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实 验 报 告　　评分：

___系___级　姓名___　日期___　NO.___　　Apr. 6th.

实验题目：~~E~~x lecture notes 2.

实验目的：

§ Counting.

1. (Burnside's Lemma). G: finite group. act on S: finite.

- $\forall s \in S$: $\sum_{t \in Orb(s)} \frac{1}{\# Orb(t)} = 1$. (Naive)

- $f(x) = \#\{s \in S : xs = s\}$, then

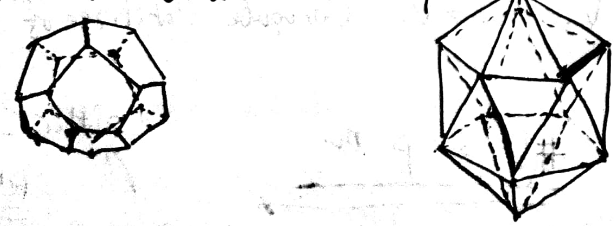
△ $$\sum_{x \in G} f(x) = \# G \cdot \#\{\text{orbits}\}$$

pf: $\sum_{x \in G} f(x) = \sum_{x \in G} \#\{s \in S : xs = s\} = \sum_{x \in G} \sum_{s \in S} 1_X(x, s)$

$= \sum_{s \in S} \sum_{x \in G} 1_X(x, s) = \sum_{s \in S} \sum_{x \in G} G_s = \cdots = \# G \, \#\{\text{orbits}\}$

2. counting isometric group of tetrahedron, hexahedron, octahedron, dodeca hedron, icosahedron.

① How to draw the picture?



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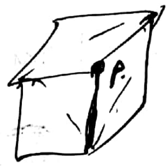
Observation: Rotation group and Symmetric Group act Transitively on the hedron. Thus we can use the counting formula: $\#G = \#\Sigma \, \# G_p$. where p is an arbitrary point. For example:

a hexihedron has 8 p vertices.



~~G_p is~~ Note that p is fixed $\Leftrightarrow$ the triangle with it is fixed.

Hence G_p is just rotation group or symmetric group of Δ

Thus order of rotation/synetric group of hexahedron is $\begin{cases} 8\times 3 = 24 \\ 8 \times 6 = 48 \end{cases}$

3. G: p^n-group. $\mathbb{F}_p$; V: f.d.v.s. over $\mathbb{F}_p$. G acts Linearly on V. i.e. $g(u+v) = gu + gv$, $g\lambda u = \lambda g u$

(1). $\exists x \in V$, $x \neq 0$ s.t. $gx = x$.

(2). $GL(V)$ act on V. $v \in V$. Compute stabilizer of v. (not interesting).

pf of (1): $\#G = p^n$, $\#V = p^m$.

$$\underset{\substack{\uparrow \\ p^m}}{\#V} = \sum_{\substack{x \text{ runs} \\ \text{over representatives}}} \mathrm{Orb}(x) \overset{?}{=} 1 + \sum_{\substack{x \text{ runs} \\ \text{over representatives} \\ x \neq 0}} \mathrm{Orb}(x)$$

either 1 or p^k, $k \leq n$

$\because$: $g(\vec{0}) = g(\vec{0}) + g(\vec{0})$. $\Rightarrow$ $g\vec{0} = \vec{0}$. $\forall g \in G$.

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In the same method, you can do many exercises.

4. G: finite group, S: finite $\#S \geq 2$. G act transitively.

$\Rightarrow \exists x \in G$: $xs \neq s$ $\forall s \in S$.

pf: By Burnside's Lemma, we have:

$$\sum_{g \in G} f(g) = \#G \, \#\{\text{orbits}\} = \#G$$

$$2 + \sum_{\substack{g \in G \\ g \neq 1}} f(g) \leq$$

5. Use four colors to paint the four vertices of tetrahedron. How many ~~plans~~ different chemes do we have?

These kind of problems, see Rotman Advanced Modern Algebra, second edition. Page 109-111

Topics about Symmetric Groups and Normal Subgroups

1. g, h conjugates in G, do they conjugates in $H \leq G$?

