# Hints to Exercises in Chapter 1

**Ex1.2.6** By $\varepsilon(G) = \varepsilon(G^c)$ and $\varepsilon(G) + \varepsilon(G^c) = \varepsilon(K_v)$.

**Ex1.3.5** By Theorem 1 and $d_D^+(x) + d_G^-(x) = v - 1$ for any $x \in V(D)$.

**Ex1.3.6** (a) By the equalities (1,3).
(b) By $d_D^+(x) + d_G^-(x) = v - 1$ for any $x \in V(D)$.

**Ex1.4.2** A generalization of Example 1.4.2.

**Ex1.4.4** By definition of the line graph.

**Ex1.4.5** By definition of the Cartesian product.

**Ex1.5.4** (c) By Example 1.4.1

**Ex1.5.6** (a) There are two ways to prove it. The one way is to consider the number $v_i$ of the component $G_i$ and obtain

$$\epsilon(G) = \sum_{i=1}^{\omega} \epsilon(G_i) \le \frac{1}{2}\sum_{i=1}^{\omega} v_i(v_i - 1) = \frac{1}{2}\sum_{i=1}^{\omega} v_i^2 - \frac{1}{2}v \le \frac{1}{2}(v-\omega)(v-\omega+1).$$

Another way is to consider $G$ as a graph with edges as large as possible, and to prove that all components of $G$ is trivial except one.

(b) By contradiction.

**Ex1.5.7** The proof of (a) is similar to Ex1.5.6(a). To prove (b), define a function

$$f(\omega) = (v-\omega)(v-\omega+1) + \frac{1}{2}(\omega - 1)(2v - \omega).$$

It is convex on the interval $[2, v]$ for $v \ge 3$.

**Ex1.6.3** There are some wrongs in this exercise. Should add that the condition " if $G$ is strongly connected then" to (b) and delete "and Theorem 1.4" from (c).

The proof of (b). The faces that $v(G) = v(L(G)) = \varepsilon(G) = \varepsilon(L(G))$ and

$$\varepsilon = \sum_{x \in V(G)} d_G^+(x) d_G^-(x) \ge \sum_{x \in V(G)} 1 = \nu = \varepsilon,$$

implies $d_G^+(x) = d_G^-(x) = 1$ for any $x \in V(G)$.

**Ex1.6.4** Similar to Example 1.6.4.

**Ex1.6.6** There is flaw in this exercise. Should add that the condition " if $G$ is an undirected graph" to (a).

**Ex1.7.4** First prove that $v(T) \ge 2k+1$. Without loss of generality, assume $k = \delta^+$. By Theorem 1.1,

$$k \cdot v \le \sum_{x \in V(T)} d_T^+(x) = \sum_{x \in V(T)} d_T^-(x).$$

Thus,

$$2kv \le \sum_{x \in V(T)} [d_T^+(x) + d_T^-(x)] = \sum_{x \in V(T)} (v-1) = v(v-1),$$

which means that $v(T) \ge 2k+1$ and the equality holds if and only if $T$ is $k$-regular.

We now show that $T$ contains a directed cycle of length $\ge 2k+1$ by induction on $v \ge 2k+1$. If $v = 2k+1$, then $T$ is $k$-regular and, hence, $T$ is balanced. By the exercise 1.5.4 (c), $T$ is strongly connected. By Theorem 1.5, $T$ contains a directed cycle of length $2k+1$.

Now assume that the conclusion is true if $v = j \ge 2k+1$ and let $v = j+1$. If $T$ is strongly connected, then $T$ contains a directed cycle of length at least $2k+1$ by Theorem 1.5. Now assume that $T$ is not strongly connected and let $H$ be a strongly connected component of $T$ with vertex-set $S$ and $(S, \overline{S}) = \emptyset$. Then $\delta^+(H) \ge \delta^+(D)$, and so $\max\{\delta^+(H),\ \delta^-(H)\} \ge k$. Since $H$ is a tournament, $|S| \ge 2k+1$. By the induction hypothesis, $H$ contains a directed cycle of length $\ge 2k+1$.

(b) By Theorem 1.5, for any $k$ $(3 \le k \le v)$, every vertex in $T$ is contained in directed $k$-cycle. Let $x \in V(T)$ and $C$ be a directed $(v-1)$-cycle containing $x$ in $T$. Then, there is $y \in V(T) \setminus V(C)$ such that $T - y$ is strongly connected.

On the other hand, let $C'$ be a directed $(v-1)$-cycle containing $y$ in $T$. Then there is $z \in V(T) \setminus V(C')$ such that $T - z$ is strongly connected. Since $z \ne y$, the set $S = \{y, z\}$ is required.

(c) Let $C$ be a directed $k$-cycle. Then $T[C]$ is a strongly connected tournament. By Theorem 1.5, the assertion is true.

**Ex1.7.6** Let $P = (x_0, x_1, \cdots, x_{k-1}, x_k)$ be a longest path in $G$. By contradiction. Assume $k < 2\delta$ and let

$$S = \{x_i :\ x_0 x_{i+1} \in E(G)\}, \quad T = \{x_i :\ x_k x_i \in E(G)\}.$$

Then, $|S| = d_G(x_0) \ge \delta$, $|T| = d_G(x_k) \ge \delta$ and $x_k \notin S \cup T$.

First prove that $G$ contains a cycle of length $k+1$. Since $x_k \notin S \cup T$, then $|S \cup T| \le k < 2\delta$, and so $S \cap T \ne \emptyset$. Let $x_i \in S \cap T$. Then $C = (x_0, x_1 \cdots, x_i, x_k, x_{k-1}, \cdots, x_{i+1}, x_0)$ is a cycle of length $k+1$ in $G$.

