Available online at www.sciencedirect.com
SCIENCE DIRECT®
Discrete Mathematics 302 (2005) 22–31
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Variations on a theme of Kuratowski

Dan Archdeacon
Department of Mathematics and Statistics, University of Vermont, Burlington, VT 05401-1455, USA

Received 7 December 2001; received in revised form 15 October 2002; accepted 22 July 2004

Abstract

Kuratowski's Theorem characterizes planar graphs in terms of two excluded subgraphs. In this paper we survey variations of Kuratowski's Theorem. We examine both finite and infinite graphs, surfaces and pseudosurfaces, and generalizations of outer-planarity.

Keywords: Obstructions; Forbidden subgraphs; Planarity; Kuratowski's Theorem

1. Introduction

The most oft-cited result in graph theory [10] is Kuratowski's Theorem: *A graph is planar if and only if it does not contain a subdivision of* K_5 *or of* $K_{3,3}$. These graphs are shown in Fig. 1. Kuratowski's original proof is given in [21]. Other nice proofs are given in [17,18,22].

There are many possible variations on this basic theorem. For example, what if we allow infinite graphs? What if we embed on surfaces other than the plane? What if we consider only embeddings with special properties, such as having all vertices on the boundary of a distinguished face? What if we consider other partial orderings on graphs?

In this paper we examine such variations of Kuratowski's Theorem. We will give the basic theory and survey some recent results. Our survey is not intended to be complete, rather to give a flavor of recent results. In Section 2 we begin with a study of partial orders and obstruction theorems. In Section 3 we study planar embeddings with special properties. In Section 4 we study analogs of Kuratowski's Theorem for other surfaces. We turn our

E-mail address: dan.archdeacon@uvm.edu.

doi:10.1016/j.disc.2004.07.026

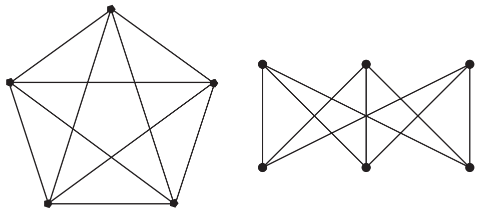

Fig. 1. The Kuratowski graphs K_5 and $K_{3,3}$.

attention to infinite graph embeddings and variations on Halin's Theorem in Section 5. Finally, we present some open problems in Section 6.

2. Partial orders and obstruction theorems

Kuratowski's Theorem can be viewed in terms of partial orders and hereditary properties. We now describe these concepts, using obstructions to planarity to illustrate them.

A *graph property* $\mathscr{P}$ is a collection of (isomorphism types of) graphs. We frequently say that *G has property* $\mathscr{P}$ in place of $G \in \mathscr{P}$. For example, we could let $\mathscr{P}$ be the class of all planar graphs.

Let $\mathscr{O}$ be a partial order on the set of all graphs. For example, this order may be the *subgraph order* formed by deleting vertices and edges. We will write $H \leqslant G$ for this order, with $H < G$ to also denote $H \neq G$. A property $\mathscr{P}$ is *hereditary* for $\mathscr{O}$ if $G \in \mathscr{P}$ and $H \leqslant G$ implies that $H \in P$. For example, any subgraph of a planar graph is planar, so planarity is hereditary under the subgraph order. Hereditary properties are also called *lower ideals* for the order.

Our goal is to find the minimal graphs without a given hereditary property, that is, we want to find graphs $G \notin \mathscr{P}$, but for any $H < G$, $H \in \mathscr{P}$. It is tempting to say that any graph G not in $\mathscr{P}$ must contain some minimal graph not $\mathscr{P}$, but in general this is not the case. There may be an infinite descending chain of graph $G_1 > G_2 > G_3 > \cdots$ of graphs all not in $\mathscr{P}$. In practice this usually is no problem, for our orders will always decrease the number of vertices or edges in the graph. When the graphs are finite, this prohibits infinite descending chains. Orders without infinite descending chains are called *Noetherian*.