(i). $H \leq G$ $x \in H$ $\Rightarrow C_H(x) = H \cap C_G(x)$

pf: just check definition.

(ii). $H \leq G$, $[G:H] = 2$. G finite. $x \in H$. Then Either $K_H(x) = K_G(x)$ (the conjugacy class of x)

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OR $\# K_H(x) = \frac{1}{2} \# K_G(x)$, or more precisely,

~~orbit break up into two with same order~~

Pf: Recall: $G \curvearrowright x$ conjugate, $\left\langle \begin{array}{l} \text{orbit} = K_G(x) \\ \text{stable group} = G_x = C_G(x) \end{array} \right.$

$H \curvearrowright x$ - - - - . $K_H(x)$, $C_H(x)$

Also Recall : $[G : H] = 2 \Rightarrow H \trianglelefteq G$.

Thus, with 2nd-isomorphism therorem, we have:

$$\frac{H\,C_G(x)}{H} = \frac{\cancel{H}}{\cancel{C_G(x)}\cap} \frac{C_G(x)}{H \cap C_G(x)} = \frac{C_G(x)}{C_H(x)}$$

• If $C_G(x) \leq H$, then $C_G(x) = C_H(x)$.

then $K_H(x) = \frac{1}{2} K_G(x)$. ~~the orbit break up into two part since the case holds for every y∈~~

• If $C_G(x) \nleq H$, then $H\,C_G(x) = G$ (why?)

$\Rightarrow \frac{C_G(x)}{C_H(x)} = 2 \Rightarrow K_G(x) = K_H(x)$.

⑧ Now we can count the conjugacy class in A_5.

(1), (12)(34), (123), (12345)

(1) ... #1 (12)(34): it ~~has commute~~ can commute with (12), which is not in A_5, thus all of this type conjugates. ... $C_5^4 \cdot 3 = 15$.

(123) commutes with (45), hence ---- $20 : C_5^3 \times 2 = 20$

For (12345): $\#K_{S_5}((12345)) = 24$ (just count)

$\Rightarrow C_{S_5}(12345) = \langle (12345) \rangle \leq A_5$. Break up.

More finely: Using Sylow's theorem, we will show that

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5-cycles x doesnot conjugate with x^2 in A_5. Thus we can show precisely how the orbits break up.

Next we turn to the proof of the ~~simpleness~~ simplicity of A_n, $n \geq 5$:

(i). all 3-cycles conjugate in A_n $n \geq 5$.

Pf: $(ijk) = \gamma(123)\gamma^{-1}$ $\gamma \in S_n$.

But $n \geq 5 \Rightarrow \exists$ $\underline{4,5}$ and $(123)(45) = (45)(123)$.

$\Rightarrow (ijk) = \gamma(45)(123)(45)\gamma^{-1}$

Either γ or $\gamma(45) \in A_n$, done.

(ii). three cycles ~~gen~~ generate A_n. In fact, this holds for $n \geq 3$

$\alpha = \tau_1\tau_2 \cdots \tau_{2k-1}\tau_{2k}$, adjacent permutation doesn't conicide.

- $(ij)(jk) = (kij)$
- $(ij)(kl) = (ij)(jk)(jk)(kl) = (jki)(jkl)$.

A_n is simple for $n \geq 5$.

(iii). $H \neq \{(1)\}$ $H \triangleleft A_5 \Rightarrow H$ contains a 3-cycle.

Trick: we consider a 'minimal' one:

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$\langle \sigma \rangle \xrightarrow[\text{on}]{\text{act}} \{1, 2, 3, \cdots n\}$ $\sigma \neq id$.

Case 1. all orbits have 2 elements or one elements.

say $\{i_1 j_1\}$ $\{i_2 j_2\}$ ......

σ even $\Rightarrow$ at least 2 orbits with order 2. $\{i_1, j_1\}$ $\{i_2, j_2\}$.

