \left\lceil \frac{\ell}{2} \right\rceil \zeta_\ell(G; x, y)$$

as required, and the theorem follows.

The bound in Theorem 14.1.4 is tight for each $\ell \in \{2, 3, 5\}$ (for $\ell = 5$ see Figure 14.1). However, Theorem 14.1.1 shows that this bound is not tight for $\ell = 4$. By Theorem 14.1.4 and the inequality (14.1.1), we have the following consequence immediately.

**Corollary 14.1.1** For any undirected graph $G$ with two non-adjacent vertices $x$ and $y$, $\kappa_\ell(G; x, y) = \zeta_\ell(G; x, y)$ for each $\ell \in \{2, 3\}$.

Itai, Perl and Shiloah [263] have showed that the problem determining $\zeta_\ell(G; x, y)$ is NP-complete for any graph $G$ and $\ell \geq 5$. Some heuristic algorithms for finding a maximum number of disjoint bounded length paths has been given by Ronen and Perl [398] and Baier et al. [15]. Thus, what one is interested in is how large the gap between $\kappa_\ell(G; x, y)$ and $\zeta_\ell(G; x, y)$. Boyles and Exoo [64] constructed a class of graphs $G_m$ with two non-adjacent vertices $x$ and $y$ satisfying

$$\zeta_{4m-3}(G_m; x, y) = 1, \quad \kappa_{4m-3}(G_m; x, y) = m,$$  \hfill (14.1.13)

which imply that

$$\kappa_{4m-3}(G_m; x, y) = m \zeta_{4m-3}(G_m; x, y).$$

For a given integer $\ell \geq 2$, we define a function

$$s(\ell) = \sup \left\{ \frac{\kappa_\ell(G; x, y)}{\zeta_\ell(G; x, y)} \right\},$$  \hfill (14.1.14)

where the supremum is taken over all graphs $G$ and two non-adjacent vertices $x$ and $y$ in $G$. It is clear from the results obtained above that

$$s(2) = s(3) = s(4) = 1, \quad s(5) = 2.$$

Theorem 14.1.4 and the equalities in (14.1.13) imply that

$$\left\lfloor \frac{1}{4} (\ell + 3) \right\rfloor \leq s(\ell) \leq \left\lfloor \frac{\ell}{2} \right\rfloor.$$

So far, the better bounds for $s(\ell)$ is
The lower bound is due to Chung [101]. For other results, the reader is referred to Pyber and Tuzá [390] and the recent article by Baier et al. [15].

We conclude this section with two open problems that deserve study further. The first of the problems is to obtain the closer bound on $s(\ell)$ than that stated above. More ideas would seem to be necessary to achieve significant improvements. The second problem is to ask the same sorts of questions for digraphs instead of undirected graphs. The techniques used in this section do not readily generalize to this situation, and it appears that the directed case of determining $s(\ell)$ may be more difficult than the undirected case.

### 14.2 Bounded Menger Number and Connectivity

In this section, we consider the global communication efficiency and fault tolerance of a network when a transmission delay is given. In other words, for a given connected graph $G$ and an integer $\ell$, we consider the Menger number and the bounded connectivity of $G$ with respect to $\ell$.

Let $G$ be a connected undirected graph or strongly connected digraph with order $n$. For a given positive integer $\ell$ with $1 \leq \ell \leq n - 1$, the Menger number of $G$ with respect to $\ell$ is defined as the parameter

$$\zeta_\ell(G) = \min \{ \zeta_\ell(G; x, y) : x, y \in V(G) \},$$

and the bounded connectivity of $G$ with respect to $\ell$ is defined as the parameter

$$\kappa_\ell(G) = \min \{ \kappa_\ell(G; x, y) : x, y \in V(G), (x, y) \notin E(G) \}.$$

To avoid the relatively trivial case in which $\ell < d(G)$ or $G$ is a complete graph, we suppose $\ell \geq d(G) \geq 2$.

It is clear for a given graph $G$ and an integer $\ell$ with $2 \leq d(G) \leq \ell \leq n - 1$ that $\zeta_\ell(G)$ and $\kappa_\ell(G)$ are well-defined and

$$\zeta_d(G) \leq \zeta_{d+1}(G) \leq \cdots \leq \zeta_{n-1}(G),$$

$$\kappa_d(G) \leq \kappa_{d+1}(G) \leq \cdots \leq \kappa_{n-1}(G),$$

where $d = d(G)$. In the literature, $\kappa_d(G)$ is called the persistence of $G$, which is an important measure of vulnerability of diameter of $G$, proposed by Boesch, Harary and Kabell [54].

It is also clear that for $\ell = n - 1$, there is no restriction on the length of paths. Thus, by Menger’s theorem,

$$\zeta_{n-1}(G) = \kappa_{n-1}(G) = \kappa(G).$$
14.2. BOUNDED MENGER NUMBER AND CONNECTIVITY

Generally,
\[ \zeta_\ell(G) \leq \kappa_\ell(G) \leq \kappa(G). \]  \hspace{1cm} (14.2.1)

It has been ever claimed that \( \zeta_d(G) = \kappa_d(G) \) for \( d = d(G) \). However, many counterexamples to it are given by several authors, for example, Exoo [147], Bondy and Hell [60]. The graph \( G \) shown in Figure 14.3 is one of such counterexamples, \( d(G) = 5 \). It is easy to verify that \( \zeta_5(G) = 1 \), but \( \kappa_5(G) = 2 \), in which two vertices \( x \) and \( y \) attain these values.

