Exercises on Chapter 3

3.1 Hypercube Networks

3.1.1 Prove directly that

(1) Q_n is both vertex-transitive and edge-transitive;

(2) $Q_n - (N_{Q_n}(x) \cup \{x\})$ is connected for any vertex x in Q_n.

3.1.2 Prove that the assertions (b) and (d) in Theorem 3.1.10.

3.1.3 Prove that for any two vertices x and y in Q_n with distance d there exists an xy-path of length l with $d \le l \le 2^n - 1$ such that l and d have the same parity. (Li *et al* [4])

3.1.4 Prove that (Kulasinghe and Bettayeb [3], Huang and Xu [2])

(**a**) CQ_n is not vertex-transitive for $n \ge 5$;

(b) CQ_n is vertex-transitive for $n \le 4$;

(c) CQ_n is not edge-transitive for $n \ge 3$;

(d) CQ_4 is a Cayley graph $C_\Gamma(S)$ and write out the group Γ and a generating set S.

3.1.5 Prove that for all $n \ge 2$ and all values of l with $4 \le l \le 2^n$, CQ_n contains cycles of length l. (Efe [1])

3.1.6 Prove that

(a) FQ_n is bipartite if and only if n is odd;

(b) FQ_n is a Cayley graph $C_\Gamma(S)$, write out the group Γ and a generating set S;

(c) $d(FQ_n) \le \lceil \frac{n}{2} \rceil$;

(d) FQ_n is $(n+1)$-regular and $(n+1)$-connected;

(**e**) FQ_n is edge-transitive.

3.2 De Bruijn Networks

3.2.1 An edge in $UB(d,n)$ is said to be singular if it corresponds a pair of symmetric edges in $B(d,n)$. Prove the following results.

(a) Directed distance between two end-vertices in different pairs of symmetric edges in $UB(d,n)$ is equal to either $n-1$ or n. Moreover, two end-vertices in different pairs of symmetric edges have no vertex in common if and only if $n \ge 2$.

(b) Let $e = xy$ be any singular edge in $UB(d,n)$. If $n \ge 2$, then there exist $2d-1$ internally disjoint xy-paths in $UB(d,n)$, one of length one, and $2d-2$ of length three except $UB(d,2)$, in which one of length one, two of length two, and $2d-4$ of length three.

(c) Let F be an edge-cut of $UB(d,n)$. If $n \geq 2$ and F contains a singular edge, then $|F| \geq 2d-1$.

(d) Let xy and uv be two distinct singular edges in $UB(d,n)(d \geq 2,\ n \geq 3)$. Then $|N(\{x,y\}) \cap N(\{u,v\})| = \begin{cases} 2 \text{ or } 0, & n=3; \\ 0, & n \geq 4. \end{cases}$

3.2.2 Let $\mathbf{A}$ be the adjacency matrix of the de Bruijn digraph $B(d,n)$ and $\mathbf{J}$ a square matrix all of whose entries are 1.

(a) Prove that $\mathbf{A}^n = \mathbf{J}$.

(b) Find all eigenvalues of $\mathbf{A}$.

3.3 Kautz Networks

3.3.1 Prove that the exercise 3.2.1 is valid for $UK(d,n)$.

3.3.2 Prove that if xy and uv are two nonadjacent edges in $K(d,n)$ $(d \geq 2, n \geq 2)$ and (x,y) is a symmetric edge, then, there exist $(2d-2)$ internally disjoint directed paths from $\{x,y\}$ to $\{u,v\}$ in $K(d,n)$.

References

[1] K. Efe, A variation on the hypercube with lower diameter. *IEEE Transactions on Computers*, **40**(11)(1991), 1312-1316.

[2] Huang Jia and Xu Jun-Ming, Multiply-twisted hypercube with four or less dimensions is vertex-transitive. A manuscript.

[3] P. Kulasinghe and S. Bettayeb, Multiply-twisted hypercube with five of More dimensions is not vertex-transitive, *Information Processing Letters*, **53**(1995), 33-36.

[4] L. K. Li, C. H. Tsai, J. M. Tan and L. H. Hsu, Bipanconnectivity and edge-fault-tolerant bipancyclicity of hypercubes, *Information Processing Letters* , **87** (2003), 107-110.