Then let $k \neq i_1, j_1, i_2, j_2$, let $\tau = (i_2 j_2 k)$.

$\sigma' = \tau \sigma \tau^{-1} \sigma^{-1}$.

check ~~$\sigma(k)$~~ $\sigma'(i_1) = i_1$, $\sigma'(j_1) = j_1$.

and for $r \neq k$, r is fixed by σ, then r is fixed by σ'.

Contradiction.

~~Case 2. At least~~

Case 2: $\exists$ orbits ≥ 3, say $\{i, j, k, \cdots\}$.

if $\sigma \neq (ijk)$, then since $\sigma \neq (ijkr)$, σ moves

at least 2 other points, say r, s.

$\tau = (krs)$.

$\sigma' = \tau \sigma \tau^{-1} \sigma^{-1}$ $\begin{cases} \sigma'(i) = i \\ \sigma'(q) = q, \text{ if } \sigma(q) = q. \end{cases}$, contradiction!

Now we can do some exercises using the above facts.

1. $N \trianglelefteq S_n$ $n \geq 5$ $\Rightarrow$ $N = \{(1)\}$ or A_n or S_n.

$N \neq A_n$: $\frac{S_n}{N} = \frac{NA_n}{N} = \frac{A_n}{N \cap A_n}$ $N \cap A_n \trianglelefteq A_n$ $\Rightarrow$ $\begin{cases} N \cap A_n = A_n \Rightarrow \checkmark \\ N \cap A_n = 1 \Rightarrow \frac{S_n}{N} = A_n, \text{ absurd} \end{cases}$

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2. $n \geq 3 \Rightarrow A_n$ is the only subgroup of S_n with order $\frac{1}{2}n!$

3. n S_5 has no subgroup of order 30. , say H.

Pf: $\varphi: S_5 \longrightarrow S_4$.

$\ker\varphi \neq 0$, $\ker\varphi \neq S_5$. $\Rightarrow \ker\varphi = A_5$.

What does this mean? $A_5 \geq H$, absurd!

4. S_5 contains no subgroup of order 40.

Pf: $\varphi: S_5 \longrightarrow S_3$ $40 \nmid 60$, done.

5. $G \leq S_n$

(i) $G \cap A_n = \{1\} \Rightarrow \#G \leq 2$

$G \not\leq A_n \Rightarrow \frac{S_n}{A_n} = \frac{GA_n}{A_n} = \frac{G}{G \cap A_n} = G \Rightarrow \#G = 2$.

(ii). G is simple, $\#G > 2 \Rightarrow G \leq A_n$.

the same method.

6. (i) $n \geq 5 \Rightarrow$ S_n has no subgroup of index r, $2 < r < n$.

same as 4.

(ii). $n \geq 5 \Rightarrow A_n$ has no subgroup of index $2 \leq r < n$.

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$\varphi: A_n \longrightarrow S_r \quad 2<r<n.$

$\# A_n > \# S_n \Rightarrow \ker\varphi \neq 0 \Rightarrow \ker\varphi = A_n$, absurd.

7. (i). G simple , G has a subgroup & index >1

$\Rightarrow G$ isomorphic to a subgroup of S_n.

(ii) ~~#G~~ infinite & simple $\Rightarrow$ G has no subgroup of finite order $n>1$.

8. G : $\#|G| = mp$ $\quad 1<m<p$ prime $\Rightarrow G$ is not simple.

pf: By Cauchy's Theorem (See P104 of Rotman),

$\# H = p$, $H \leq G$, the same method.

Some more Exercises.

1. Aut (S_3) 的求法:

内自同构.

①. $S_3 \longrightarrow \mathrm{Inn}(S_3) \leq \mathrm{Aut}(S_3)$

"$\rightarrow$" $\Rightarrow$ "$\hookrightarrow$". 因为 $Z(S_3)$ 平凡.

②. $\mathrm{Aut}(S_3)$ act on $\{(12), (13), (23)\}$,

$\mathrm{Aut}(S_3) \rightarrow S_3$.