By Menger’s theorem, Theorem 14.1.1 and Corollary 14.1.1, for any connected undirected graph \( G \) with order \( n \) and diameter at least two we immediately have
\[ \zeta_\ell(G) = \kappa_\ell(G) \quad \text{for each } \ell \in \{2, 3, 4, n - 1\} \]  \hspace{1cm} (14.2.2)

Clearly, determining \( \zeta_\ell(G) \) is also difficult for any graph \( G \) since determining \( \zeta_\ell(G; x, y) \) has been shown to be NP-complete. Thus it is of interest to determine the exact values of \( \zeta_\ell(G) \) and \( \kappa_\ell(G) \) for some well-known graphs \( G \) and for given \( \ell \geq d(G) \geq 2 \). We will introduce some such results below.

First, we state a result on the relations between \( \kappa_\ell(G) \) and \( D_w(G) \).

**Theorem 14.2.1** For any connected graph \( G \) with diameter \( d(G) \geq 2 \),

(a) \( \kappa_\ell(G) = w \Leftrightarrow D_w(G) \leq \ell < D_{w+1}(G) \) if \( G \) is \( w + 1 \)-connected, or

(b) \( D_w(G) = \ell \Leftrightarrow \kappa_{\ell-1}(G) < w \leq \kappa_\ell(G) \) if \( G \) is \( w \)-connected.

**Proof.** (a) Suppose \( \kappa_\ell(G) = w \). On the one hand, there are two vertices \( x \) and \( y \) of \( G \) with \( (x, y) \notin E(G) \) such that \( \kappa_\ell(G; x, y) = w \). Thus, there is a subset \( F \subset V(G) \setminus \{x, y\} \) with \( |F| = w \) such that there are no \((x, y)\)-paths of length at most \( \ell \) in \( G - F \). This implies \( d(G - F; x, y) > \ell \), i.e., \( D_{w+1}(G; x, y) > \ell \), and hence
\[ D_{w+1}(G) \geq D_{w+1}(G; x, y) > \ell. \]
On the other hand, by the definition of \( \kappa_\ell(G) = w \), for any two vertices \( x \) and \( y \) in \( G \), and any subset \( S \subset V(G) \setminus \{x, y\} \) with \( |S| < w \), there is at least one \((x, y)\)-path of length at most \( \ell \) in \( G - S \). This implies \( D_w(G) \leq \ell \). Thus, \[
\kappa_\ell(G) = w \implies D_w(G) \leq \ell < D_{w+1}(G).
\]

Conversely, if \( D_w(G) \leq \ell \), then any two vertices \( x \) and \( y \), \( D_w(G; x, y) \leq D_w(G) \leq \ell \). Thus, there is at least one \((x, y)\)-path of length at most \( \ell \) in \( G - S \) for any subset \( S \subset V(G) \setminus \{x, y\} \) with \( |S| < w \). This means \( \kappa_\ell(G; x, y) \geq w \) and, hence, \( \kappa_\ell(G) \geq w \) by the randomicity of \( x \) and \( y \).

Also if \( D_{w+1}(G) > \ell \), then there are two vertices \( x \) and \( y \), and a subset \( F \subset V(G) \setminus \{x, y\} \) with \( |F| = w \) such that \( G - F \) has no \((x, y)\)-path of length at most \( \ell \). This means \( \kappa_\ell(G; x, y) \leq w \), and hence \( \kappa_\ell(G) \leq \kappa_\ell(G; x, y) \leq w \). Thus, \[
D_w(G) \leq \ell < D_{w+1}(G) \implies \kappa_\ell(G) = w.
\]

Similarly, we can prove the assertion (b), left to the reader as an exercise for the details.

Although we can not find more \( \ell \)'s such that \( \zeta_\ell(G) = \kappa_\ell(G) \) for any graph \( G \), we can find some graphs \( G \) such that \( \zeta_\ell(G) = \kappa_\ell(G) \) for any \( \ell \). The following three results are due to Tao and Xu [429].

**Theorem 14.2.2** For the hypercube \( Q_n \), and \( 2 \leq n \leq 2^n - 1 \), \[
\zeta_\ell(Q_n) = \kappa_\ell(Q_n) = \begin{cases} 
  n - 1, & \text{for } \ell = n; \\
  n, & \text{for } \ell > n. 
\end{cases}
\]

**Proof.** When \( n \leq \ell \leq 2^n - 1 \), \( \zeta_\ell(Q_n) \) and \( \kappa_\ell(Q_n) \) are well-defined as \( d(Q_n) = n \). Let \( x \) and \( y \) be any two non-adjacent vertices of \( Q_n \). By Theorem 6.3.1 there are \( n \) internally disjoint \( xy \)-paths of length at most \( n + 1 \) in \( Q_n \), which implies that \( \zeta_{n+1}(Q_n; x, y) \geq n \). Thus, for \( \ell \geq n + 1 \), by (14.2.1) we have \[
n \leq \zeta_\ell(Q_n) \leq \kappa_\ell(Q_n) \leq \kappa(Q_n) = n.
\]

