The Final Exercises

1. Prove $d(G) \leq \frac{3n-\delta-3}{\delta+1}$ for a connected graph G with order n.

2. Prove that for a connected undirected graph G of order n,

$$\varepsilon(G) \leq k + \tfrac{1}{2}(n-k+4)(n-k-1),$$

where k is the diameter of G.

3. Let G be a undirected 2-connected graph and for a vertex x in G let

$$\sigma_x(G) = \sum_{y \in V \setminus \{x\}} d(G; x, y).$$

Prove that $\sigma_x(G) \leq \left\lfloor \frac{1}{4} v^2 \right\rfloor$ for any $x \in V$.

4. The lexicographic product of G_1 and G_2, denoted by $G_1[G_2]$, has $V(G_1) \times V(G_2)$ as its vertex-set, and $x = (x_1, x_2)$ is adjacent with $y = (y_1, y_2)$ whenever x_1 is adjacent with y_1 in G_1 or $x_1 = y_1$ and x_2 is adjacent with y_2 in G_2. Prove or disprove that the lexicographic product of two vertex (resp. edge)-transitive graphs is vertex (resp. edge)-transitive; the lexicographic product of two Cayley graphs is a Cayley graph.

5. Prove that for any two vertices x and y in Q_n with distance d there exists an xy-path of length l with $d \leq l \leq 2^n - 1$ such that l and d have the same parity.

6. Let $\mathbf{A}$ be the adjacency matrix of the de Bruijn digraph $B(d, n)$ and $\mathbf{J}$ a square matrix all of whose entries are 1.

(a) Prove that $\mathbf{A}^n = \mathbf{J}$.

(b) Find all eigenvalues of $\mathbf{A}$.

7. Prove that if G is a 2-connected graph of order n then the forwarding index $\tau(G) \leq \frac{1}{2}(n-2)(n-3)$ and this bound is best possible in view of $K_{2,n-2}$.

8. Let $F(t, d)$ denote the minimum diameter of an altered graph obtained by adding t extra edges to a graph with diameter d. Prove that

(a) $F(t, d) \leq F(t, d')$ and $g(t, d) \leq g(t, d')$ for $d \leq d'$;

(b) $F(t, g(t, d)) \leq d \leq g(t, F(t, d))$.

9. Prove that for any graph G,

(a) $\zeta_l(G) = w \Leftrightarrow d_w(G) \leq l < d_{w+1}(G)$ if G is $w+1$-connected, or

(b) $d_w(G) = l \Leftrightarrow \zeta_{l-1}(G) < w \leq \zeta_l(G)$ if G is w-connected.

10. Let G be a λ'-nonoptimal and vertex-transitive graph of degree k. Then $\lambda'(G) = k$ if and only if the induced subgraph $G[X]$ is a complete graph of order k for any λ'-atom X of G.