The Bondage Number of Graphs with Crossing Number Less than Four *

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Abstract

The bondage number $b(G)$ of a graph G is the smallest number of edges whose removal results in a graph with domination number greater than the domination number of G. Kang and Yuan [Bondage number of planar graphs. Discrete Math. 222 (2000), 191-198] proved $b(G) \leqslant \min\{8, \Delta+2\}$ for every connected planar graph G, where Δ is the maximum degree of G. Later Carlson and Develin [On the bondage number of planar and directed graphs. Discrete Math. 306 (8-9) (2006), 820-826] presented a method to give a short proof for this result. This paper applies this technique to generalize the result of Kang and Yuan to any connected graph with crossing number less than four.

Keywords: Bondage number; Domination number; Crossing number; Planar graphs

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1 Introduction

Let $G=(V,E)$ be a finite, undirected and simple graph. For a vertex $x \in V(G)$ and a subset $X \subset V(G)$, let $N_G(x) = \{y \in V(G) : \; xy \in E(G)\}$ and $N_G(X) = \{y \in V(G) \setminus X : \; xy \in E(G)\}$. We denote the degree of x by $d_G(x) = |N_G(x)|$, the maximum and the minimum degree of G by $\Delta(G)$ and $\delta(G)$, respectively.

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The *crossing number* of G, denoted by $cr(G)$, is the smallest number of pairwise intersections of its edges when G is drawn in the plane. If $cr(G) = 0$, then G is a planar graph.

A subset D of $V(G)$ is called a *dominating set*, if $D \cup N_G(D) = V(G)$. The minimum cardinality of all dominating sets in G is called the *domination number* of G, and denoted by $\gamma(G)$. The *bondage number* of a nonempty graph G, denoted by $b(G)$, is the smallest number of edges whose removal from G results in a graph with domination number strictly larger than $\gamma(G)$.

The first result on bondage numbers was obtained by Bauer et al. [1] in 1983. Fink et al. [5] first introduced the concept of the bondage number in 1990, and conjectured that $b(G) \leqslant \Delta(G) + 1$ for a nonempty graph G. Later, Teschner [10] found a counterexample to this conjecture. Then Dunbar et al. [4] proposed the following conjecture.

Conjecture 1.1 [4] *If G is a planar graph, then $b(G) \leqslant \Delta(G) + 1$.*

In 2000, Kang and Yuan [9] confirmed this conjecture for $\Delta(G) \geqslant 7$ by proving that $b(G) \leqslant 8$, and proved $b(G) \leqslant \Delta(G) + 2$ for any planar graph G. Later, Carlson and Develin [2] presented a method to give short proofs of these results. In this paper, we generalize their elegant technique to more general graphs and show that $b(G) \leqslant \Delta(G) + 2$ for any connected graph G with $cr(G) \leqslant 3$. Combining this with our result in [8] (see Lemma 3.2 below), we obtain that $b(G) \leqslant \min\{8, \Delta(G) + 2\}$ for any connected graph G with $cr(G) \leqslant 3$, which generalizes the result of Kang and Yuan.

The rest of the paper is organized as follows. Section 2 presents some necessary lemmas. Our main results and their proofs are in Section 3.

2 Some Lemmas

We begin this section with some necessary definitions and notations. For terminology and notation on graph theory not given here, the reader is referred to any standard textbook on graph theory, for example, [3] or [14].

Let $G = (V, E)$ be a graph. Use $\upsilon(G)$ and $\varepsilon(G)$ to denote the numbers of vertices and edges in G, respectively. The *distance* between two vertices x and y of G, denoted by $d_G(x, y)$, is the length of a shortest xy-path in G. The parameter $\omega(G)$ is the *number of connected components* of G. If $\omega(G) = 1$, then G is called to be *connected*; otherwise *disconnected*. The *edge connectivity* of a connected graph G, denoted by $\lambda(G)$, is the minimum number of edges whose deletion results in a disconnected graph. Let $V_i(G)$ denote the set of vertices of degree i in G and let $\upsilon_i(G) = |V_i(G)|$ for $i = 1, 2, \ldots, \Delta(G)$.