Also by Theorem 6.3.1, there are at least \( n - 1 \) internally disjoint \((x, y)\)-paths of length at most \( n \) between any two vertices \( x \) and \( y \) in \( Q_n \). This implies \( \zeta_n(Q_n) \geq n - 1 \). On the other hand, from Section 13.6, we have \( D_{n-1}(Q_n) = n < n+1 = D_n(Q_n) \). By Theorem 14.2.1 we have \( \kappa_n(Q_n) = n - 1 \). Thus we have \[
n - 1 \leq \zeta_n(Q_n) \leq \kappa_n(Q_n) = n - 1.
\]

The theorem follows.
Theorem 14.2.3 For the de Bruijn digraph $B(d, n)$ and $2 \leq n \leq \ell \leq d^n - 1$, 

$$\zeta_\ell(B(d, n)) = \kappa_\ell(B(d, n)) = \begin{cases} 1, & \text{for } \ell = n; \\ d - 1, & \text{for } \ell > n. \end{cases}$$

Proof. When $n \leq \ell \leq d^n - 1$, $\zeta_\ell(B(d, n))$ and $\kappa_\ell(B(d, n))$ are well-defined as $d(B(d, n)) = n$. By Theorem 8.2.4 there are $d - 1$ internally disjoint $(x, y)$-paths of length at most $n + 1$ for any two vertices $x$ and $y$ in $B(d, n)$. Thus, for $\ell \geq n + 1$, by (14.2.1) we have 

$$d - 1 \leq \zeta_\ell(B(d, n)) \leq \kappa_\ell(B(d, n)) \leq \kappa(B(d, n)) = d - 1.$$

Consider two vertices $x$ and $y$ with distance $d(B(d, n); x, y) = n$. Clearly, $(x, y) \notin E(B(d, n))$ since $n \geq 2$. By Theorem 8.2.2, there is the unique $(x, y)$-path of length $n$, which means $\kappa_n(B(d, n); x, y) = \zeta_n(B(d, n); x, y) = 1$. On the other hand, $\zeta_n(B(d, n)) \geq 1$ clearly. Thus, we have 

$$1 \leq \zeta_n(B(d, n)) \leq \kappa_n(B(d, n)) \leq \kappa_n(B(d, n); x, y) = 1.$$ 

The theorem follows.

Theorem 14.2.4 For the Kautz digraph $K(d, n)$ and $2 \leq n \leq \ell \leq d^n + d^{n-1} - 1$, 

$$\zeta_\ell(K(d, n)) = \kappa_\ell(K(d, n)) = \begin{cases} 1, & \text{for } \ell = n; \\ d - 1, & \text{for } \ell = n + 1; \\ d, & \text{for } \ell > n + 1. \end{cases}$$

Proof. For $n \leq \ell \leq d^n + d^{n-1} - 1$, $\zeta_\ell(K(d, n))$ and $\kappa_\ell(K(d, n))$ are well-defined as $d(K(d, n)) = n$. Let $x$ and $y$ be two vertices of $K(d, n)$ such that the distance $d(K(d, n); x, y) = n$. By Theorem 9.1.4, there is the unique $(x, y)$-path of length $n$, which means $\kappa_n(K(d, n); x, y) = \zeta_n(K(d, n); x, y) = 1$. On the other hand, $\zeta_n(K(d, n)) \geq 1$ clearly. Thus, we have 

$$1 = \zeta_n(K(d, n)) \leq \kappa_n(K(d, n)) \leq \kappa_n(K(d, n); x, y) = 1.$$ 

By Theorem 9.1.5, in $K(d, n)$ there are $d$ internally disjoint $(x, y)$-paths of length at most $n + 2$ for any two vertices $x$ and $y$ of $K(d, n)$. Thus, for $\ell \geq n + 2$, by (14.2.1) we have 

$$d \leq \zeta_\ell(K(d, n)) \leq \kappa_\ell(K(d, n)) \leq \kappa(K(d, n)) = d.$$

Also, by Theorem 9.1.5, there are $d - 1$ internally disjoint $(x, y)$-paths of length at most $n + 1$ for any two distinct vertices $x$ and $y$ in $K(d, n)$. This implies $\zeta_{n+1}(K(d, n)) \geq d - 1$. On the other hand, from Section 13.6 we have 

$$D_{d-1}(K(d, n)) = n + 1 < n + 2 = D_{d}(K(d, n)).$$
By Theorem 14.2.1 we have \( \kappa_{n+1}(K(d, n)) = d - 1 \). Thus
\[
d - 1 \leq \zeta_{n+1}(K(d, n)) \leq \kappa_{n+1}(K(d, n)) = d - 1
\]
as required.

Recently, Ma et al. [336] have presented a result on the Menger number of the Cartesian product of graphs. We state this result as follows, the proof is left the reader as an exercise.

**Theorem 14.2.5** For any two connected graphs \( G_1 \) and \( G_2 \), if \( \ell_i \geq 2 \) for \( i = 1, 2 \), then \( \zeta_{\ell_1+\ell_2}(G_1 \times G_2) \geq \zeta_{\ell_1}(G_1) + \zeta_{\ell_2}(G_2) \).

As an application of Theorem 14.2.5, combining Theorem 5.2.1 with Theorem 5.2.4, we immediately obtain the Menger number of the grid network.