"$\rightarrow$" $\Rightarrow$ "$\hookrightarrow$"

2. (Normal subgroup): G : finite ; $N \triangleleft G$. ~~$N, G/N$ has relative~~ $(\# N, \# G/N) = 1$.

a). $H \leq G$ $\quad \# H = \# \frac{G}{N} \Rightarrow G = HN$

thus we have an easy semidirect product.

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实验题目：

实验目的：

(b)* $g \in Aut(G) \Rightarrow g(N) = N$.

(a). $m^{\#H}=1$, $m^{\#N}=1$ $(\#H, \#N)=1 \Rightarrow m=1$.

(b). $G = N \rtimes H$. (Refer to my previous Exlecture notes)

observation: Let $g(n,1) = (n_1', h)$.

$(n_1', h)(n_1', h) = (n_1 n_1^h, h^2)$ $n_1^h = h n_1 h^{-1}$

.....

But $(n,1)^{\#N} = 1 \Rightarrow 1 = g((n,1)^n) = [g(n,1)]^n = (n_1 n_1^h \cdots n_1^{h^{\#N-1}}, h^{\#N})$

$h^{\#N} = 1 \Rightarrow h = 1$ since $(\#H, \#N) = 1$. done

Exercises 2.3.7, 2.3.8, 2.3.11 are somehow difficult and tricky, you can refer to Mr. Feng (Keqin Feng's 300 exercises in Modern Algebra). But I will ~~give~~ show them at request.

Now I can ~~give my~~ talk about a topic that I like.

$PSL_2(\mathbb{Z})$

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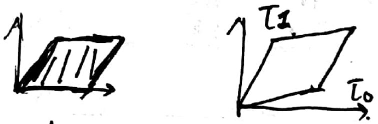
we add an equivalent relation: $(\tau_0,\tau_0)\sim(\tau_1',\tau_0')$ if ... or ... "integral basis" $\frac{\tau_1}{\tau_0}=\frac{\tau_1'}{\tau_0'}$

Λ : Lattice in $\mathbb{C}$ ... the complex plane.

$\dfrac{\mathbb{C}}{\Lambda}$:



(τ_0,τ_1) : integral basis

$SL_2(\mathbb{Z}) \xrightarrow{\text{act on}} \{\text{integral bases}\}/\Delta \quad \mathcal{H}$

i.e. the orbits

$\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix}\tau_1 \\ \tau_0\end{pmatrix} = \phi\begin{pmatrix} a\tau_1+b\tau_0 \\ c\tau_1+d\tau_0\end{pmatrix}$, $\dfrac{\mathcal{H}}{SL_2(\mathbb{Z})} = \mathcal{M}$.

Our aim is to check out ~~the~~ explictly what different elements (~~kind~~ equivalent class of Riemannian Surface) are in $\mathcal{M}$, and we will draw a picture to make it clear. ($\phi\ \frac{\tau_0}{\tau_1}$ in $\mathbb{C}$).

Thus, the action is just $\dfrac{az+b}{cz+d}$ or Möbius Map.

1. Reduce to $\tau_0=1$. $\tau_1\in H$ ... the upper half plane.

Note that $-I$ act trivially, thus

$$\mathcal{M}=\frac{H}{PSL_2(\mathbb{Z})}.$$

Define: fundamental domain for G act on X : (→ topolgical space or $\mathbb{C}$ here)

$\Omega\subseteq X$ s.t. $\forall$ Gx orbit $\cap\,\Omega\neq\phi$ &

no orbit meets Ω more than once.

Theorem: $\Omega=\{z\in H : |z|>1,\ -\frac{1}{2}<\mathrm{Re}(z)<\frac{1}{2}\}$ is a fundamental domain for $PSL(2,\mathbb{Z})$ on $H=\{z\in\mathbb{C}: \mathrm{Im}\, z>0\}$

Pf: Choose an arbitrary $z_0\in H$.