For a planar graph G, the following inequality is well known (see, for example, Corollary 3.4.1 in [14])

$$\varepsilon(G) \leqslant 3\upsilon(G) - 6. \tag{2.1}$$

Let G be a graph. A spanning subgraph H of G is called a *maximum planar subgraph* of G if H is planar and contains as many edges of G as possible. It is clear that $\omega(H) = \omega(G)$ and

$$0 \leqslant \varepsilon(G) - \varepsilon(H) \leqslant cr(G). \tag{2.2}$$

Lemma 2.1 *Let G be a graph with $cr(G) \leqslant 5$. Then $\delta(G) \leqslant 5$.*

Proof. Let $\Delta = \Delta(G)$. We can easily verify the following equalities

$$\begin{aligned} \upsilon(G) &= \upsilon_1(G) + \upsilon_2(G) + \cdots + \upsilon_\Delta(G), \\ 2\varepsilon(G) &= \upsilon_1(G) + 2\upsilon_2(G) + \cdots + \Delta\upsilon_\Delta(G). \end{aligned} \tag{2.3}$$

Let H be a maximum planar subgraph of G. By (2.1), we have

$$\varepsilon(H) \leqslant 3\upsilon(H) - 6. \tag{2.4}$$

Combining (2.4) with $\upsilon(G) = \upsilon(H)$, we have

$$\varepsilon(G) - 3\upsilon(G) + 6 \leqslant \varepsilon(G) - \varepsilon(H). \tag{2.5}$$

Combining (2.5) with (2.2), we have

$$\varepsilon(G) - 3\upsilon(G) + 6 \leqslant cr(G). \tag{2.6}$$

Substituting (2.3) into the inequality (2.6) yields

$$\begin{aligned} & 5\upsilon_1(G) + 4\upsilon_2(G) + \cdots + \upsilon_5(G) \\ \geqslant\ & \upsilon_7(G) + 2\upsilon_8(G) + \cdots + (\Delta - 6)\upsilon_\Delta(G) + 12 - 2cr(G). \end{aligned} \tag{2.7}$$

Since $cr(G) \leqslant 5$, from (2.7), we have $\delta(G) \leqslant 5$. ∎

Suppose that G is a connected graph. We say that G has *genus* ρ if G can be embedded in a surface S with ρ handles such that edges are pairwise disjoint except possibly for end-vertices. The boundary of every region contains at least three edges and every edge is on the boundary of at most two regions (the two regions are identical when e is a cut-edge). For any edge e of G, let $r_G^1(e)$ and $r_G^2(e)$ be the numbers of edges comprising the regions which the edge e borders. For convenience, we always assume $r_G^1(e) \leqslant r_G^2(e)$ in this paper. It is clear that if $e = xy \in E(G)$ then

$$\begin{cases} r_G^2(e) \geqslant r_G^1(e) \geqslant 4 & \text{if } |N_G(x) \cap N_G(y)| = 0, \\ r_G^2(e) \geqslant 4,\ r_G^1(e) \geqslant 3 & \text{if } |N_G(x) \cap N_G(y)| = 1, \\ r_G^2(e) \geqslant r_G^1(e) \geqslant 3 & \text{if } |N_G(x) \cap N_G(y)| \geqslant 2. \end{cases} \tag{2.8}$$

Following Carlson and Develin [2] and Fischermann et al. [6], for any edge $e = xy$ of G, we define

$$D_G(e) = \frac{1}{d_G(x)} + \frac{1}{d_G(y)} + \frac{1}{r_G^1(e)} + \frac{1}{r_G^2(e)} - 1. \tag{2.9}$$

Let G be a connected with an embedding in a surface S of genus $\rho(G)$ and having $\phi(G)$ regions, including the unbounded region. The well-known Euler's Formula (see, for example, Theorem 4.22 in [3]) is stated as follows.

$$\upsilon(G) - \varepsilon(G) + \phi(G) = 2 - 2\rho(G). \tag{2.10}$$

From the definition (2.9) of $D_G(e)$ and Euler's Formula (2.10), it is easy to see that

$$\sum_{e \in E(G)} D_G(e) = \upsilon(G) - \varepsilon(G) + \phi(G) = 2 - 2\rho(G). \tag{2.11}$$