Since $G$ is connected, $v > 2\delta \ge k+1$, thus there are a vertex $x \in V(G) \setminus V(C)$ and a vertex in $C$, say $x_j$ $(j \ne 0, k)$ such that $x x_j \in E(G)$. However, $C - x_i x_{i+1} + x x_j$ contains a longer path than $P$, a contradiction. Therefore, $k \ge 2\delta$.

**Ex1.7.7** Let $C$ be a shortest odd cycle of length $n$ in $G$. Assume $n \ge 5$ and $n \ge 2k+1$. Let $S = V(C)$. From the proof of Example 1.7.5, $|(S, \overline{S})| \le 2(v-n)$. Thus,

$$\delta(G)\, n \le \sum_{x \in S} d_G(x) \le 2\varepsilon(G[S]) + 2(v-n) = 2n + 2(v-n) = 2v,$$

from which a contradiction is deduced as follows: $\delta(G) \le \left\lfloor \frac{2v}{2k+1} \right\rfloor$.

**Ex1.9.7** (a) By contradiction. Let $G$ be maximal counterexample, that is, $G$ is a graph that satisfies the given conditions but $G + xy$ contains a Hamilton cycle for any two nonadjacent vertices $x \in X$, $y \in Y$, where $X = \{x_1, x_2, \cdots, x_n\}$, $Y = \{y_1, y_2, \cdots, y_n\}$. Let $C = (x_1, y_1, x_2, y_2, \cdots, x_n, y_n, x_1)$ be a Hamilton cycle in $G + x_1y_1$ and let

$$I = \{i:\ 2 \le i \le n,\ x_1y_i \in E(G)\}.$$

Then $I = N_G(x_1)$ and $x_{i-1}y_1 \notin E(G)$ for any $i \in I$. Thus, $N_G(y_1) \subseteq X \setminus \{x_{i-1}:\ i \in I\}$, and so $d_G(y_1) \le n - |I| = n - d_G(x_1)$, from which a contradiction is deduced as follows. $d_G(x_1) + d_G(y_1) \le n$.

(b) From (a), it is sufficient to prove that $d_G(x) + d_G(y) > n$ for any two nonadjacent vertices $x \in X$ and $y \in Y$. By contradiction. Assume that there exist two nonadjacent vertices $x \in X$ and $y \in Y$ such that $d_G(x) + d_G(y) \le n$. Since $G$ is bipartite and $|X| = |Y| = n$, $G$ can be viewed as a graph obtained from $K_{n,n}$ by deleting $h$ edges. On the one hand,

$$h = n^2 - \varepsilon(G) < n^2 - (n^2 - n + 1) = n - 1.$$

On the other hand, for $x \in X$ and $y \in Y$,

$$h \ge [n - d_G(x)] + [n - d_G(y)] - 1\ = 2n - [d_G(x) + d_G(y)] - 1 \ge n - 1.$$

This is a contradiction.

**Ex1.10.5** (d) Let $\mathbf{A}^* = \mathbf{B} - \mathbf{A}$ and $\mathbf{X} = (x_1, x_2, \cdots, x_v\}$ be any vector. Then

$$\mathbf{X}^T\mathbf{A}^*\mathbf{X} = \sum_{ij \in E(G)} (x_i - x_j)^2 \ge 0,$$

the equality holds $\Longleftrightarrow \mathbf{X}^T = (1, 1, \cdots, 1)$. Thus, $\mathbf{A}^*$ semi-positive.

($\Longrightarrow$) By induction on $v \ge 1$. If $v = 1$, $\mathbf{A} = \mathbf{O}, \mathbf{A}^* = \mathbf{O}$, and so, rank $\mathbf{A}^* = 0 = 1-1$. Assume the assertion holds for any connected graph of order less than $< v$, and let $G$ be a simple undirected graph of order $v$, $d_i = d_G(x_i)\,(i = 1, 2, \cdots, v)$. Choose a non-cut-vertex of $G$ (there are at least two such vertices by Example 1.5.3). Without loss of generality, let $x_1$ be such a vertex and $N_G(x_1) = \{x_2, x_3, \cdots, x_d + 1\}$. There are two cases.

*Case 1* $d_1 < v - 1$. In this case, $\mathbf{A}$ and the adjacency matrix of $G - x_1$ can be expressed as

$$\mathbf{A} = \begin{bmatrix} 0 & \mathbf{J}_{1,d_1} & \mathbf{O}_{1,v-d_1-1} \\ \mathbf{J}_{d_1,1} & \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{O}_{v-d_1-1,1} & \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \qquad \mathbf{A}_{v-1} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}.$$

Define two diagonal matrices as follows.

$$\mathbf{P}_1 = \mathrm{diag}\,(d_2 - 1, \cdots, d_{d_1+1} - 1)\mathbf{I}_{d_1} \qquad \mathbf{P}_2 = \mathrm{diag}\,(d_{d_1+2}, \cdots, d_v)\mathbf{I}_{v-d_1-1}.$$