**Corollary 14.2.1** Let \( G = G(m_1, m_2, \ldots, m_n) \) be a grid network. If \( \ell \geq \sum_{i=1}^{n} m_i \) and \( m_i \geq 2 \) for each \( i = 1, 2, \ldots, n \), then
\[
\zeta_{\ell}(G) = \zeta_{m_1}(P_{m_1}) + \zeta_{m_2}(P_{m_2}) + \cdots + \zeta_{m_n}(P_{m_n}) = n.
\]

Finally, we mention a significant function. For two positive integers \( \ell \) and \( w \), define a function
\[
h(\ell, w) = \min\{ h : \kappa_{\ell}(G) \geq w \Rightarrow \zeta_{h}(G) \geq w, \ G \}.
\]
Lovász, Neumann-Lara and Plummer [326] have shown the existence of \( h(\ell, w) \) for any \( \ell \) and \( w \). Pyber and Tuza [390] established an upper bound
\[
h(\ell, w) < \left( \frac{\ell + w - 2}{\ell - 2} \right) + \left( \frac{\ell + w - 3}{\ell - 2} \right).
\]
In particular, they determined
\[
h(\ell, 2) = \left\lfloor \frac{1}{4}(\ell - 1)^2 \right\rfloor + 2,
\]
and obtained an upper bound
\[
h(\ell, 3) < \frac{4}{27}(\ell + 1)^3.
\]

Apart from these we have known nothing about the function \( h(\ell, w) \) as far.

We conclude this section with other types of conditions which insure the existence of a given number of internally disjoint \((x, y)\)-paths of bound length are presented by Faudree et al. [158, 159, 160]. In fact, they studied the property \( \mathcal{P}_{\ell, w} \), for which a graph \( G \) having the property \( \mathcal{P}_{\ell, w} \) implies \( \zeta_{\ell}(G) \geq w \).
14.3 Bounded Edge-Connectivity

The edge version of disjoint paths for bounded length has been also studied by several authors, for example, Exoo [147, 148], Boyles and Exoo [64], Caccetta [69], Pyber and Tuza [390].

Let $x$ and $y$ be any two distinct vertices in $G$. The \((x, y)\)-Menger number with respect to $\ell$ for edges, denoted by $\eta_\ell(G; x, y)$, is the maximum number of edge-disjoint \((x, y)\)-paths of length at most $\ell$ in $G$.

Analogously, the bounded \((x, y)\)-edge-connectivity, denoted by $\lambda_\ell(G; x, y)$, is the minimum number of edges in $G$ whose deletion destroys all \((x, y)\)-paths of length at most $\ell$.

For example, considering the undirected graph $G$ shown in Figure 14.4, we have that $\eta_4(G; x, y) = 1$, $\lambda_4(G; x, y) = 2$.

![Figure 14.4: An example explaining edge disjoint bounded paths](image)

Since the length of any path in $G$ of order $n$ does not exceed $n - 1$, we suppose $\ell \leq n - 1$. If $\ell = n - 1$, then we have that

$$\eta_{n-1}(G; x, y) = \eta(G; x, y), \text{ and } \lambda_{n-1}(G; x, y) = \lambda(G; x, y).$$

By Menger’s theorem, we have the equality

$$\eta_{n-1}(G; x, y) = \lambda_{n-1}(G; x, y).$$

Clearly, the following inequality holds for any integer $\ell$ ($d(G; x, y) \leq \ell \leq n - 1$)

$$\eta_\ell(G; x, y) \leq \lambda_\ell(G; x, y). \quad (14.3.1)$$

Like the vertex case, the parameters $\eta_\ell(G; x, y)$ and $\lambda_\ell(G; x, y)$ are important measurements for efficiency of a real-time system. The following result, due to Exoo [147], and Pyber and Tuza [390], establishes a link between the parameters $\zeta_\ell(G; x, y)$, $\eta_\ell(G; x, y)$, and $\kappa_\ell(G; x, y)$, $\lambda_\ell(G; x, y)$ by using the concept of line graphs.

For any two distinct vertices $x$ and $y$ of $G$, we consider a new graph $G'$ obtained from $G$ by appending a loop $e_x$ at $x$ and a loop $e_y$ at $y$. Let $L_G(x, y)$ be the line graph of $G'$. 
**Theorem 14.3.1** For any undirected graph $G$ with two distinct vertices $x$ and $y$, there is a one-to-one correspondence between $xy$-paths of length $\ell$ in $G$ and chordless $e_xe_y$-paths of length $\ell + 1$ in $L_G(x, y)$.

**Proof.** Suppose that $P = x_1e_1x_2\cdots e_{\ell-1}x_{\ell}e_{\ell}y$ is an $xy$-path of length $\ell$ in $G$. Then $\ell + 2$ edges $e_x, e_1, \ldots, e_{\ell}, e_y$ are vertices of $L_G(x, y)$, which induce a chordless $e_xe_y$-path of length $\ell + 1$ in $L_G(x, y)$.

Conversely, let $P_L = (e_x, e_1, e_2, \ldots, e_{\ell-1}, e_{\ell}, e_y)$ be a chordless $e_xe_y$-path of length $\ell + 1$ in $L_G(x, y)$, where the set of vertices $\{e_x, e_1, e_2, \ldots, e_{\ell-1}, e_{\ell}\}$ defines a sequence of edges in $G$, which induces an $xy$-walk $W$ in $G$. If $W$ is not a path, then at least three edges in $\{e_x, e_1, e_2, \ldots, e_{\ell-1}, e_{\ell}, e_y\}$ are incident to the same vertex of $G$. In this case, the path $P_L$ is not chordless in $L_G(x, y)$.