Lemma 2.2 *Let G be a connected graph with crossing number $cr(G)$ and let H be a maximum planar subgraph of G. Then*

$$\sum_{e \in E(H)} D_G(e) \geqslant 2 - \frac{2cr(G)}{\delta(G)}.$$

Proof. Let H be a graph obtained from G by deleting $E' \subseteq E(G)$. For each $i = \delta(G), \ldots, \Delta(G)$ and an integer h with $0 \leqslant h \leqslant i-1$, let

$$V_i^{(h)} = \{x \in V_i(G) : \ d_H(x) = i - h\} \quad \text{and} \quad \upsilon_i^{(h)} = |V_i^{(h)}|.$$

Then summing all $h\upsilon_i^{(h)}$ and applying (2.2), we have

$$\sum_{i=\delta}^{\Delta} \sum_{h=0}^{i-1} h\upsilon_i^{(h)} = 2|E'| \leqslant 2cr(G). \tag{2.12}$$

By the definition, we observe that $r_H^i(e) \geqslant r_G^i(e)$ for every $e = xy \in E(H)$ and $i = 1, 2$. Using this fact, along with (2.9) and (2.12), we have

$$\begin{aligned}
\sum_{e \in E(H)} D_H(e) &= \sum_{xy \in E(H)} \left\{ \frac{1}{d_H(x)} + \frac{1}{d_H(y)} + \frac{1}{r_H^1(e)} + \frac{1}{r_H^2(e)} - 1 \right\} \\
&\leqslant \sum_{xy \in E(H)} \left\{ \frac{1}{d_H(x)} + \frac{1}{d_H(y)} + \frac{1}{r_G^1(e)} + \frac{1}{r_G^2(e)} - 1 \right\} \\
&= \sum_{xy \in E(H)} \left\{ D_G(e) + \frac{1}{d_H(x)} - \frac{1}{d_G(x)} + \frac{1}{d_H(y)} - \frac{1}{d_G(y)} \right\}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{xy\in E(H)} D_G(e) + \sum_{v\in V(H)} d_H(v)\left(\frac{1}{d_H(v)} - \frac{1}{d_G(v)}\right) \\
&= \sum_{xy\in E(H)} D_G(e) + \sum_{i=\delta(G)}^{\Delta(G)} \sum_{h=0}^{i-1} v_i^{(h)}(i-h)\left(\frac{1}{i-h} - \frac{1}{i}\right) \\
&= \sum_{xy\in E(H)} D_G(e) + \sum_{i=\delta(G)}^{\Delta(G)} \frac{1}{i} \sum_{h=0}^{i-1} h v_i^{(h)} \\
&\leqslant \sum_{xy\in E(H)} D_G(e) + \frac{1}{\delta(G)} \sum_{i=\delta(G)}^{\Delta(G)} \sum_{h=0}^{i-1} h v_i^{(h)} \\
&\leqslant \sum_{e\in E(H)} D_G(e) + \frac{2cr(G)}{\delta(G)},
\end{aligned}
$$

that is,

$$
\sum_{e\in E(H)} D_G(e) \geqslant \sum_{e\in E(H)} D_H(e) - \frac{2cr(G)}{\delta(G)} \tag{2.13}
$$

Since H is connected and $\rho(H) = 0$, combining (2.13) with (2.11) for the planar graph H yields the required result immediately. ∎

To prove our main results in this paper, we need three basic upper bounds of $b(G)$ for a graph G.

Lemma 2.3 [1, 5, 13] *For any two distinct vertices x and y with $d_G(x, y) \leqslant 2$ in G, we have $b(G) \leqslant d_G(x) + d_G(y) - 1$.*

Lemma 2.4 [7, 13] *If the edge connectivity $\lambda(G) \geqslant 1$, then $b(G) \leqslant \Delta(G) + \lambda(G) - 1$.*

Lemma 2.5 [7] *For any two adjacent vertices x and y in G, we have $b(G) \leqslant d_G(x) + d_G(y) - 1 - |N_G(x) \cap N_G(y)|$.*

3 Main Results

In [2], Carlson and Develin used the function $D_G(e)$ defined in (2.9) to give a short proof of Kang and Yuan's result [9] that $b(G) \leqslant \min\{8, \Delta(G) + 2\}$. In this section, we show that this bound is valid for graphs with small crossing number.