By the induction hypothesis,

$$\mathbf{A}^*_{v-1} = \begin{pmatrix} \mathbf{P}_1 - \mathbf{A}_{11} & -\mathbf{A}_{12} \\ -\mathbf{A}_{21} & \mathbf{P}_2 - \mathbf{A}_{22} \end{pmatrix}$$

has rank $v-2$. Since $\mathbf{A}^*$ is semi-positive, $\mathbf{P}_1 - \mathbf{A}_{22}$ is positive and $\mathbf{Q} = \mathbf{P}_2 - \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{P}_1^{-1}\mathbf{A}_{21}$ is semi-positive. Thus, $\mathbf{Q} + \mathbf{I}_{d_1}$ is invertible. On the other hand, let

$$\mathbf{S} = \begin{pmatrix} \mathrm{diag}\,(d_2, \cdots, d_{d_1+1})\mathbf{I}_{d_1} - \mathbf{A}_{11} & -\mathbf{A}_{12} \\ -\mathbf{A}_{21} & \mathrm{diag}\,(d_{d_1+2}, \cdots, d_v)\mathbf{I}_{v-d_1-1} - \mathbf{A}_{22} \end{pmatrix},$$

the th-$(v-1)$ sub-matrix of $\mathbf{A}_v^*$, its determinant $\det \mathbf{S} = \det \mathbf{P}_1 \cdot \det(\mathbf{Q} + \mathbf{I}_{d_1}) \neq 0$, and $\mathbf{A}_v^* \cdot \mathbf{J}_{v,1} = \mathbf{O}_{v,1}$, that is, rank $(\mathbf{A}^*) = v-1$.

*Case 2* $d_1 = v-1$. In this case,

$$\mathbf{A} = \begin{pmatrix} 0 & \mathbf{J}_{1,v-1} \\ \mathbf{J}_{v-1,1} & \mathbf{A}_{v-1} \end{pmatrix},$$

where $\mathbf{A}_{v-1}$ is the adjacency matrix of $G - x_1$. $\mathrm{diag}(d_2 - 1, \cdots, d_{v-1}) - \mathbf{A}_{v-1}$ is semi-positive and $\mathrm{diag}(d_2, \cdots, d_v) - \mathbf{A}_{v-1}$ is the th-$(v-1)$ sub-matrix of $\mathbf{A}^*$, which is invertible. Since $\mathbf{A}^* \cdot \mathbf{J}_{v-1} = \mathbf{O}_{v,1}$, rank $(\mathbf{A}^*) = v-1$.

$(\Longleftarrow)$ Assume rank $(\mathbf{A}^*) = v-1$ and $G$ is disconnected. Let $G_1, G_2, \cdots, G_\omega$ be connected components of $G$ with orders $v_1, v_2, \cdots, v_\omega$, respectively. Then

$$\mathbf{A}^* = \mathrm{diag}\,(\mathbf{A}_{v_1}^*, \mathbf{A}_{v_2}^*, \cdots, \mathbf{A}_{v_\omega}^*),$$

where $\mathbf{A}_{v_i}^*$ is the Laplace matrix of $G_i$ $(i = 1, 2, \cdots, \omega)$. Since $G_i$ is connected, thus,

$$\mathrm{rank}\,(\mathbf{A}_{v_i}^*) = v_i - 1, \quad i = 1, 2, \cdots, \omega.$$

It follows that

$$v - 1 = \sum_{i=1}^{\omega}(v_i - 1) = v - \omega,$$

from which, $\omega = 1$, that is, $G$ is connected.

# Hints to Exercises in Chapter 2

**Ex2.1.2** The assertion (c) can be proved structurally.

**Ex2.1.3** $v_1 = v_3 + 2v_4 + \cdots + (\Delta - 2)v_\Delta + 2 = 2 + \sum_{i=3}^{\Delta}(i-2)v_i = 2 + \sum_{x \in U}(d_G(x) - 2)$.

**Ex2.1.4** There are some wrongs in this exercise. It should be corrected as " Prove that if $\{X, Y\}$ is a bipartition of $T$ with $|X| = |Y| = k$, then there are at least $(k+1)$ vertices of degree one in $X$. " By induction on $k \geq 0$.

**Ex2.1.15** Since any two spanning trees of $X$ have the same number of edges, the symmetric difference of their edge sets is even, say $2m$. To prove that $T(X)$ is connected, it is sufficient to show that any two spanning trees of $X$, as two vertices, are connected in $T(X)$. We can do this by induction on $m \geq 1$. If $m = 1$, then, by the definition of $T(X)$, the two spanning trees are adjacent in $T(X)$, and so are connected. Assume any two spanning trees of $X$ are connected in $T(X)$ if the symmetric difference of their edge sets is less than $2m$.

Let $T$ and $T'$ be two spanning trees in $X$ that the symmetric difference of their edge sets is $2m$ with $m \geq 2$. let $E_1 = E(T) \setminus E(T')$ and $E_2 = E(T') \setminus E(T)$. Then $|E_1| = |E_2| = m$. Since $T'$ is a spanning tree of $X$, for an edge $e \in E_1$, $T' + e$ contains only cycle, denoted by $C_e$. Since $T'$ contains no cycle, there exists exactly one edge $e' \in E_2 \cap E(C_e)$ such that $T'' = T' - e' + e$ is a spanning three of $X$. It is clear that the symmetric difference of $E(T')$ and $E(T'')$ is 2, and thus, they are adjacent in $T(X)$. Also, the symmetric difference of $E(T)$ and $E(T'')$ is $2(m-1)$, and thus, by the induction hypothesis, they are connected in $T(X)$. It follows that $T$ and $T'$ are connected in $T(X)$.