Clearly, the correspondence given above is one-to-one.

**Corollary 14.3.1** For any undirected graph $G$ with two non-adjacent vertices $x$ and $y$,

$$
\eta_{\ell-1}(G; x, y) = \zeta_{\ell}(L_G(x, y); e_x, e_y), \\
\lambda_{\ell-1}(G; x, y) = \kappa_{\ell}(L_G(x, y); e_x, e_y).
$$

**Proof.** Two paths are edge-disjoint in $G$ if and only if the corresponding paths in $L_G(x, y)$ are internally disjoint. This fact implies, by Theorem 14.3.1,

$$
\eta_{\ell-1}(G; x, y) \leq \zeta_{\ell}(L_G(x, y); e_x, e_y).
$$

On the other hand, if $P'_1, P'_2, \ldots, P'_w$ are $w$ internally disjoint $e_xe_y$-paths in $L_G(x, y)$, then there exist $w$ chordless $e_xe_y$-paths $P_1, P_2, \ldots, P_w$ such that $P_i \subseteq P'_i$, and, hence, are internally disjoint for each $i = 1, 2, \ldots, w$. This implies

$$
\eta_{\ell-1}(G; x, y) \geq \zeta_{\ell}(L_G(x, y); e_x, e_y).
$$

It follows from the first assertion that $\lambda_{\ell-1}(G; x, y) = \kappa_{\ell}(L_G(x, y); e_x, e_y)$ holds since a vertex-set meets all $e_xe_y$-paths of length at most $\ell$ in $L_G(x, y)$ if and only if (as an edge in $G$) meets all chordless $xy$-paths in $G$.

**Corollary 14.3.2** For any undirected graph $G$ with vertices $x$ and $y$,

$$
\eta_{\ell}(G; x, y) = \lambda_{\ell}(G; x, y) \text{ for each } \ell \in \{1, 2, 3\}.
$$

**Proof.** By Theorem 14.1.1, Corollary 14.1.1 and Corollary 14.3.1, we have

$$
\zeta_{\ell}(L_G(x, y); e_x, e_y) = \kappa_{\ell}(L_G(x, y); e_x, e_y) \text{ for each } \ell \in \{2, 3, 4\}.
$$

Combining this with Theorem 14.3.1,

$$
\eta_{\ell-1}(G; x, y) = \lambda_{\ell-1}(G; x, y) \text{ for each } \ell \in \{2, 3, 4\}.$$
Niepel and Šafáriková [360] have shown that Theorem 14.1.3 is true for the edge case. Similar to the vertex case, we can define a function

$$s'(\ell) = \sup \left\{ \frac{\lambda_\ell(G; x, y)}{\eta_\ell(G; x, y)} \right\},$$

where the supremum is taken over all graphs $G$ and vertices $x$ and $y$ in $G$. Combining (14.1.14) with (14.3.2), by Corollary 14.3.1, we have $s'(\ell) \leq s(\ell + 1)$ for any $\ell \geq 2$. Niepel and Šafáriková [360], and Exoo [148] obtained the lower bound and the upper bounds on $s'(\ell)$, respectively, as follows.

$$\left\lfloor \frac{\ell}{3} \right\rfloor + 1 \leq s'(\ell) \leq \left\lfloor \frac{\ell}{2} \right\rfloor.$$

We now define the Menger number for edges and bounded edge-connectivity of $G$ with respect to a given integer $\ell$ as follows.

$$\eta_\ell(G) = \min \left\{ \eta_\ell(G; x, y) : x, y \in V(G) \right\},$$

$$\lambda_\ell(G) = \min \left\{ \lambda_\ell(G; x, y) : x, y \in V(G) \right\},$$

respectively, where $\ell \geq d(G)$. By Corollary 14.3.2 and Menger’s theorem, we have

$$\eta_\ell(G) = \lambda_\ell(G)$$

for each $\ell \in \{2, 3, n - 1\}$. Like the vertex case, the above equality does not hold for $\ell = d(G)$. See the graph $G$ shown in Figure 14.5, $d(G) = 4$, $\eta_4(G) = 1$ and $\lambda_4(G) = 2$, two vertices $x$ and $y$ attain these values. In the literature, $\lambda_d(G)$ is called the edge-persistence of $G$, where $d = d(G)$.

Figure 14.5: A graph $G$ with $d(G) = 4$ to show $\eta_4(G) \neq \lambda_4(G)$

There are two questions that have not been yet clarified. The one is to ask whether or not there is a result similar to Theorem 14.1.2 for the edge case, that is, whether or not the following equality holds

$$\eta_d(G; x, y) = \lambda_d(G; x, y),$$

where $d = d(G; x, y)$. 

Exoo [147] showed that \( \lambda_2(G) \leq \kappa_2(G) \) for any undirected graph \( G \) with diameter 2 and, for any two given integers \( a \) and \( b \) with \( a \leq b \), there is a graph \( G \) such that \( \lambda_2(G) = a \) and \( \kappa_2(G) = b \), moreover for any three positive integers \( \ell (\geq 3) \), \( a \) and \( b \), there is a graph \( G \) such that

\[
d(G) = \ell, \quad \kappa_\ell(G) = a \quad \text{and} \quad \lambda_\ell(G) = b.
\]

Thus, the other question is to ask whether or not that the following inequality is true

\[
\kappa_\ell(G) \leq \lambda_\ell(G)
\]

for any graph \( G \) and given \( \ell \) with \( d(G) < \ell \leq n - 2 \).