Theorem 3.1 *Let G be a connected graph. Then $b(G) \leqslant \Delta(G) + 2$ if G satisfies one of the following conditions:*

(a) $cr(G) \leqslant 3$,
(b) $cr(G) = 4$ *and* G *is not* 4*-regular,*
(c) $cr(G) = 5$ *and* G *contains no vertices of degree* 4.

Proof. The theorem holds if $\lambda(G) \leqslant 3$ by Lemma 2.4. Assume $\lambda(G) \geqslant 4$; this implies $\delta(G) \geqslant 4$. By this fact and Lemma 2.1 we have

$$4 \leqslant \delta(G) \leqslant 5. \tag{3.1}$$

Suppose to the contrary that $b(G) > \Delta(G) + 2$ under our hypothesis. We will derive a contradiction.

Since $b(G) > \Delta(G) + 2$, by Lemma 2.5, for any edge xy in G, we have

$$d_G(x) + d_G(y) - 1 - |N_G(x) \cap N_G(y)| > \Delta(G) + 2.$$

Assume, without loss of generality, that $d_G(x) \leqslant d_G(y)$. Then, rearranging the above, we have

$$|N_G(x) \cap N_G(y)| \leqslant d_G(x) - 4. \tag{3.2}$$

Let G be embedded on a surface of genus $\rho(G)$. Then $\rho(G) \leqslant cr(G)$. We first prove that

$$D_G(e) \leqslant 0 \quad \text{for any } e = xy \in E(G), \tag{3.3}$$

where the equality holds only if either $d_G(x) = d_G(y) = 4$ and $r^1_G(e) = r^2_G(e) = 4$ or $d_G(x) = d_G(y) = 6$ and $r^1_G(e) = r^2_G(e) = 3$.

We prove this by considering the following three cases respectively.

If $d_G(x) = 4$, then $|N_G(x) \cap N_G(y)| = 0$ by (3.2), and so $r^2_G(e) \geqslant r^1_G(e) \geqslant 4$ by (2.8). Thus, $D_G(e) \le \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} - 1 = 0$. Clearly, $D_G(e) = 0$ only if $d_G(x) = d_G(y) = 4$ and $r^1_G(e) = r^2_G(e) = 4$.

If $d_G(x) = 5$, then $|N_G(x) \cap N_G(y)| \leqslant 1$ by (3.2), and so $r^1_G(e) \geqslant 3$, and $r^2_G(e) \geqslant 4$ by (2.8). Thus, $D_G(e) \leqslant \frac{1}{5} + \frac{1}{5} + \frac{1}{3} + \frac{1}{4} - 1 < 0$.

If $d_G(x) \geqslant 6$, since $r^2_G(e) \geqslant r^1_G(e) \geqslant 3$ by (2.8), then $D_G(e) \leqslant \frac{1}{6} + \frac{1}{6} + \frac{1}{3} + \frac{1}{3} - 1 = 0$. Clearly, $D_G(e) = 0$ only if $d_G(x) = d_G(y) = 6$ and $r^1_G(e) = r^2_G(e) = 3$.

Thus, (3.3) holds for any edge $e = xy$ in G.

Let H be the maximum planar subgraph of G. Substituting (3.3) into Lemma 2.2, we have

$$cr(G) \geqslant \delta(G) \left(1 - \frac{1}{2} \sum_{e \in E(H)} D_G(e)\right) \geqslant \delta(G). \tag{3.4}$$

Thus, $4 \leqslant cr(G) \leqslant 5$ since $\delta(G) \geqslant 4$.

If $cr(G) = 4$ then $\delta(G) = 4$ and $D_G(e) = 0$ for any $e \in E(G)$ by (3.4). By (3.3), G contains only vertices of degree 4 and/or 6, that is, $V(G) = V_4(G) \cup V_6(G)$. Clearly, $V_4(G) \neq \emptyset$ since $\delta(G) = 4$. If $V_6(G) \neq \emptyset$, then there is an edge $e = xy$ in G with $x \in V_4(G)$ and $y \in V_6(G)$ since G is connected. However, $D_G(e) \leq \frac{1}{4} + \frac{1}{6} + \frac{1}{4} + \frac{1}{4} - 1 < 0$, a contradiction. Thus, $d_G(x) = 4$ for any $x \in V(G)$, which contradicts our hypothesis that G is not 4-regular.