**Ex2.3.2** See Example 1.10.2.

**Ex2.3.3** The proofs of (a) and (b) are similar to Example 2.3.2. As (c), if $G$ is not bipartite, then $G$ contains an odd cycle. Let $C$ be a shortest odd cycle. The determinant of the sub-matrix of $M$ responding to the vertices and edges in $C$ is equal to two, a contradiction. Conversely, by induction on $k \geq 1$, which is order of a sub-matrix of $M$. In the induction step, assume $P_{\ell+1}$ is a sub-matrix of $M$. If there is exactly one non-zero entry in some column, then it is easy to prove $\det P_{\ell+1} = 0, \pm 1$. Assume there are exactly two non-zero entries in $P_{\ell+1}$ below. It is clear that $\det P_{\ell+1} = 0$ since the sum of any column is two.

**Ex2.3.6** For each vertex $y \neq x$, select one in-coming edge of $y$. Let $T$ be the induced subgraph by these edges. Then $T$ is an in-tree rooted at $x$, for $T$ has $v - 1$ edges, contains no directed cycles, $d_T^-(x) = 0$, $d_T^-(y) = 1$ for any $y \neq x$ (see Exercise 2.1.1). Conversely, every in-tree rooted at $x$ occurs in this way. Hence the number of such in-trees rooted at $x$ is $\varsigma_x(G) = \prod_{y \in V \setminus \{x\}} d_G^-(y)$.

# Hints to Exercises in Chapter 3

**Ex3.1.4** Using the following three equalities:

$$\begin{aligned}
&3v - \varepsilon = 6,\\
&v = v_3 + v_4 + v_5 + v_6 + v_7 + v_8 + \cdots + v_\Delta,\\
&2\varepsilon = 3v_3 + 4v_4 + 5v_5 + 6v_6 + 7v_7 + 8v_8 + \cdots + \Delta v_\Delta.
\end{aligned}$$

**Ex3.1.6** Using the following three equalities:

$$\begin{aligned}
&3v = \phi_1 + 2\phi_2 + 3\phi_3 + 4\phi_4 + 5\phi_5 + 6\phi_6 + 7\phi_7 + 8\phi_8 + \cdots,\\
&2\varepsilon = \phi_1 + 2\phi_2 + 3\phi_3 + 4\phi_4 + 5\phi_5 + 6\phi_6 + 7\phi_7 + 8\phi_8 + \cdots,\\
&\phi = \phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5 + \phi_6 + \phi_7 + \phi_8 + \cdots,
\end{aligned}$$

and Euler's formula $v - \varepsilon + \phi = 2$.

**Ex3.1.7** Without loss of generality, assume that $G$ is a plane graph. By Theorem 3.2 and Euler's formula,

$$2\varepsilon(G) = \sum_{f\in F(G)} d_G(f) \geq g\phi = g(2 - v + \varepsilon) = -g(v-2) + g\varepsilon.$$

**Ex3.1.8** From $\varepsilon \leq 3v - 6$ and $\varepsilon(G) + \varepsilon(G^c) = \frac{1}{2}v(v-1)$, we have

$$\varepsilon(G^c) \geq \frac{1}{2}v(v-1) - 3(v-3) = \frac{1}{2}(v^2 - 7v + 12) > 3v - 6 \text{ for } v \geq 11.$$

**Ex3.1.14** Without loss of generality, suppose that $G$ is maximal and that $\{d_1, d_2, \cdots, d_v\}$ is the degree-sequence of $G$. Then

$$\sum_{i=1}^{v} d_i = 2\varepsilon = 6n - 12.$$

Let

$$f(d_1, d_2, \cdots, d_v) = \sum_{i=1}^{v} d_i^2.$$

(a) For $3 \leq d_i \leq v - 1$ and $v \geq 4$,

$$\begin{aligned}
f(d_1, d_2, \cdots, d_v) \leq\ & f(3, 3, 4, 4, \cdots, 4, v-1, v-1)\\
=\ & 18 + 16(v-4) + 2(v-1)^2 = 2(v+3)^2 - 62.
\end{aligned}$$

(b) Note that $\delta \geq 4$ implies $v > 5$. For $4 \leq d_i \leq v - 1$ and $v > 5$,

$$\begin{aligned}
f(d_1, d_2, \cdots, d_v) \leq\ & f(4, 4, \cdots, 4, v-2, v-2) = 16(v-2) + 2(v-2)^2\\
=\ & 2(v+3)^2 - 4v - 42 < 2(v+3)^2 - 62.
\end{aligned}$$

**Ex3.2.3** Let $V(G) = \{x, y, z, x_4, x_5, \cdots, x_v\}$. By Kuratowski's theorem, $G$ contains no $K_{3,3}$. There exists at most two $x_i$, which all are adjacent to $x, y, z$. The number of edges between $\{x, y, z\}$ and $V \setminus \{x, y, z\}$ is at most $2 \cdot 3 + (n-5) \cdot 2 = 2n - 4$. It follows that $d_G(x) + d_G(y) + d_G(z) \leq 6 + (2n-4) = 2n + 2$.

# Hints to Exercises in Chapter 4

**Ex4.1.2** (b) For any two distinct vertices $u, v \in S$, if $a = (u, v) \in E(G)$, then $\mathbf{f}^+(u)$ contains $\mathbf{f}(a)$ and $\mathbf{f}^-(v)$ contains $\mathbf{f}(a)$, that is, $\mathbf{f}(a)$ does not contribute to the sum. In other words, any edge $a$ such that $\mathbf{f}(a)$ contributes to the sum has exactly one end-vertex in $S$.

