Exercises on Chapter 4

4.1 Routings in Interconnection Networks

4.1.1 Prove that $\tau(G) \le \tau(T)$ for any connected spanning subgraph T of a connected G.

4.1.2 Count the vertex-forwarding index and the edge-forwarding index of Petersen graph.

4.1.3 Count the vertex-forwarding and edge-forwarding indices of a path P_n and a star $K_{1,n-1}$ for $n \ge 3$.

4.1.4 Prove that if G is a 2-connected graph of order n then $\tau(G) \le \frac{1}{2}(n-2)(n-3)$ and this bound is best possible in view of $K_{2,n-2}$.

4.1.5 The symbols $\tau_{\delta,n}$ and $\pi_{\delta,n}$ denote the minimum of $\tau(G)$ and $\pi(G)$, respectively, taken over all graphs G of order n with minimum degree δ. Prove that $\tau_{\delta,n} = \left\lceil \frac{2(n-1-\delta)}{\delta} \right\rceil$ and $\pi_{\delta,n} = \left\lceil \frac{2(n-1)}{\delta} \right\rceil$ for any n and δ with $n > \delta \ge 1$.

4.1.5 Prove that for each $i = 1, 2$, if $G_i = (V_i, E_i)$ is a connected graph with $|V_i| = n_i$ and $|E_i| = \varepsilon_i$, then

$$\begin{aligned}
\tau(G_1 \times G_2) &\ge \frac{n_2}{n_1} \sum_{u_1 \in V_1} \left(\sum_{v_1 \in V_1 \setminus \{u_1\}} (d_{G_1}(u_1, v_1) - 1) \right) \\
&\quad + \frac{n_1}{n_2} \sum_{u_2 \in V_2} \left(\sum_{v_2 \in V_2 \setminus \{u_2\}} (d_{G_2}(u_2, v_2)) - 1) \right) + (n_1 - 1)(n_2 - 1) \\
\pi(G_1 \times G_2) &\ge \min \left\{ \frac{n_2}{\varepsilon_1} \sum_{(u_1, v_1) \in V_1 \times V_1} d_{G_1}(u_1, v_1),\ \frac{n_1}{\varepsilon_2} \sum_{(u_2, v_2) \in V_2 \times V_2} d_{G_2}(u_2, v_2) \right\}.
\end{aligned}$$

4.1.6 Prove that $\tau(G) \le (n-1)\lceil \frac{(n-k-1)}{k} \rceil$ if G is a k-connected undirected graph of order n, where $k \ge 1$.

4.2 Fault-Tolerant Diameter

4.2.1 Let $F(t, d)$ denote the minimum diameter of an altered graph obtained by adding t extra edges to a graph with diameter d. Prove that

(a) $F(t, d) \le F(t, d')$ and $g(t, d) \le g(t, d')$ for $d \le d'$;

(b) $F(t, g(t, d)) \le d \le g(t, F(t, d))$.

4.2.2 Let $P(t, d)$ denote the minimum diameter of an altered graph G obtained from a single path of diameter d plus t extra edges. Prove that

(a) $P(t, (2k-1)t + h + 1) \le 2k$ for any integers t, k and h with $0 \le h \le 2k - 1$;

(b) $P(4, 5(2k-1)+h) \leq 2k+1$ for any integers k and h with $2 \leq h \leq 5$;

(c) $P(5, 6(2k-1)+h) \leq 2k+1$ for any integers k and h with $2 \leq h \leq 6$.

4.2.3 Prove that $F(t,d) = P(d,t)$ for any positive integers t and d.

4.2.4 Prove that for given positive integers t and d,

$$\left\lceil \frac{d}{t+1} \right\rceil \leq P(t,d) \leq \left\lfloor \frac{d-2}{t+1} \right\rfloor + 3.$$

In particular, $P(t,(2k-1)(t+1)+1) = 2k$ for any positive integer k and $F(t,d) \leq \left\lfloor \frac{d}{t} \right\rfloor + 1$ if k is enough large.