Consider the (common) case that every graph not in $\mathscr{P}$ must contain a minimal graph not in $\mathscr{P}$. Let $\mathrm{Obs}(\mathscr{P}, \mathscr{O})$ denote the set of all graphs that are minimal in order $\mathscr{O}$ without property $\mathscr{P}$. We call this the *obstruction set* for this property under the partial order. We will use the notation $\mathrm{Obs}(\mathscr{P})$ when the order is understood from context. We have the following general lemma.

Lemma 2.1. *A graph G has property $\mathscr{P}$ if and only if there does not exist an $H \in \mathrm{Obs}(\mathscr{P})$ with $H \leqslant G$.*

We now turn to other partial orders on graphs. We say that a graph G is a *subdivision* of H if we can form G by deleting an edge e of H, adding in a new vertex v, and edges joining v to each of the two old vertices incident with e. The reader is invited to picture placing the new vertex in the middle of the old edge. We also say that we can get H from

G by *smoothing* the degree-two vertex v. The *topological order* on graphs has $G \geqslant H$ if we can create H from G by a sequence of edge deletions, vertex deletions, and smoothing degree-two vertices. Graphs with a common subdivision are called *homeomorphic*. Hence in the topological order, $G \geqslant H$ if and only if G contains a subgraph homeomorphic to H.

Subdividing an edge does not change the planarity of a graph, that is, if G is planar and H is formed by smoothing a degree-two vertex, then H is also planar. Thus, planarity is hereditary under the topological order. In the subgraph order there are infinitely many obstructions to planarity: any subdivision of K_5 or of $K_{3,3}$ is such an obstruction. In contrast, Kuratowski's Theorem asserts that there are exactly two obstructions to planarity under the topological order.

The topological order is especially important for the class of all cubic graphs. We cannot delete edges in a cubic graph without violating the property that every vertex is of degree 3. However, we can delete an edge and then smooth the resulting degree-two vertices. The *cubic order* is the topological order on the class of cubic graphs.

The preceeding discussion illustrates a recurring idea in relating partial orders and obstructions. We describe our partial orders by a set of elementary reductions (such deleting a single edge or vertex) and then extend it to a partial order by transitive closure. Suppose that we add another elementary reduction (such as smoothing a degree-two vertex). We first have to check that the property is hereditary under this added reduction. If so, then we can relate the obstructions for a property under the two orders. We make this more formal in the following lemma.

Lemma 2.2. *Suppose that $\mathscr{P}$ is a hereditary property under both partial orders $\mathscr{O}$ and $\mathscr{O}'$. Suppose that $\mathscr{O} \subseteq \mathscr{O}'$, that is, $\mathscr{O}'$ is finer than O. Then* $\mathrm{Obs}(\mathscr{P}, \mathscr{O}') \subseteq \mathrm{Obs}(\mathscr{P}, \mathscr{O})$.

We continue our study of different partial orders. Let e be an edge of G joining vertices u, v. Make a new graph H by deleting u, v and all incident edges, adding a new vertex x, and an edge wx for each wu and each wv in $E(G)$. We say that H is formed by *contracting* the edge e. The reader is invited to picture gradually making e shorter and shorter until its two ends merge into a single vertex.

The *minor order* on graphs is generated by the two subgraph operations and the contraction of edges. Notice that smoothing a degree-two vertex is the same as contracting one of its incident edges. If G is planar and H is an edge contraction of G, then H is planar. The converse is not necessarily true: unlike subdivisions, we can have a non-planar G but a planar contraction H. One example is contracting an edge in $K_{3,3}$.

The planarity property is hereditary under the minor order. What are the obstructions to planarity under the minor order? They are again K_5 and $K_{3,3}$. This result is commonly known as Wagner's Theorem [28].

In a remarkable sequence of papers, Robertson and Seymour have shown that in any infinite set of finite graphs one must be a minor of another [24]. This implies:

Theorem 2.1. *Let $\mathscr{P}$ be a property of finite graphs that is hereditary under the minor order. Then* $\mathrm{Obs}(\mathscr{P})$ *is finite*.