We can state a result similar to Theorem 14.2.1, the proof is left to the reader as an exercise.

**Theorem 14.3.2** For any connected graph \( G \),

\[
(a) \quad \lambda_t(G) = t \iff D'_1(G) \leq \ell < D'_{t+1}(G) \quad \text{if} \quad G \text{ is } t+1\text{-edge-connected}, \quad \text{or} \\
(b) \quad D'_t(G) = \ell \iff \lambda_{t-1}(G) < t \leq \lambda_t(G) \quad \text{if} \quad G \text{ is } t\text{-edge-connected}.
\]

Finally, we state a result on the bounded edge-connectivity for the Cartesian product graphs obtained by Lu, Xu and Hou [327], the proof is left the reader as an exercise.

**Theorem 14.3.3** For any connected undirected graphs \( G_i \), if \( \ell_i \geq 2 \) for \( i = 1,2 \), then \( \lambda_{\ell_1+\ell_2}(G_1 \times G_2) \geq \lambda_{\ell_1}(G_1) + \lambda_{\ell_2}(G_2) \).

As an immediate consequence of Theorem 14.3.3, we obtain a lower bound on the edge-persistence of the Cartesian products of two graphs.

**Corollary 14.3.3** If \( d_i = d(G_i) \geq 2 \) for each \( i = 1,2 \), then \( \lambda_{d_1+d_2}(G_1 \times G_2) \geq \lambda_{d_1}(G_1) + \lambda_{d_2}(G_2) \).

According to this result, Lu, Xu and Hou [327] determined the edge-persistence of some Cartesian product graphs such as \( C_n \times P_m \), \( C_n \times C_m \), \( Q_n \times P_m \) and \( Q_n \times C_m \), the proofs are left the reader as exercises.

### 14.4 Rabin Numbers of Networks

In this section, we discuss other kind of measurement of message delay and fault tolerance motivated by the work on fault-tolerant parallel routing using information dispersal algorithm by Rabin [393], who proposed a fault-tolerant communication scheme for the hypercube \( Q_n \).
To route a packet $P$ from $x$ to $y$ in $Q_n$, Rabin first breaks it into $n$ smaller pieces. Then, choose $n$ distinct vertices $y_1, y_2, \ldots, y_n$ randomly, called the random intermediate vertices, and route the pieces along $n$ vertex-disjoint paths of length at most $n+1$ to the random intermediate vertices $y_1, y_2, \ldots, y_n$. In the second phase, symmetrically, the pieces are routed from the intermediate vertices $y_1, y_2, \ldots, y_n$ to their destination $y$, again along other $n$ vertex-disjoint paths of length at most $n + 1$. The use of random intermediate vertices, an idea due to Valiant [445], is to alleviate congestion and that of vertex-disjoint paths tolerate loss of pieces due to faults.

Valiant’s idea can be used in many kinds of communication patterns, particularly, in one-to-many. For example, the multicasting belongs to the class of one-to-many communication. Given a source vertex and $w$ destinations vertices, the multicasting problem requires message transmission from the source vertex to each of the $w$ destinations vertices. Undoubtedly, the paths used to route the message should have as small length as possible.

Such considerations led us to study the following parameter, Rabin number, suggested by Hsu and Lyuu [245], because Rabin first determined this number for the hypercube.

Let $G$ be a $w$ ($\geq 1$)-connected graph, the $w$-Rabin number, denoted by $r_w(G)$, is defined as the minimum number $r$ such that for $w+1$ distinct vertices $x, y_1, \ldots, y_w$ there exist $w$ internally disjoint $(x, y_i)$-paths in $G$ of length at most $r$ for each $i = 1, 2, \ldots, w$. In other words, $r_w(G)$ is the minimum integer $r$ for which there exists an $(x, Y)$-fan $F_w(x, Y)$ such that every path in $F_w(x, Y)$ is of length at most $r$ for any vertex $x$ in $G$ and any set $Y$ consisting of $w$ vertices in $G - x$.

It is clear that if $1 \leq w \leq \kappa(G)$, then $r_w(G)$ is well-defined by Theorem 1.5.4 and

$$d(G) = r_1(G) \leq r_2(G) \leq \cdots \leq r_{w-1}(G) \leq r_w(G).$$

For example, $r_2(C_n) = n - 2$ for $n \geq 3$ and $r_w(K_n) = 1$ for any $1 \leq w \leq n - 1$ and $n \geq 2$, where $C_n$ and $K_n$ are an undirected cycle and a complete graph, respectively.

The following result, obtained by to Duh and Chen [129], independently, Liaw et al. [315], reveals the relations between $D_w(G)$ and $r_w(G)$.

**Theorem 14.4.1** For any $w$ ($\geq 1$)-connected graph $G$,

$$D_w(G) \leq r_w(G). \quad (14.4.1)$$

**Proof.** Since $G$ is $w$ ($\geq 1$)-connected, $D_w(G)$ and $r_w(G)$ are well-defined. Choose $Y' = \{y_1, y_2, \ldots, y_{w-1}\} \subset V(G)$ and $x, y_w \in V(G) \setminus Y'$ such that

$$D_w(G) = D_w(G - Y'; x, y_w) = d(G - Y'; x, y_w).$$
Let \( Y = \{ y_1, y_2, \ldots, y_{w-1}, y_w \} \). By the definition of \( r_w(G) \), there is an \((x, Y)\)-fan \( \mathcal{F}_w(x, Y) = \{ P_1, P_2, \ldots, P_w \} \) such that the length of \( P_i \) is at most \( r_w(G) \) for each \( i = 1, 2, \ldots, w \). Without loss of generality, suppose that \( P_w \) is of the maximum length in \( \mathcal{F}_w(x, Y) \). It follows that

\[
D_w(G) = d(G - Y'; x, y_w) \leq \varepsilon(P_w) \leq r_w(G)
\]
as required.