Step 1. We use $T(z)=z+1\in PSL(2,\mathbb{Z})$ to make $-\frac{1}{2}\le \mathrm{Re}\, z\le\frac{1}{2}$

Step 2. For $|z_0|<1$, Let $S(z)=-\frac{1}{z}$ to make it outside the semi-circle.

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实验题目：

实验目的：

Now comes the hard part of the proof. We have to Show each orbit meet Ω at most one time.

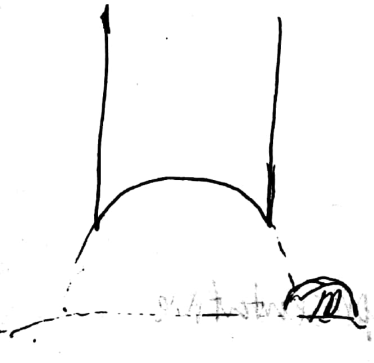
① Lemma: $\forall g \in PSL(2,\mathbb{Z})$, $g \neq id$ then $g(\Omega) \cap \Omega = \phi$.

pf: ① $g(z) = \frac{az+b}{cz+d}$ $\quad$ $c=0$.

Then $g = T^k$ in $PSL_2(\mathbb{Z})$, $T = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$.

obviously, $g(\Omega) \cap \Omega = \phi$.

$$g(z) = \frac{az+b}{cz+d} = \frac{az + \frac{ad}{c} - \frac{ad}{c} + b}{cz+d} = \frac{a}{c} - \frac{\frac{ad-bc}{c}}{cz+d} = \frac{a}{c} - \frac{c^{-2}}{z - \frac{d}{c}}$$



$A_g = \{ z \in H : |z - \frac{d}{c}| \geq \frac{1}{|c|} \}$

$\Rightarrow gA_g = ?$

$g(z) = \frac{a}{c} - \frac{c^2}{w}$ $\quad \Rightarrow w\left(\frac{a}{c} - g(z)\right) = c^2$

$\rightarrow w = \left| \frac{c^{-2}}{\frac{a}{c} - g(z)} \right| \geq \frac{1}{|c|} \Rightarrow$

$|g(z) - \frac{a}{c}| \leq \frac{1}{|c|}$

This $gA_g = \{ z : |z - \frac{a}{c}| \leq \frac{1}{|c|} \}$

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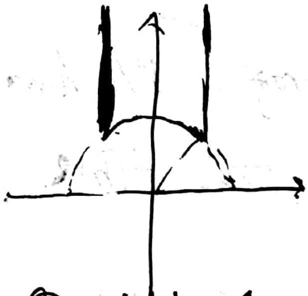
$\Rightarrow \overline{\Omega} \cap \overline{g(\Omega)} = \emptyset$

(PSL!)

Case 2: $|c|<2$ $\Rightarrow |c|=1$, Let $c=1$.

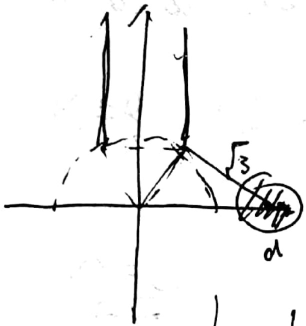


$\cancel{|z-d| \geq 1}$

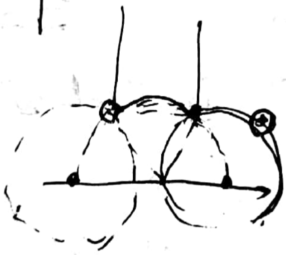
$|z-a|=1$

① $|d|>1$

$d \geq 2$



② $|a|>1$ $|d|=1$:



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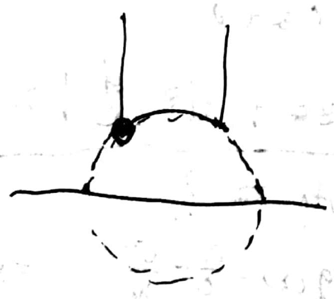
③ $|a|=1$, $|d|=1$



Exercise: Determine exactly a representative

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