**Ex4.1.4** See the proof of Theorem 4.1.

**Ex4.2.3** (a) Note that the condition "$d_G^+(x) - d_G^-(y) = k$" should be replaced by "$d_G^+(x) - d_G^-(x) = k$".

One of the ways is to use Exercise 1.8.3. directly. The other way is use Menger's theorem 4.2. In fact, let $(S, \overline{S})$ be a $\lambda$-cut of $G$. Then $\eta_G(x, y) = \lambda_G(x, y) = |(S, \overline{S}| \geq |(S, \overline{S})| - |(\overline{S}, S)| = \sum\limits_{u \in S}(d_G^+(u) - d_G^-(u)) = d_G^+(x) - d_G^-(x) = k$.

**Ex4.2.4** The assertion can be proved structurally. Let $k = d_G(x) \leq d_G(y)$, $A = N_G(x) \cap N_G(y) = \{u_1, u_2, \cdots, u_h\}$, $X = N_G(x) - A - \{y\} = \{x_1, x_2, \cdots, x_a\}$, $X = N_G(y) - A - \{x\} = \{y_1, y_2, \cdots, y_b\}$. Then $a \leq b$ and $k = h + a + \delta_{xy}$, where $\delta_{xy}$ is equal to one if $x$ and $y$ adjacent, equal to zero otherwise.

For each $u_i \in A$, $P_i = (x, u_i, y)$ is an $xy$-path of length two. $P_{\delta_{xy}} = xy$ if $xy$ exists.

Since $d(G) \leq 2$, there are $a$ edge disjoint $x_j y_j$-paths $Q_j$ $(j = 1, 2, \cdots, a)$ of length at most two. Let $P_{h+j} = xx_j + Q_j + y_j y$ for $j = 1, 2, \cdots, a$. Thus, $P_1, P_2, \cdots, P_{h+a+\delta_{xy}}$ are $k$ edge disjoint $xy$-paths of length at most 4.

**Ex4.2.5** Note that the condition "$k \geq 2$" should be added to the exercise.

Let $x$ and $y$ be two vertices in $G$ such that $d_G(x, y) = d(G)$. Since $G$ is $k$-regular and $k \geq 2$, there exists $z \in N_G^-(y)$ different from $x$. Consider $k$ $(x, z)$-paths $P_1, P_2, \cdots, P_k$, one of them, say $P_i$, must use the vertex $y$, whose length $\varepsilon(P_i) \geq d_G(x, y) + 1 = d(G) + 1$.

**Ex4.3.3** Reduce it to Example 4.3.2.

**Ex4.3.10** Apply Menger's theorem (4.3) to the graph $H$ obtained from by adding a new vertex $y$ and $k$ edges from $x_i$ to $y$ for each $i = 1, 2, \cdots, k$.

**Ex4.3.11** Without loss of generality, assume $k \geq 3$. Let $S$ be a set of $k$ vertices and $C$ be a cycle that contain vertices in $S$ as large as possible. Let $m = |V(C) \cap S|$. Then $m \geq 2$. Want to prove $m = k$ by contradiction. Let $x$ be a vertex in $S$ not in $C$. Label the vertices in $S \cap V(C)$ as $s_1, s_2, \cdots, s_m$ in some given direction of $C$. By Exercise 4.3.10, there are $m$ internally disjoint $(x, s_i)$-paths $P_i\,(i = 1, 2, \cdots, m)$ in $G$. If $m < k$, then $s_1, s_2, \cdots, s_m$ partitions $C$ into $m$ sections, of which at least one contains no vertex in $S$ except end-vertices. Assume $(s_1, s_2)$-section $C'$ contains no vertex in $S$. Then, $C' \cup P_1 \cup P_2$ forms a cycle in $G$, which contains vertices in $S$ is more than $C$ does since $x \notin V(C)$, a contradiction.

**Ex4.3.12** Let $x \in V(G)$ and let $B = \{y_1, y_2, \cdots, y_k\} \subseteq N_G^-(x)$. By exercise 4.3.10, there are $k$ internally disjoint $(x, y_i)$-paths $P_i\,(i = 1, 2, \cdots, k)$ in $G$. Thus, $v(P_i) \geq g =$

$g(G)$ for each $i = 1, 2, \cdots, k$. It follows that

$$v = v(G) \geq k(g-1) + 1 \Longrightarrow g(G) = g \leq \left\lfloor \frac{v+k-1}{k} \right\rfloor = \left\lceil \frac{v}{k} \right\rceil.$$

**Ex4.3.13** Assume $\kappa(G) = k \geq 1$ and $d(G) \geq 3$. Let $x, y \in V(G)$ such that $d_G(x, y) = d(G)$. By Menger's theorem, $\zeta_G(x, y) \geq \kappa(G) = k \geq 1$ and $\zeta_G(y, x) \geq \kappa(G) = k \geq 1$. Let $P_1, P_2, \cdots, P_k$ be $k$ internally disjoint $(x, y)$-paths in $G$. Then

$$d_G^+(x) - k \geq \delta^+(G) - k, \quad d_G^-(y) - k \geq \delta^-(G) - k.$$

Since $d(G) \geq 3$, $N_G^+(x) \cap N_G^-(y) = \emptyset$. It follows that

$$\begin{aligned} v \geq\ & \sum_{i=1}^{k}(v(P_i) - 1) + 2 + \delta^+(G) - k + \delta^-(G) - k \\ \geq\ & k(d-1) + 2 + \delta^+ + \delta^- - 2k = k(d-3) + \delta^+ + \delta^- + 2. \end{aligned}$$