The above theorem implies that under the given conditions $\mathrm{Obs}(\mathscr{P})$ under the topological order is also finite.

Suppose that we are given a graph G with a cubic vertex v adjacent to vertices a, b, c. Form H from G by deleting v and its incident edges and adding in new edges ab, bc, ca. The deleted edges formed a *wye*, or $K_{1,3}$, and the new edges form a *Delta*, or K_3. We say that H is formed from G by a *$Y\Delta$-transformation*.

The *$Y\Delta$-ordering* is formed (under transitive closure) from the minor ordering by adding in relations $G \geqslant H$ if H is a $Y\Delta$-transformation of G. If G is planar, then so is H. The converse need not apply; a $Y\Delta$-transform of $K_{3,3}$ is the planar graph $K_5 - K_2$. What are the obstructions to planarity under the $Y\Delta$-ordering? Again, they are the two Kuratowski graphs.

There is one more operation and its corresponding order to examine. Suppose that u, v are two adjacent degree-three vertices in a graph G. Let a, b be the other neighbors of u, and c, d be the other neighbors of v. Create H by deleting u, v and their incident edges, adding a new vertex x, and edges xa, xb, ab and xc, xd, cd. The deleted edges formed an H, the added edges form a *bowtie* ($\bowtie$), so we call this the $H\bowtie$-transformation. This transformation can also be described as first subdividing the edge joining two degree-three vertices, then making two $Y\Delta$-transformations. The *$H\bowtie$-ordering* is formed by adding the $H\bowtie$-transformation to the $Y\Delta$-ordering.

Again, planarity is a hereditary property under the $H\bowtie$-ordering. Now, however, we have a reduction in the size of the obstruction set for planar graphs. Applying an $H\bowtie$-transformation to $K_{3,3}$ gives K_5. Hence there is just a single graph, K_5, in the obstruction set for planarity under the $H\bowtie$-ordering.

There are many other graph operations. Depending on the particular property studied, it may or may not be hereditary under a partial order formed by these operations. The ones presented here are among the most common, especially for topological properties.

We close with the observation that if we can find $\mathrm{Obs}(\mathscr{P})$ under any one of the topological, minor, $Y\Delta$, or $H\bowtie$ orders, then we can find the obstructions under the other three orders. In particular, if the property $\mathscr{P}$ is hereditary under the minor order, then the minor-obstruction set is finite and hence so is each of the other obstruction sets.

3. Planarity with restrictions

Our first variation of Kuratowski's Theorem is based on restrictions of "planarity". The most famous of these are based on the idea of outer-planarity. A graph is *outerplanar* if it embeds in the plane so that all vertices lie on the boundary of one distinguished face. Traditionally, this face is taken to be the outside, or unbounded face.

Theorem 3.1 (*Chartrand and Harary [16]*). *A graph is outer-planar if and only if it does not contain a subdivision of K_4 or of $K_{2,3}$.*

It is easy to see that the two graphs cited are not outer-planar; the difficulty, as usual, arises from showing that these are the only two minimal non-outer-planar graphs under the topological order.

Consider the graph property that "*A graph G embeds in the plane such that there are two faces with every vertex incident with at least one of them*". Such small face covers were first considered in [11]. If the two distinguished faces are vertex disjoint, then deleting their interiors form a closed *cylinder*, or homeomorphically an *annulus*. The above property corresponds to embedding on an annulus such that every vertex lies on the boundary. If the faces are not disjoint, then deleting their interiors forms a *pinched cylinder*. In either case, we refer to the above property as being *outer-cylindrical*.

We invite the reader to prove that being outer-cylindrical is hereditary under all of the orders described in Section 2. Hence it makes sense to talk of obstructions to being outer-cylindrical. These obstructions are known.

Theorem 3.2 (*Archdeacon et al. [6], Cáceres [15], Revuelta [23] and Scott [25]*). *A graph is outer-cylindrical if and only if it does not contain a subdivision of one of* 56 *graphs. A graph is outer-cylindrical if and only if it does not contain a minor of one of* 38 *graphs. It is outer-cylindrical if and only if it does not contain a subgraph isomorphic to one of* 17 *graphs under the* $Y\Delta$*-ordering.*

Precise descriptions of these obstruction sets are available in [3].