There are a number of graphs, some of which are listed in the end of this section, to show that the equality in (14.4.1) holds.

Recently, Niu, Shi and Lu [361] have shown that for any 2-connected undirected graph \( G \) with diameter \( d \),

\[
r_2(G) \leq \max\{d(D_2(G) - \frac{1}{2}d - \frac{1}{2}), D_2(G)\}.
\]

Liaw and Chang [313] proposed a more general concept called the strong \( w \)-Rabin number. The strong \( w \)-Rabin number of a \( w \)-connected graph \( G \), denoted by \( r^*_w(G) \), is the minimum \( \ell \) such that for any vertex \( x \) in \( G \) and any \( w \) (not necessarily distinct) vertices \( y_1, y_2, \ldots, y_w \) there exist \( w \) internally disjoint paths of length at most \( \ell \) from \( x \) to \( y_1, y_2, \ldots, y_w \). The strong \( w \)-Rabin number is called the \( w \)-star diameter by Gao and Hsu [179].

Clearly, 

\[
r^*_w(G) \geq r_w(G)
\]
for any \( w \)-connected graph \( G \). Moreover, if 

\[
r^*_w(G) = \ell,
\]
then \( \zeta(G) \geq w \) when \( y_1 = y_2 = \cdots = y_w \). For more information on strong Rabin numbers or star diameters, see Hsu and Lyuu [242, 245], Liaw and Chang [312], Lai and Chen [287], Gao and Hsu [179].

Hsu and Lyuu [245] proved that the problem of determining \( r_w(G) \leq \ell \) is NP-complete for any \( w \)-connected graph \( G \) and any integer \( \ell \). Thus, the problem we are concerned with is determining the exact value of \( r_w(G) \) for a given \( w \)-connected graph \( G \). We will below introduce the Rabin numbers for some well-known networks. All proofs will be here omitted because of the limitations of space.

1. **Hypercube** \( Q_n \) \( (n \geq 2) \)

We have known that \( \kappa(Q_n) = n = d(Q_n) \). Then \( r_w(Q_n) \) is well-defined if \( w \leq n \). Rabin [393] showed \( r_n(Q_n) = n + 1 \), Liaw and Chang [312] proved that (this result is slightly improved in [180])

\[
r_w(Q_n) = D_w(Q_n) = \begin{cases} n, & \text{for } 1 \leq w \leq n - 1; \\ n + 1, & \text{for } w = n. \end{cases}
\]

2. **Directed de Bruijn network** \( B(d, n) \) \( (d \geq 2, n \geq 1) \)

By Theorem 8.1.2, we have that \( \kappa(B(d, n)) = d - 1 \). Thus, \( r_w(B(d, n)) \) is well-defined if \( w \leq d - 1 \). Du, Lyuu and Hsu [127] showed
14.4. RABIN NUMBERS OF NETWORKS

\[ r_w(B(d, n)) = D_w(B(d, n)) = n + 1, \text{ for } 1 \leq w \leq d - 1. \]

3. Directed Kautz network \( K(d, n) \) \((d \geq 2, n \geq 1)\)

By Theorem 9.1.2, we have \( \kappa(K(d, n)) = d \). Thus \( r_k(K(d, n)) \) is well-defined if \( w \leq d \). Du, Lyuu and Hsu [127] showed

\[ r_d(K(d, n)) = D_d(K(d, n)) = n + 2. \]

The value of \( r_w(K(d, n)) \) for \( 2 \leq w \leq d - 1 \) remains an open question.

4. Circulant digraph \( G(d^n; 1, d, \ldots, d^{n-1}) \)

The circulant digraph \( G(d^n; S) \) with \( d \geq 2 \) has the connectivity \( n \) (due to Hamidoune [203]), where \( S = \{1, d, \ldots, d^{n-1}\} \). Liaw and Chang [312] proved that

\[
\begin{align*}
    r_w(G(d^n; S)) &= r_w^*(G(d^n; S)) = D_w(G(d^n; S)) \\
    &= \begin{cases} 
    n(d-1), & \text{for } 1 \leq w \leq n-1; \\
    n(d-1) + 1, & \text{for } w = n.
    \end{cases}
\end{align*}
\]

5. Directed toroidal mesh \( \overrightarrow{C}(d_1, \ldots, d_n) \)

Use \( \overrightarrow{C}_n(d) \) to denote \( \overrightarrow{C}(d_1, d_2, \ldots, d_n) \) for \( d_1 = d_2 = \cdots = d_n = d \geq 3 \). Hsu and Lyuu [245] have shown that \( r_n(\overrightarrow{C}_n(d)) = n(d-1) + 1 \). Liaw and Cheng [312] showed that

\[
\begin{align*}
    r_w(\overrightarrow{C}_n(d)) &= r_w^*(\overrightarrow{C}_n(d)) = D_w(\overrightarrow{C}_n(d)) \\
    &= \begin{cases} 
    n(d-1), & \text{for } 1 \leq w \leq n-1; \\
    n(d-1) + 1, & \text{for } w = n.
    \end{cases}
\end{align*}
\]