# Hints to Exercises in Chapter 5

**Ex5.1.1** Without loss of generality, assume $G$ is connected. Clearly, $\alpha'(G) \le \lfloor \frac{1}{2} \rfloor$. We now show $\alpha'(G) \le \lfloor \frac{1}{1+\Delta} \rfloor$ by induction on $\varepsilon \ge 1$. Assume $\varepsilon(G) = k+1 \ge 2$. If $G$ contains a cycle $C$, then choose an edge $e$ in $C$. By the induction hypothesis,

$$\alpha'(G) \ge \alpha'(G-e) \ge \frac{v}{\Delta(G-e)} \ge \frac{v}{1+\Delta(G)}.$$

If $G$ contains no cycle, then $G$ is a tree. If $G$ is a star, then the conclusion holds clearly. Assume $G$ is not a star below. Thus, there is an edge $e$ in $C$. Let $G_1$ and $G_2$ be two components of $G-e$, and let $v_i = v(G_i)$ for each $i = 1, 2$. By the induction hypothesis,

$$\alpha'(G) \ge \alpha'(G_1) + \alpha'(G_2) \ge \frac{v_1}{\Delta(G_1)} + \frac{v_2}{\Delta(G_2)} \ge \frac{v_1+v_2}{1+\Delta(G)} = \frac{v}{1+\Delta(G)}.$$

**Ex5.1.3** Note that this exercise has an erratum, that is, $\frac{2}{3}\,v$ should be replaced by $\frac{2}{3}\,\varepsilon$.

Let $G$ be a plane triangulation of order $v$ ($\ge 4$). By the exercise 4.3.8, $G^*$ is a 3-regular 2-edge-connected simple graph. By Corollary 5.2.1, $G^*$ has a perfect matching $M^*$. $G$ is connected since $G$ a plane triangulation. Thus, by the exercise 3.3.1, $(G^*)^* \cong G$, and so $(G^* - M^*)^*$ is isomorphic to some subgraph of $G$. By Euler's formula,

$$\varepsilon(G^* - M^*) = \varepsilon(G^*) - \frac{1}{2}\,v^* = \frac{3}{2}\,v^* - \frac{1}{2}\,v^* = v^* = \phi = 2 + \varepsilon - v.$$

Since $G$ is a plane triangulation, by Theorem 3.4, $\varepsilon = 3v - 6$, that is, $v = \frac{1}{3}\,\varepsilon + 2$. It follows that

$$\varepsilon(G^* - M^*) = 2 + \varepsilon - v = 2 + \varepsilon - \frac{1}{3}\,\varepsilon - 2 = \frac{2}{3}\,\varepsilon.$$

Note that $(G^* - M^*)^*$ is equivalent to a graph obtained from $G$ removing some common edges in two triangles. Thus, $G^* - M^*$ contains only cycles of length four, hence is bipartite. Thus, $G$ contains a bipartite subgraph with $\frac{2}{3}\,\varepsilon$ edges.

**Ex5.1.6** By Tutte's theorem, $o(G-x) \le 1$ for any $x \in V(G)$. If there is a vertex $x$ such that $o(G-x) = 0$, then $G$ has odd order, a contradiction.

Conversely, by induction on $v \ge 2$. Let $G$ be a tree of order $v \ge 3$. Choose a vertex $x$ of degree one in $G$. Since $o(G-x) = 1$, $v$ is even. Let $y$ be the unique neighbor of $x$. Since $o(G-y) = 1$, $\{x\}$ is the only odd component of $G-y$, and other components $G_1, G_2, \cdots, G_p$ are even. Then $o(G_i - z) = 1$ for any $z \in V(G_i)$ and each $i = 1, 2, \cdots, p$ (Why?). By the induction hypothesis, $G_i$ has perfect matching $M_i$ for each $i = 1, 2, \cdots, p$. Then $M = M_i \cup M_2 \cup \cdots \cup M_p \cup \{xy\}$ is a perfect matching of $G$.

**Ex5.1.10** By definitions of permanent $\mathbf{A} = (a_{ij})_{m \times n}$ and $\mathrm{Per}(\mathbf{A})$,

$$\mathrm{Per}(\mathbf{A}) = \sum_{f \in F} a_{1f(1)} a_{2f(2)} \cdots a_{mf(m)},$$

where $F = \{f: \ f$ is an injective from $\{1,2,\cdots,m\}$ to $\{1,2,\cdots,n\}$ $\}$, and $|F| = n(n-1)\cdots(n-m+1)$. Let $X = \{x_1, x_2, \cdots, x_m\}$ and $Y = \{y_1, y_2, \cdots, y_m\}$.

If $a_{1f(1)}a_{2f(2)}\cdots a_{mf(m)} \neq 0$, then $a_{1f(1)}a_{2f(2)}\cdots a_{mf(m)} \neq 0$ denotes that the set of edges

$$E_G(x_1, y_{f(1)}) \neq \emptyset, \ E_G(x_2, y_{f(2)}) \neq \emptyset, \cdots, E_G(x_m, y_{f(m)}) \neq \emptyset.$$

Thus, $a_{1f(1)}a_{2f(2)}\cdots a_{mf(m)}$ is the number of the matchings that saturate $X$ and consist of edged in the set of edges

$$E_G(x_1, y_{f(1)}) \cup E_G(x_2, y_{f(2)}) \cup \cdots \cup E_G(x_m, y_{f(m)}).$$

It follows that the number of the matchings that saturate $X$ is equal to $\mathrm{Per}(\mathbf{A})$.

**Ex5.2.5** (a) By Theorem 5.6, it is sufficient to show $\alpha(G) \leq \kappa(G)$. By contradiction, if $\alpha(G) > \kappa(G) = k$, then there is an independent set $I$ such that $|I| = k+1$. A contradiction can be deduce as follows. $|(\bar{I}, I)| \leq k^2 < k(k+1) = |(I, \bar{I})|$.