A natural further generalization would be to characterize the graphs that have an embedding with all vertices on the boundary of *three* distinguished faces. Such graphs are called *outer-pants* graphs, because if the boundary of the faces are pairwise disjoint, then the surface resembles a pair of pants (the three boundary regions form the waist and the two leg cuffs of the pants). However, the obstruction set for these graphs is unknown, even for cubic graphs. It is known that there is a unique non-outer-pants graphs of order 8 [6].

There is another generalization of outer-planar graphs. A face *sees* each vertex it is incident with, and vice versa. A face 2-*sees* another face if both are incident with a common vertex, and a vertex 2-*sees* another vertex if they are incident with a common face. We extend this definition as follows; a face *k-sees* another face or vertex if there is a sequence $x_1, \ldots, x_k$ such that each x_i is incident with x_{i-1}, $i = 1, \ldots, k-1$. The reader is invited to show that both the properties "*G has an embedding such that some vertex* (*face*) *k-sees all other vertices* (*faces*)", and "*G has an embedding such that some vertex* (*face*) *k-sees all faces* (*vertices*)" are hereditary under each of the orders described in Section 2. Hence we look for obstructions for these properties.

These obstruction sets are known for small values of *k*. They were studied in [9] under the guise of nesting points in the sphere. They also consider fixed embeddings of graphs.

Theorem 3.3. *There are exactly* 2 *minor-minimal planar graphs such that no planar embedding has a face seeing all faces. There are exactly* 3 *minor-minimal planar graphs such that no planar embedding has a vertex seeing all faces. There are exactly* 9 *minor-minimal planar maps such that no planar embedding has a face that* 2*-sees all other faces.*

The first two results are not quite dual to each other; one of the graphs in the second result is disconnected.

4. Other surfaces

In this section, we expand planar embeddings (or equivalently spherical embeddings) to other surfaces. A *surface* is a Hausdorff topological space such that every point has a neighborhood homeomorphic to the real Euclidean plane $\mathscr{R}^2$. The compact surfaces fall into two infinite classes: the sphere with h handles attached, called the *orientable surface of genus h*, and the sphere with k crosscaps attached, called the *non-orientable surface of genus k*. One class of non-compact surfaces arise from deleting a finite set of points from a compact surface. For example, deleting one point from the sphere creates the Euclidean plane. For a complete classification of non-compact surfaces see [22].

Let S be an orientable or non-orientable surface and let $\mathscr{P}$ be the property that *G embeds on S*. Again, it is easy to see that this property is hereditary under each of the orders described in Section 2. What are their obstruction sets?

The complete obstruction set is known only for the projective plane, that is for graphs of non-orientable genus one. This is the excluded subgraph theorem for projective-planar graphs. These graphs were first exhibited in [19]; Archdeacon [1] announces that the set is complete as proven in [2].

Theorem 4.1. *There are exactly* 103 *obstructions to embedding in the projective plane under the topological order. There are exactly* 35 *obstructions under the minor order. There are exactly* 14 *obstructions under the* $Y\Delta$ *order*.

This set of graphs is available in [3].

We now vary the concept of surface. A *pseudosurface*, or *pinched surface*, is the quotient space of a surface under an equivalence relation on the points, where there are a finite number of non-trivial equivalence classes and each class contains a finite number of points. The classic example is the *spindle surface* (misnamed, because it is not really a surface), formed from the sphere by identifying two different points commonly referred to as the *north pole* and the *south pole*. Another common example is the 2-*banana* surface, formed by distinguishing two points (the north and south poles) of two spheres, and identifying the two different north poles with a single point and the two different south poles with a second point. The resulting surface (again misnamed) resembles two bananas joined at their respective stems and base points.

We examine obstruction theorems for embeddings on pseudo-surfaces. There is one main positive result and one main negative result.