6. Generalized hypercube \( Q(d_1, \ldots, d_n) \)

We have known that \( Q(d_1, d_2, \ldots, d_n) \) has the connectivity \( m = d_1 + d_2 + \cdots + d_n - n \). Duh, Chen [129] showed that \( r_n(Q_n(d)) = n + 1 \). This result is improved by Liaw and Chang [312] as

\[
\begin{align*}
    r_w(Q(d_1, \ldots, d_n)) &= r_w^*(Q(d_1, \ldots, d_n)) = D_w(Q(d_1, \ldots, d_n)) \\
    &= \begin{cases} 
    n, & \text{for } 1 \leq w \leq n-1; \\
    n, & \text{for } w = n \text{ and the existence of at least two } d_i \geq 3; \\
    n + 1, & \text{for } w = n \text{ and the existence of at least one } d_i \geq 3; \\
    n + 1, & \text{for } n + 1 \leq w \leq m.
    \end{cases}
\end{align*}
\]

7. Butterfly network \( BF(n) \)

It has been known that the \( n \)-dimensional butterfly \( BF(n) \) has the connectivity \( 2 \). Cao, Du, Hsu and Wan [74] showed that \( 2n + 2 \leq r_2(BF(n)) \leq 2n + 4 \) for \( n \geq 2 \). Liaw and Chang [314] showed that

\[ r_2(BF(n)) = D_2(BF(n)) = 2n + 2, \text{ for } n \geq 2. \]
8. Folded hypercube $FQ_n$

By Theorem 3.1.11, $FQ_n$ has the connectivity $n + 1$. Lai et al. [286, 287, 288] determined that

$$r_w(FQ_n) = r_w^*(FQ_n) = D_w(FQ_n)$$

$$= \begin{cases} \left\lceil \frac{1}{2} n \right\rceil, & \text{for } 1 \leq w \leq \left\lfloor \frac{1}{2} n \right\rfloor - 1; \\ \left\lfloor \frac{1}{2} n \right\rfloor + 1, & \text{for } \left\lfloor \frac{1}{2} n \right\rfloor \leq w \leq n + 1. \end{cases}$$

Ranbin numbers of other well-known networks are also studies or determined, for example, Liaw and Chang [312], Duh and Chen [129], and Park [375] for WK-recursive networks, Chen and Chen [86], Gu and Peng [195] for star networks, Xiang and Stewart [486] for $(n - k)$-star graphs, Wu et al. [484] for hierarchical hypercube networks, and so on.

Exercises

14.1 Prove that for any $w$-connected graph $G$, if $d(G) \geq 2$ then $D_w(G) = \ell \Leftrightarrow \kappa_{\ell-1}(G) < w \leq \kappa_{\ell}(G)$ (i.e., Theorem 14.2.1 (b)).

14.2 Prove that (i.e., Theorem 14.2.5) for any two connected graphs $G_1$ and $G_2$, if $\ell_i \geq 2$ for $i = 1, 2$, then $\zeta_{\ell_1 + \ell_2}(G_1 \times G_2) \geq \zeta_{\ell_1}(G_1) + \zeta_{\ell_2}(G_2)$.

14.3 Prove that for any connected graph $G$ (i.e., Theorem 14.3.2)

(a) $\lambda_t(G) = t \Leftrightarrow \lambda'_t(G) \leq \ell < \lambda'_{t+1}(G)$ if $G$ is ($t + 1$)-edge-connected,

(b) $\lambda'_t(G) = \ell \Leftrightarrow \lambda_{\ell-1}(G) < t \leq \lambda_{\ell}(G)$ if $G$ is $t$-edge-connected.

14.4 Prove $\lambda_2(G) \leq \kappa_2(G)$ for any undirected graph $G$ with diameter 2.

14.5 For any given three integers $\ell, a$ and $b$, construct an undirected graph $G$ such that $d(G) = \ell, \kappa_{\ell}(G) = a$ and $\lambda_{\ell}(G) = b$. (Exoo [147])

14.6 Construct an undirected graph $G$ such that $\zeta'_t(w) = \left\lceil \frac{w}{\ell - 1} \right\rceil$.

14.7 Prove that for any undirected graph $G$ with two vertices $x$ and $y$, if $m = \ell - d(G; x, y) \geq 0$ for some integer $\ell$, then $\lambda_{\ell}(G; x, y) \leq (m + 1)\eta_\ell(G; x, y)$. (Niepel and Šafáriková [300])

14.8 Prove Theorem 14.3.3, that is, for any connected undirected graphs $G_i$, if $\ell_i \geq 2$ for $i = 1, 2$, then $\lambda_{\ell_1 + \ell_2}(G_1 \times G_2) \geq \lambda_{\ell_1}(G_1) + \lambda_{\ell_2}(G_2)$, and determine the edge-persistence of $C_n \times P_m$, $C_n \times C_m$, $Q_n \times P_m$ and $Q_n \times C_m$. (Lu, Xu and Hou [327])

14.9 Prove that $D_w(G) \leq r_w^*(G)$ and $r_w(G) \leq r_w^*(G)$ for any $k$-connected graph and any $w$ with $1 \leq w \leq k$. 