4.2.5 Prove that for any integer $d\,(\geq 4)$,

$$\left\lceil \frac{d}{5} \right\rceil \leq P(4,d) \leq \left\lceil \frac{d}{5} \right\rceil + 1 \quad \text{and} \quad \left\lceil \frac{d}{6} \right\rceil \leq P(5,d) \leq \left\lceil \frac{d}{6} \right\rceil + 1.$$

4.3 Menger-Type Problems in Parallel Systems

4.3.1 Prove that

$$\begin{aligned} r_w(Q_n) = &\ D_w(Q_n) \\ = &\left\{ \begin{array}{ll} n, & \text{for } 1 \leq w \leq n-1; \\ n+1, & \text{for } w = n. \end{array} \right. \end{aligned}$$

4.3.2 Prove that

$$r_w(B(d,n)) = D_w(B(d,n)) = n+1, \text{ for } 1 \leq w \leq d-1.$$

4.4 Wide Diameter of Networks

4.4.1 Let x and y be two vertices in Q_n with distance l (> 0). Then, for any integer t with $0 \leq t \leq n-2$,

$$d_{n-t}(Q_n; x, y) = \left\{ \begin{array}{ll} l, & \text{if } n-t \leq l; \\ l+2, & \text{if } n-t > l. \end{array} \right.$$

4.5 (l,w)-Independence and -Dominating Numbers

4.5.1 Prove that (a) $\alpha_{2,n}(Q_n) = \left\{ \begin{array}{ll} 2, & \text{if } n = 2; \\ 2^n, & \text{if } n \geq 3; \end{array} \right.$

(b) $\alpha_{3,n}(Q_n) = 2^{n-1}$ if $n \geq 3$;

(c) $\alpha_{n,n}(Q_n) = \left\{ \begin{array}{ll} 4, & \text{if } n = 3; \\ 2, & \text{if } n > 3; \end{array} \right.$

(d) $\alpha_{n-1,n}(Q_n) = \left\{ \begin{array}{ll} 8, & \text{if } n = 3,\ 4; \\ 4, & \text{if } n = 5,\ 6; \\ 2, & \text{if } n \geq 7; \end{array} \right.$

(e) $\alpha_{d,n-t}(Q_n) = \alpha_{d,n}(Q_n)$, where $0 \leq t \leq n-2$ and $1 \leq d \leq n-t-1$.

4.6 Restricted Fault-Tolerance of Networks

4.6.1 Prove that if (x,y) is a symmetric edge in Kautz digraph $K(d,n)\,(d\geq 2,\ n\geq 2)$, then the set $E^+(\{x,y\})$ of out-edges from $\{x,y\}$ is a restricted edge-cut in $K(d,n)$.

4.6.2 Prove by using the exercises 2.3.1, 2.3.2 and the exercise 4.6.1 that (Fan and Xu [1])

(a) $\lambda'(K(d,n)) = \begin{cases} \text{not exist} & n=1,\ d=2; \\ 2d-2, & d\geq 3,\ n=1 \text{ or } d\geq 2,\ n\geq 2. \end{cases}$

(b) $4d-5\leq \lambda'(UK(d,n))\leq 4d-4,\quad d\geq 3,\ n\geq 3$

4.6.3 Let G be a non-optimal and vertex-transitive graph of degree k. Then $\lambda'(G)=k$ if and only if the induced subgraph $G[X]$ is a complete graph of order k for any λ'-atom X of G.

4.6.4 Let G be a non-optimal, k-regular and connected graph. If G contains a complete graph K_k, then $X=V(K_k)$ is a λ'-atom of G, and, hence $\lambda'(G)=k$.

4.6.5 Let G be a non-optimal and connected vertex-transitive graph with degree $k\,(\geq 3)$. Then $\lambda'(G)=k$ if and only if G contains a complete graph of order k.

4.6.6 Let G be a non-optimal and connected vertex-transitive graph. Then G has a prefect matching, and hence G has even order.

4.6.7 Complete the proof of Theorem 4.6.10 (b).

References

[1] Fan Ying-Mei and Xu Jun-Ming, Restricted edge-connectivity of Kautz graphs (in Chinese), to appear in *Applied Math* (2004) No.3.