Theorem 4.2 (*Archdeacon and Bonnington [4]*). *There are exactly* 21 *minimal graphs under the cubic ordering that do not embed on the spindle surface*.

Theorem 4.3 (*Širáň and Gvozdjak [26]*). *There are infinitely many minor-minimal graphs that do not embed on the* 2-*banana surface*.

One infinite class of graphs in Theorem 4.3 are the line graphs of the Möbius ladders. This class is also described by taking a $2n$-cycle on vertices $1, \ldots, 2n$, and for $i = 1, \ldots, n$, adding vertices x_i adjacent to $i, i+1, i+n$, and $i+n+1$.

At first glance Theorem 4.3 seems to contradict the Robertson–Seymour proof. It does not, as the property of embedding on the 2-banana surface is not hereditary under minors, in particular, it is not hereditary under edge contractions.

We now combine projective-planarity and outer-planarity. A graph is *outer-projective-planar* if it embeds on the projective plane with all vertices on the boundary of a single distinguished face.

Theorem 4.4 (*Archdeacon et al.* [8], *Cáceres* [15] *and Revuelta* [23]). *There are exactly* 45 *topologically minimal non-outer-projective-planar graphs. There are exactly* 32 *such minor-minimal graphs, and exactly* 9 *under the* $Y\Delta$ *ordering*.

Similarly, a graph to be *outer* 2-*banana* if there is an embedding of the graph on the 2-banana surface with every vertex on the boundary of a single distinguished face. This property is now hereditary under the minor order [13].

Theorem 4.5 (*Boza et al.* [13]). *There are exactly* 38 *minor-minimal non-outer* 2-*banana graphs*.

We return to embeddings on the projective plane. A *signed graph* is a graph together with a signature $+$ or $-$ on each edge. A cycle is *balanced* in a signed graph if and only if it has an even number of negative edges. A *signed embedding* of a signed graph is an embedding on a surface such that a cycle is orientation-reversing if and only if it is non-balanced. This surface will be orientable if and only if every cycle is balanced. In the projective plane, this corresponds to a cycle being non-contractable if and only if it is non-balanced. The theory of minors extends to signed graphs (see [29]) and look for the obstruction set to signed obstructions for projective planar graphs.

Theorem 4.6 (*Zaslavsky* [29]). *There are exactly eight minor-minimal signed graphs that do not have a signed embedding on the projective plane*.

5. Infinite graphs

We turn our attention to obstructions to embeddings of infinite graphs. These graph embeddings present some different problems. First, the number of points on any surface is the continuium: the cardinality of the real line. Thus, if an infinite graph contains more than a continuium number of points, then it cannot embed on any surface.

We are primarily interested in embeddings that are *vertex-accumulation-point free*, that is, those where the subset in the (non-compact) surface of vertex points does not have an accumulation point. This class of graphs are the most natural, as explained in [27]. Such graphs are necessarily *locally finite*, that is, that each vertex has a finite degree, and they have a countable number of components. These implies that the vertex set of these graphs is countable.

A classical result, usually attributed to Erdös (see e.g. [27]), is that a graph on a countable vertex set embeds on a (pseudo)-surface if and only if every finite subgraph embeds on that surface. Let H be any finite graph. We call a graph G *residually finite* if G is created from H by adding a finite number of infinite-one-ended-rays each rooted at a vertex of H. A nice theorem [12] shows the following for surfaces; the techniques therein extend easily to pseudo-surfaces. For related work see [14,15].

Theorem 5.1. *If G is an obstruction to embedding in a pseudo-surface with a finite subset of points removed, then G is residually finite.*

The first generalization is to examine connected, locally finite graphs that embed in the plane without accumulation points. Such graphs are characterized by Halin's Theorem.

Theorem 5.2 (*Halin [20]*). *A possibly infinite graph embeds in the plane without accumulation points if and only if it does not contain a topological subgraph shown in Fig*. 2.

An equivalent form of Halin's Theorem is to examine which locally finite countable graphs embed on the sphere with just a single accumulation point. We examine variations on Halin's Theorem.

