Exercises on Chapter 1

1.1 Prove the inequalities (a) $\sim$ (c) in Theorem 1.2.2.

1.2 Prove Theorem 1.2.3 and Theorem 1.2.5.

1.3 Prove that $\gamma(G) \leq \lfloor \frac{n}{2} \rfloor$ and $\gamma \leq \alpha(G)$ for a connected graph G of order $n \geq 2$.

1.4 Count the number of vertices of a complete complete k-ary tree with height h.

1.5 Give an embedding of a complete binary tree with height two into a cube Q_3 with no slowdown.

1.6 Prove $d(G) \leq \frac{3n-\delta-3}{\delta+1}$ for a connected graph G with order n. (Soares [4])

1.7 Prove Theorem 1.4.4 by a self-contained language. (Ore [3])

1.8 Prove that the average distance of an undirected cycle C_n is equal to

$$\mu(C_n) = \begin{cases} \frac{n+1}{4}, & \text{if } n \text{ is odd;} \\ \frac{n^2}{4(n-1)}, & \text{if } n \text{ is even.} \end{cases}$$

1.9 Prove Theorem 1.4.5. Can you improve the two lower bounds?

1.10 Let G be a undirected connected graph and for a vertex x in G let

$$\sigma_x(G) = \sum_{y \in V \setminus \{x\}} d(G; x, y).$$

Prove that $\sigma_x(G) \leq \left\lfloor \frac{1}{4} v^2 \right\rfloor$ for any $x \in V$. (Plesnik [2])

1.11 Read the reference [1] and give a proof of Theorem 1.4.6.

1.12 Prove that if G is a k-connected graph of order n, then number of pairs of distinct vertices at distance at most two in G at least $\min\{2kn, n(n-1)\}$.

References

[1] Kouider, M., and Winkler, P., Mean distance and minimum degree. *Journal of Graph Theory*, **25** (1997), 95-99

[2] Plesnik, J., On the sum of all distances in a graph or digraph. *Journal of Graph Theory*, **8** (1984), 1-21

[3] Ore, O., Diameters in graphs. *Journal of Combinatorial Theory*, **5** (1968), 75-81

[4] Soares. J., Maximum diameter of regular digraphs. *Journal of Graph Theory*, **16** (5) (1992), 437-450