(b) The proof is similar to one of Theorem 5.7. Since $\alpha(G) = 1$ if and only if $G \cong K_v$, assume $\alpha(G) \geq 2$. Choose a longest cycle $C$ in $G$. Suppose to the contrary that $C$ is not a Hamilton cycle. Then, $V(G) \setminus V(C) \neq \emptyset$ and $T \subseteq V(C)$. Choose $x \in V(G) \setminus V(C)$ and let $T = \{x_1, x_2, \cdots, x_s\}$ and occurs in $C$ in this order. By the choice of $C$, any successive two vertices in $\{x_1, x_2, \cdots, x_s\}$ are not adjacent in $C$, otherwise we can construct a longer cycle than $C$.

Specify $C$ a direction to obtained a directed cycle $\overrightarrow{C}$. Let $Y = \{y_i : \ (x_i, y_i) \in E(\overrightarrow{C}), i = 1, 2, \cdots, s\}$. Then $Y \cup \{x\}$ is an independent set of $G$ and $\alpha(G) \geq |I| = s + 1 \geq |T| + 1$, contradicting to the hypothesis.

**Ex5.2.6** The proof is similar to one of Theorem 5.7. Without loss of generality, assume $\alpha(G) \geq 2$. By contradiction, assume $\alpha \geq \delta(G) + 1$. Let $I$ and $S$ be a maximum independent set and a minimum separating set of $G$, respectively. Then, for any $x, y \in I$, we have $|N_G(x) \cup N_G(y)| \leq v - \alpha$, and

$$\begin{aligned} |N_G(x) \cap N_G(y)| = & \ d_G(x) + d_G(y) - |N_G(x) \cup N_G(y)| \\ \geq & \ 2\delta - (v - \alpha) \geq 3\delta - v + 1 \geq \kappa + 1 > |S|. \end{aligned}$$

This implies that only one of all components of $G - S$, say $G_1$, may contain vertices in $I$, that is, $I \subseteq V(G_1) \cup S$. Since $\alpha \geq 2\delta + 1$, there exists $x \in I \cap V(G_1)$. Choose a vertex $z$ in other component, say $G_2$, of $G - S$. Then

$$|N_G(x) \cup N_G(z)| \leq v - 2 - |I \cap V(G_1)| + 1 = v - \alpha + |I \cap S| - 1.$$

Since $N_G(x) \cap N_G(z) \subseteq S \setminus I$, thus, $|N_G(x) \cap N_G(z)| \leq \kappa - |I \cap S|$. Thus, we should have that

$$2\delta \leq d_G(x) + d_G(z) = v - \alpha + \kappa - 1 \leq v + \kappa - \delta - 2.$$

From this we can deduce a contradiction as follows. $\delta \leq \frac{1}{3}(v + \kappa - 2) < \frac{1}{3}(v + \kappa)$.

**Ex5.2.7** Note that this exercise has an erratum. The exercise is restated as follows. Let $G$ be a loopless digraph. Prove that $G$ contains an independent set $I$ such that $d_G(I, y) \leq 2$ for any $y \in V(G) \setminus I$, where $d_G(I, y) = \min\{d_G(x, y) : \ x \in I\}$.

By induction on $v \geq 1$. Assume the conclusion holds for any digraph with order less then $v$ and let $G$ be a digraph with order $v$. Arbitrarily choose $x \in V(G)$. Then by the induction hypothesis, in the subdigraph $G - (\{x\} \cup N_G^+(x))$, there is an independent set $I'$ such that $d_G(I', y) \leq 2$ for any $y \in V(G) \setminus I'$. If there is $u \in I'$ such that $x \in N_G^+(x)$, then $d_G(I', y) \leq 2$ for any $y \in V(G) \setminus I'$ clearly. If there is no such a vertex in $I'$, then $x$ is not adjacent with any vertex in $I'$. Thus, $I = I' \cup \{x\}$ is an independent set required.

# Hints to Exercises in Chapter 6

**Ex6.1.4** Let $\chi(G)=k$. Then $G$ has a $k$-coloring $\pi=(V_1,V_2,\cdots,V_k)$, where $V_i$ is an independent set of $G$ and, hence, $v_i=|V_i|\le\alpha$ for each $i=1,2,\cdots,k$.

(a) Since $v=v_1+v_2+\cdots+v_k\le k\,\alpha$, $\chi(G)=k\ge\left\lceil\dfrac{v}{\alpha}\right\rceil$.

Choose an independent set $I$ such that $|I|=\alpha$. Then $G-I$ is $(v-\alpha)$-colorable clearly. Thus, $\chi(G)=k\le v+1-\alpha$.

(b) Note that $(V_i,V_j)\ne\emptyset$ since $\chi(G)=k$. Thus, $\varepsilon\ge\frac{1}{2}\,k(k-1)$, which implies $k\le\frac{1}{2}+\sqrt{2\,\varepsilon+\frac{1}{4}}$ .

Using Example 1.2.2 and Langrange's method of minimum multipliers, we have that

$$2\varepsilon\le\varepsilon(T_{k,v})=\sum_{i=1}^{k}v_i(v-v_i)=v^2-\sum_{i=1}^{k}v_i^2\le v^2-k\left(\frac{v}{k}\right)^2=v^2-\frac{v^2}{k}.$$

This implies $k\ge\dfrac{v^2}{v^2-2\varepsilon}$ .

