Exercises on Chapter 2

2.1 Line Graphical Method

2.1.1 Construct a digraph G to show that the term "strongly" in Theorem 2.1.2 (e) is necessary.

2.1.2 Prove Corollary of Theorem 2.1.4 and shows that it is false for undirected graphs.

2.1.3 Construct an undirected graph G to show that Theorem 2.1.5 is false.

2.1.4 Show that $\mathrm{Aut}(K_n)$ is not isomorphic to $\mathrm{Aut}(L(K_n))$ if and only if $n = 2$ or 4.

2.1.5 Show that the graph $K_5 - e$ (obtained by deleting any edge e from K_5) is not a line graph.

2.1.6 Show that $K_{1,3}$ is not an induced subgraph of a line graph.

2.1.7 Prove that any induced subgraph of a line graph is a line graph.

2.1.8 Show that $\mathrm{Aut}(G)$ and $\mathrm{Aut}(L(G))$ are isomorphic if $\delta(G) \geq 4$.

2.1.9 Show that if two trees have isomorphic line graphs, they are isomorphic.

2.1.10 A graph is *self-complementary* if it is isomorphic to its complement. Show that $L(K_{3,3})$ is self-complementary.

2.1.11 Let G be a digraph, L_1 a subgraph of $L(G)$ and $E_1 = V(L_1)$. Define $G_1 = G[E_1]$. Prove that if L_1 is a strongly connected subgraph of L with at least two vertices, then the subgraph $G_1 \subseteq G$ is strongly connected.

2.1.12 Prove or disprove that $\lambda(G) \geq \kappa(L(G))$.

2.2 Cayley Method

2.2.1 Construct a regular undirected graph or digraph that is not transitive.

2.2.2 Prove that a graph G of order n is vertex-transitive if and only if all subgraphs of order $n-1$ of G are isomorphic.

2.2.3 Prove that any edge-transitive undirected graph is edge-regular.

2.2.4 Prove Corollary 1, Corollary 2 and Corollary 3 of Theorem 2.2.3.

2.2.5 Let $L(G)$ be the line graph of a graph G. Prove or disprove the following statements.

(a) If G is edge-transitive, then $L(G)$ is vertex-transitive.

(b) If $L(G)$ is vertex-transitive, then G is edge-transitive.

2.2.6 Prove that Petersen graph is vertex-transitive, but not a Cayley graphs.

2.2.7 Let $\Gamma = Z_4 \times Z_2$ be a group and $S = \{10, 11, 20, 30, 31\}$. Prove that the Cayley graph $C_\Gamma(S)$ is isomorphic to $G(8; \pm\{1, 3, 4\})$.

2.2.8 Let Γ be the symmetric group on the set $X = \{1, 2, 3, 4\}$, $a = (12)$ and $b = (134)$ be two elements in Γ. Draw the Cayley $G = C_\Gamma(S)$, where $S = \{a, b, ab\}$.

2.2.9 There are some wrongs in the end on page 71. Find and correct them.

2.3 Cartesian Product Method

2.3.1 Construct two connected graphs G_1 and G_2 such that $\kappa(G_1 \times G_2) > \kappa(G_1) + \kappa(G_2)$.

2.3.2 Investigate the average distance of $G_1 \times G_2$ if the average distances of G_1 and G_2 are m_1 and m_2, respectively.

2.3.3 Prove that $G_1 \times G_2$ is bipartite if G_1 and G_2 are both bipartite.

2.3.4 There are some labelling wrongs in Figure 2.11. Find and correct them.

2.3.5 Prove or disprove that the cartesian product of two edge-transitive graphs is edge-transitive.

2.3.6 The lexicographic product of G_1 and G_2, denoted by $G_1[G_2]$, has $V(G_1) \times V(G_2)$ as its vertex-set, and $x = (x_1, x_2)$ is adjacent with $y = (y_1, y_2)$ whenever x_1 is adjacent with y_1 in G_1 or $x_1 = y_1$ and x_2 is adjacent with y_2 in G_2.

(a) Draw $C_n[K_2]$, $n \geq 4$.

(b) Prove or disprove that the lexicographic product of two vertex (resp. edge)-transitive graphs is vertex (resp. edge)-transitive; the lexicographic product of two Cayley graphs is a Cayley graph.

2.4 A Basic Problem in Optimal Design

2.4.1 Prove that $n(3, 3) = 20$, $n(4, 5) = 15$ and $n(5, 2) = 24$ and draw the maximum $(5, 2)$-graph.