The first generalization of Halin's Theorem would be to examine obstruction sets to the property "*A graph embeds on the plane with at most two accumulation points*". The general obstruction set is not known, see Problem 6.6. However, we do have the following positive result for cubic graphs.

Theorem 5.3 (*Archdeacon et al. [7]*). *There are exactly* 29 *cubic graphs that do not embed in a sphere with exactly two accumulation points, but such that every proper subgraph does so embed*.

We turn our attention to Halin's Theorem for the open Möbius band, that is, find the obstructions to embedding on a Möbius band without accumulation points. The one-point compactification of the Möbius band is the projective plane. So this problem is equivalent to embedding graphs in the projective plane with at most one accumulation point.

Theorem 5.4 (*Archdeacon et al. [5]*). *There are exactly* 350 *minor-minimal graphs that do not embed in the projective plane with at most one accumulation point. There are* 1235 *such topologically minimal graphs*.

Halin's graphs and Kuratowski's graphs are closely related. The former come from the latter by deleting either an edge or a vertex in the graph, and adding one-way-infinite rays to each incident or adjacent vertex. The relation between the projective-planar obstructions and the graphs of Theorem 5.4 is similar.

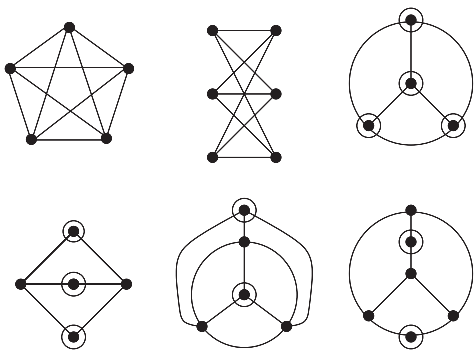

Fig. 2. The Halin graphs (add infinite rays to the circled vertices).

6. Conclusion

We end this paper with a collection of open problems. The first and perhaps the most important problem is the following.

Problem 6.1. Find the obstruction set under the minor order for embedding on a torus.

It may be easier to consider only cubic graphs, as in the following problem.

Problem 6.2. Find the obstruction set under the cubic order for embedding on a torus.

The author believes that Problem 6.2 may be within reach using techniques similar to those in [4]. We also ask the analogous questions for the Klein bottle, although we suspect this may be more difficult than the torus.

The next two problems involve generalizing outer-planar graphs. As before, the second is the restriction of the first to cubic graphs.

Problem 6.3. Find the obstruction set under the minor order for embedding on the sphere with every vertex on the boundary of one of three distinct faces.

Problem 6.4. Find the obstruction set under the cubic order for embedding on a sphere with every vertex on the boundary of one of three distinct faces.

We next consider embedding graphs on a pseudo-surface and ask:

Problem 6.5. Find the obstruction set under the minor order for embedding on the spindle surface for (non-cubic) graphs.

The next open problem is about embedding infinite graphs without accumulation points.

Problem 6.6. Find the obstruction set under the minor order for infinite graphs to embed in the sphere with exactly two accumulation points.

There are many other ways to combine the variations presented herein. In addition, we have not discussed planar two-dimensional simplicial complexes, directed graphs, hypergraphs, and so forth. These combinations yield a wealth of interesting problems that I hope the reader will enjoy.

Acknowledgements

The author would like to thank all co-authors of [4–9] for their fundamental contributions to understanding the nature of obstructions sets. He especially thanks C. Paul Bonnington and Marisa Debowsky for their helpful comments on an earlier version of this paper.