**Ex6.1.5** If $G$ contains no odd cycle, then $G$ is bipartite and, hence, $\chi(G)\le 2$. Assume now $G$ contains an odd cycle $C$. Then $G-C$ contains no odd cycle. Thus, $\chi(G)\le\chi(C)+\chi(G-C)\le 3+2=5$.

**Ex6.1.6** ($\Rightarrow$) Since $G$ is $k$-critical, $G$ is not an odd cycle. Also since $G$ is not complete, $k\le\Delta$ by Brooks' theorem. Choose $x\in V(G)$ such that $d_G(x)=\Delta$. By Corollary 6.1.1, $\delta\ge k-1$, and so we have

$$\begin{aligned}2\varepsilon=&\ \sum_{x\in V}d_G(x)=\Delta+\sum_{y\in V\setminus\{x\}}d_G(y)\ge\Delta+(v-1)\delta\\ \ge&\ \Delta+(v-1)(k-1)=(\Delta-k)+v(k-1)+1\\ \ge&\ v(k-1)+1.\end{aligned}$$

($\Leftarrow$) Let $G$ be a connected simple graph neither an odd cycle nor a complete graph, $H$ be a $\chi(G)$-critical subgraph of $G$.

Assume $H$ is an odd cycle. Since $G$ is not an odd cycle, $\chi(G)=\chi(H)=3\le\Delta(G)$.

Assume $H$ is a complete graph. Since $G$ is not a complete graph, $\chi(G)<\chi(G)+1\le\Delta(G)$.

Assume $H$ is neither an odd cycle nor a complete graph. Since $k\ge 4$, we have $v\Delta(G)\ge v\Delta(H)\ge 2\varepsilon(H)\ge v(\chi(G)-1)+1$, that is, $\Delta(G)\ge\chi(G)-1+\frac{1}{v}$, which implies $\chi(G)\le\Delta(G)$.

**Ex6.2.2** By contradiction. Suppose that $\chi(G)=\Delta$, and let $\pi'=(E_1,E_2,\cdots,E_\Delta)$ be a $\Delta$-edge-coloring of $G$. Since $|E_i|\le\alpha'(G)$ for each $i=1,2,\cdots,\Delta$, by the condition (a), we have can deduce a contradiction as follows.

$$\Delta\alpha'(G)<\varepsilon=\sum_{i=1}^{\Delta}|E_i|\le\Delta\alpha'(G).$$

Also since $|E_i| \le \left\lfloor \frac{v}{2} \right\rfloor$ for each $i = 1, 2, \cdots, \Delta$, by the condition (b), we have can deduce a contradiction as follows.

$$\Delta \left\lfloor \frac{v}{2} \right\rfloor < \varepsilon = \sum_{i=1}^{\Delta} |E_i| \le \Delta \left\lfloor \frac{v}{2} \right\rfloor .$$

(c) Since $G$ has odd order and is regular, we have $\varepsilon = \Delta \frac{v}{2} > \Delta \left\lfloor \frac{v}{2} \right\rfloor$. By (b), $G$ belongs to class two.

(d) If $G$ is odd order, the assertion holds by (b). We now assume $G$ is even order. Let $x$ be a cut-vertex of $G$. Then there are two subgraphs $H$ and $K$ such that $G = H \cup K$ and $V(H) \cap V(K) = \{x\}$. Without loss of generality, suppose that $H$ has odd order $h$. Then $1 \le d_H(x) \le \Delta(G)$, and so

$$\varepsilon(H) = \frac{1}{2}\left((h-1)\Delta + d_H(x)\right) > \frac{1}{2}(h-1)\Delta = \Delta \left\lfloor \frac{h}{2} \right\rfloor .$$

By (b), $H$ belongs to class two, and so does $G$.

**Ex6.2.5** By contradiction. Suppose that $\chi'(G) = k > \left\lfloor \frac{3}{2}\Delta \right\rfloor$. By removing sufficiently many edges from $G$ (if necessary), we may assume that $\chi'(G-e) = k-1$, for each edge $e$ of $G$. It follows from Theorem 6.3 that $k \le \Delta(G) + \mu(G)$, and so there must exist vertices $x$ and $y$ which are joined by at least $k - \Delta$ edges.

We now color all of the edges of $G$ except one of the edges joining $x$ and $y$; since $\chi'(G-e) = k-1$, this coloring can be done with $k-1$ colors. Now the number of colors missing from $x$ or $y$ (or both) can not exceed $(k-1) - (\mu - 1)$, which in turn can not exceed $\Delta$, since $k \le \Delta + \mu$. But the number of colors missing from $x$ is at least $(k-1) - (\Delta - 1) = k - \Delta$, and similarly the number of colors missing from $y$ is at least $k - \Delta$. It follows that the number of colors missing from both $x$ and $y$ is at least $(2k - \Delta) - \Delta$, which is positive since $k > \left\lfloor \frac{3}{2}\Delta \right\rfloor$. By assigning one of these missing colors to the un-colored edge joining $x$ and $y$, we have colored all of the edges of $G$ using only $k-1$ colors. thereby contradicting the fact that $\chi'(G) = k$.

**Ex6.2.6** Let $L$ be the line of a simple graph $G$. By Exercise 6.2.3, $\chi'(G) = \chi(L)$. Since $\Delta(G) \ge 3$, $L$ is nether an odd cycle nor a complete graph, and $\Delta(L) \le 2\Delta(G) - 2$. Thus, By Brooks' theorem, $\chi'(G) = \chi(L) \le \Delta(L) \le 2\Delta(G) - 2$.