If $cr(G) = 5$ then $\delta(G) = 5$ since G contains no vertices of degree 4. It follows from (3.4) that $D_G(e) = 0$ for any $e \in E(G)$. However, $D_G(e) = 0$ for any $e \in E(G)$ implies that G does not contain vertices of degree 5 by (3.3), which contradicts $\delta(G) = 5$.

The theorem follows. ∎

Lemma 3.2 [8] *If G is a connected graph with $cr(G) \leqslant 3$, then $b(G) \leqslant 8$.*

Combating Theorem 3.1 with Lemma 3.2, we immediately obtain the following result.

Corollary 3.3 $b(G) \leqslant \min\{8, \Delta(G) + 2\}$ *for any connected graph G with $cr(G) \leqslant 3$.*

Fischermann et al. [6] used the method of Carlson and Develin [2] to show Conjecture 1.1 valid for planar graphs with some conditions on triangles. We also generalize their results to graphs with small crossing number.

Theorem 3.4 *Let G be a connected graph with $\Delta(G) \geqslant 6$ and $cr(G) \leqslant 3$. If $\Delta(G) \geqslant 7$ or if $\Delta(G) = 6$, $\delta(G) \neq 3$ and every edge $e = xy$ with $d_G(x) = 5$ and $d_G(y) = 6$ is contained in at most one triangle, then $b(G) \leqslant \min\{8, \Delta(G) + 1\}$.*

Proof. By Corollary 3.3, $b(G) \leqslant 8 \leqslant \Delta(G) + 1$ when $\Delta(G) \geqslant 7$. Now suppose $\Delta(G) = 6$. If $\lambda(G) \leqslant 2$, then $b(G) \leqslant 7$ by Lemma 2.4. So we assume $\lambda(G) \geqslant 3$ below. Thus, $\delta(G) \geqslant 4$ since $\delta(G) \neq 3$. Suppose to the contrary that $b(G) \geqslant \Delta(G) + 2 = 8$. Then, by Lemma 2.5, we have

$$d_G(y) \geqslant 9 + |N_G(x) \cap N_G(y)| - d_G(x) \quad \text{for any } e = xy \in E(G). \tag{3.5}$$

Let G be embedded on a surface of genus $\rho(G)$. Then $\rho(G) \leqslant cr(G)$. We first prove

$$D_G(e) \leqslant 0 \quad \text{for any } e = xy \in E(G). \tag{3.6}$$

Let $e = xy$ be any edge in G.

If $d_G(x) = 4$, then $5 \leqslant d_G(y) \leqslant 6$ by (3.5). If $d_G(y) = 5$, then $|N_G(x) \cap N_G(y)| = 0$ by (3.5), and so $r_G^2(e) \geqslant r_G^1(e) \geqslant 4$. Hence $D_G(e) \leqslant \frac{1}{4} + \frac{1}{5} +$

$\frac{1}{4} + \frac{1}{4} - 1 < 0$. If $d_G(y) = 6$, then $|N_G(x) \cap N_G(y)| \leqslant 1$ by (3.5), and so $r_G^1(e) \geqslant 3$ and $r_G^2(e) \geqslant 4$. Hence $D_G(e) \leqslant \frac{1}{4} + \frac{1}{6} + \frac{1}{3} + \frac{1}{4} - 1 = 0$.

If $d_G(x) = 5$, then $5 = d_G(x) \leqslant d_G(y) \leqslant \Delta(G) = 6$. If $d_G(y) = 5$, then $|N_G(x) \cap N_G(y)| \leqslant 1$ by (3.5), which implies that $r_G^1(e) \geqslant 3$ and $r_G^2(e) \geqslant 4$. Thus $D_G(e) \leqslant \frac{1}{5} + \frac{1}{5} + \frac{1}{3} + \frac{1}{4} - 1 < 0$. If $d_G(y) = 6$, then $|N_G(x) \cap N_G(y)| \leqslant 1$ by the hypothesis. Thus $r_G^1(e) \geqslant 3$, and $r_G^2(e) \geqslant 4$. It follows that $D_G(e) \leqslant \frac{1}{5} + \frac{1}{6} + \frac{1}{3} + \frac{1}{4} - 1 < 0$.