References

[1] D. Archdeacon, A Kuratowski theorem for the projective plane, J. Graph Theory 5 (1981) 243–246.
[2] D. Archdeacon, A Kuratowski Theorem for the projective plane, Ph.D. Thesis, The Ohio State University, 1980.
[3] D. Archdeacon, A gallery of graphs, emphasizing obstruction sets, http://www.emba.uvm.edu∼archdeac/graphs/.
[4] D. Archdeacon, C.P. Bonnington, Obstructions for embedding cubic graphs on the spindle surface, J. Combin. Theory. Ser. B 91 (2) (2004) 229–252.
[5] D. Archdeacon, C.P. Bonnington, M. Debowsky, M. Prestridge, Halin's theorem for the Möbius strip, Ars Combin. 68 (2003) 243–256.
[6] D. Archdeacon, C.P. Bonnington, N. Dean, N. Hartsfield, K. Scott, Obstruction sets for outer-cylindrical graphs, J. Graph Theory 38 (1) (2001) 42–64.
[7] D. Archdeacon, C.P. Bonnington, J. Širáň, Halin's Theorem for cubic graphs on an annulus, Discrete Math. 281 (1–3) (2004) 13–25.
[8] D. Archdeacon, N. Hartsfield, C.H.C. Little, B. Mohar, Obstruction sets for outer-projective-planar graphs, Ars Combin. 49 (1998) 113–127.
[9] D. Archdeacon, F. Sagols, Nesting points in the sphere, Discrete Math. 244 (1–3) (2002) 5–16.
[10] G. Berman, Frequently cited publications in pure graph theory, J. Graph Theory 1 (1977) 175–180.
[11] D. Bienstock, N. Dean, On obstructions to small face covers in planar graphs, J. Combin. Theory Ser. B 55 (1992) 163–189.
[12] C.P. Bonnington, R.B. Richter, Graphs embedded in the plane with a bounded number of accumulation points, J. Graph Theory 44 (2) (2003) 132–147.
[13] L. Boza, E. Fedriani, J. Núñez, Obstruction sets for outer-bananas-surface graphs, Ars Combin. 73 (2004) 65–77.
[14] L. Boza, M. Gugúndez, A. Márquez, P. Revuelta, Embedding graphs without accumulation points in generalized cylinders, manuscript (2001).
[15] J. Cáceres, Diversos tipos de planaridad de grafos, Tesis doctoral, Universidad de Almería, 1996.
[16] G. Chartrand, F. Harary, Planar permutation graphs, Ann Inst Henri Poincaré Sec B 3 (1967) 433–438.
[17] G.A. Dirac, S. Schuster, A theorem of Kuratowski, Indag. Math. 16 (1954) 343–348.
[18] G.A. Dirac, S. Schuster, Corrigendeum: a theorem of Kuratowski, Indag. Math. 23 (1961) 360.
[19] H. Glover, J.P. Huneke, C.S. Wang, 103 graphs that are irreducible for the projective plane, J. Combin. Theory Ser. B 27 (3) (1979) 332–370.
[20] R. Halin, Zur häufungspunktfreien Darstellung abzählbarer Graphen in der Ebene, Arch. Math. (Basel) 17 (1966) 239–243.
[21] K. Kuratowski, Sur le probléme des courbes gauches en topologie, Fund. Math. 5 (1930) 271–283.
[22] B. Mohar, C. Thomassen, Graphs on Surfaces, Johns Hopkins University Press, Baltimore, MD, 2001.
[23] M.P. Revuelta, Immersiones de grafos en superficies tubulares de género finito, Ph.D. Dissertation, Dpto. de Matemática Aplicada I. Univ. de Sevilla, 1999.
[24] N. Robertson, P.D. Seymour, Graph Minors XX: Wagner's conjecture, J. Combin. Theory Ser. B 92 (2) (2004) 325–357.
[25] K. Scott, Outermobius and cylindrical graphs, Senior Thesis, Princeton University, 1997.
[26] J. Širáň, P. Gvozdjak, Kuratowski-type theorems do not extend to pseudosurfaces, J. Combin. Theory Ser. B 54 (1992) 209–212.
[27] C. Thomassen, Infinite graphs, in: L.W. Beineke, R.J. Wilson (Eds.), Selected Topics in Graph Theory 2, Academic Press, New York, 1983, pp. 129–160.
[28] K. Wagner, Über eine Eigenschaft der ebenen Komplexe, Math. Ann. 114 (1937) 570–590.
[29] T. Zaslavsky, The projective-planar signed graphs, Discrete Math. 113 (1993) 233–247.