If $d_G(x) = 6$, then $d_G(y) = 6$. Since $r_G^2(e) \geqslant r_G^1(e) \geqslant 3$, we have $D_G(e) \leqslant \frac{1}{6} + \frac{1}{6} + \frac{1}{3} + \frac{1}{3} - 1 = 0$.

Therefore (3.6) holds. Let H be the maximum planar subgraph of G. Substituting (3.6) and $\delta(G) \geqslant 4$ into Lemma 2.2, we have

$$3 \geqslant cr(G) \geqslant 4 - 2 \sum_{e \in E(H)} D_G(e) \geqslant 4.$$

This contradiction completes the proof of the theorem. ∎

The *girth* of G, $g(G)$, is the length of the shortest cycle in G. If G has no cycles we define $g(G) = \infty$.

Theorem 3.5 [8] *Let G be a connected graph with girth $g(G) \geqslant 4$. Then $b(G) \leqslant 6$ if $cr(G) \leqslant 3$, or if $cr(G) = 4$ and G is not 4-regular.*

Using Theorem 3.5 we obtain the following.

Theorem 3.6 *Let G be a connected graph with $\Delta(G) = 5$ and $cr(G) \leqslant 4$. If no triangles contain two vertices of degree 5, then $b(G) \leqslant 6 = \Delta(G) + 1$.*

Proof. If $\lambda(G) \leqslant 2$, then $b(G) \leqslant 5 + 2 - 1 = 6$ by Lemma 2.4. We assume $\lambda(G) \geqslant 3$ below. Suppose to the contrary that $b(G) \geqslant \Delta(G) + 2 = 7$. Then $g(G) = 3$ by Theorem 3.5.

Let (x, y, z) be a triangle in G. Then $d_G(x) + d_G(y) \leqslant 9$ by the hypothesis. By Lemma 2.5 we have $b(G) \leqslant 6$ if $d_G(x) + d_G(y) \leqslant 8$. It follows that the degree-sum of any two vertices in the triangle is equal to 9, which implies the degree of every vertex in the triangle is at least four. We can, without loss of generality, assume $d_G(x) = 4$ and $d_G(y) = 5$. Then $d_G(z) = 5$ since $d_G(x) + d_G(z) = 9$. Thus, the triangle (x, y, z) contains two vertices of degree 5, which contradicts the hypothesis. The theorem follows. ∎

4 Remarks

Motivated by these results for graphs with small crossing number, we suggest the following conjecture.

Conjecture 4.1 *$b(G) \leqslant \Delta(G)+1$ for any connected graph G with $cr(G) \leqslant 3$.*

It has been mentioned in the introduction that this conjecture is not valid if there are no constraints $cr(G)$. Furthermore, Hartnell and Rall [7] and Teschner [12] independently proved that $b(G)$ can be much greater than $\Delta(G)$ by showing that $b(G_n) = \frac{3}{2}\Delta(G_n)$ for the cartesian product $G_n = K_n \times K_n$. Thus, there exists no upper bound of the form $b(G) \leqslant \Delta(G) + c$ for any integer c. Teschner [11] proved that $b(G) \leqslant \frac{3}{2}\Delta(G)$ for graphs with $cr(G) \leqslant 3$, and posed the following conjecture.

Conjecture 4.2 [11] *$b(G) \leqslant \frac{3}{2}\Delta(G)$ for any graph G.*

As far as we know, there is no much work on this conjecture. We pose the following questions.

Question 1: Is there a cubic 3-edge-connected graph with $cr(G) \leqslant 4$ and $b(G) = 5$?

Question 2: Is there a 4-regular 4-edge-connected graph with $cr(G) = 4$ and $b(G) = 6$?

If the answers to the two questions are negative, Conjecture 4.2 is true for all graphs with $cr(G) \leqslant 4$.

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