Stochastic Stability Analysis of Packet-Based Networked Control Systems

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Abstract—By taking advantage of the packet-based transmission in networked control systems (NCSs), a packet-based control approach is proposed for NCSs. Using this approach, the control law can be designed with explicit compensation for network-induced delay, data packet dropout and data packet disorder simultaneously. The sufficient and necessary condition for the stochastic stability of the closed-loop system is obtained, by modeling the closed-loop system as a Markov jump system. A numerical example is also considered to illustrate the effectiveness of the proposed approach.

I. INTRODUCTION

Networked Control Systems (NCSs) is an emerging research area which has attracted considerable attentions in recent years. In NCSs, data is exchanged through a communication network which inevitably introduces communication constraints to the control systems, e.g., network-induced delay, data packet dropout, data packet disorder, data rate constraint, etc.. These communication constraints in NCSs present great challenges for conventional control theory [1]–[4].

It is noticed that in the early research work on NCSs, the latency of optimizing the system performance by taking advantage of the network characteristics is somewhat neglected. The network is simply treated as a negative parameter to the system and as a result a conventional control system instead of an NCS is actually considered [5], [6]. However, though the communication constraints in NCSs normally degrade the system performance, the network is not necessarily negative to the system. Based on this observation, preliminary work has been done recently under the so called “co-design” framework, where the characteristics of the network are analyzed and utilized further [7]–[13].

Most of the work under the co-design framework is motivated by the observation of the packet-based transmission in NCSs [14]. This characteristic can mean whether an NCS sends one single bit or several hundreds bits of data consumes the same amount of network resource. More specifically, it can be concluded that the same network resource is required to send either a single step control signal or multiple steps of forward control signals within a certain length. Based on this observation, a packet-based control approach is proposed in this paper, where by designing a packet-based controller and a corresponding comparison rule at the actuator side, this approach can actively compensate for the communication constraints including the network-induced delay, data packet dropout and data packet disorder simultaneously in both the forward and backward channels. This merit can not be achieved using conventional control approaches as in, e.g., [15], [16].

The NCSs setup in Fig. 1 is considered in this paper, where \( \tau_{ac,k} \) and \( \tau_{ca,k} \) are the backward channel delay and forward channel delay respectively and the plant is linear in discrete-time,

\[
x(k+1) = Ax(k) + Bu(k)\tag{1}
\]

where \( x(k) \in \mathbb{R}^n, u(k) \in \mathbb{R}^m, A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times m} \). The full state information is assumed to be available for measurements for simplicity.

It is necessary to point out that the forward channel delay \( \tau_{ca,k} \) is not available for the controller when the control action is calculated at time \( k \), since \( \tau_{ca,k} \) occurs after the determination of the control action. Therefore, when applying conventional design techniques such as in Time Delay Systems (TDSs) to NCSs, the active compensation for the forward channel delay can not be provided. That is, the conventional design techniques are conservative in the networked control environment, which however can be dealt with using the packet-based controller presented in the next section.

The control law based on the packet-based controller can be obtained as follows with explicit compensation for the communication constraint,

\[
u(k) = K_\tau(x) = K(\tau^*_k) x(k - \tau_k^*)\tag{2}
\]

where \( \tau^*_k = \tau_{ac,k}^* + \tau_{ca,k}^* \) and \( \tau_{ac,k}^* \) and \( \tau_{ca,k}^* \) are the network-induced delays of the control action that is actually applied to the plant at time \( k \). These delays are different from \( \tau_{ac,k} \), \( \tau_{ca,k} \) and \( \tau_k = \tau_{ac,k} + \tau_{ca,k} \) in Fig. 1 due to the introduction of a comparison process at the actuator side, which will be explained in detail in the following Section.
The remainder of this paper is organized as follows. In Section II, the packet-based control approach is proposed, and the stochastic stability of the corresponding closed-loop system is then analyzed in Section III. A numerical example is considered in Section IV to illustrate the effectiveness of the proposed approach and Section V concludes the paper.

II. PACKET-BASED CONTROL

In order to design the packet-based control approach for NCSs, the following assumptions are required.

**Assumption 1:** The data packets sent from the sensor are time-stamped.

**Assumption 2:** The sum of the maximum network-induced delay and the maximum number of continuous data packet dropout in the round trip is upper bounded by \( \tilde{\tau} \) and

\[
\tilde{\tau} \leq \frac{B_p}{B_c} - 1 \quad (3)
\]

where \( B_p \) is the size of the effective load of the data packet and \( B_c \) is the bits required to encode a single step control signal.

**Remark 1:** From Assumption 1, the network-induced delay that each data packet experiences can be known by the actuator on its arrival.

The block diagram of the packet-based control structure is illustrated in Fig. 2. It is distinct from conventional control structure in two aspects: the specially designed packet-based controller and the corresponding Control Action Selector (CAS) at the actuator side.

In order to implement the control law in (2), we take advantage of the packet-based transmission of the network to design a packet-based controller instead of trying to obtain directly current forward channel delay as this is actually impossible in practice. The packet-based controller determines a sequence of forward control actions as follows and sends them together in one data packet to the actuator,

\[
U(k|\tau_{sc,k}) = [u(k-\tau_{sc,k}) \ldots u(k-\tau_{sc,k} + \tilde{\tau}| k - \tau_{sc,k})]^{T} \quad (4)
\]

where \( u(k-\tau_{sc,k} + i| k - \tau_{sc,k}), i = 0, 1, \ldots, \tilde{\tau} \) are the forward control action predictions based on information up to time \( k - \tau_{sc,k} \).

When a data packet arrives at the actuator, the designed CAS compares its time stamp with the one already in CAS and only the one with the latest time stamp is saved. The comparison process is introduced because different data packets may experience different delays thus producing a situation where a packet sent earlier may arrive at the actuator later, that is, data packet disorder. After the comparison process, only the latest available information is used.

CAS also determines the appropriate control action from the forward control sequence \( U(k|\tau_{sc,k}) \) (after comparison process) at each time instant as follows

\[
u(k) = u(k|\tau_{sc,k}^*) \quad (5)
\]

It is necessary to point out that the appropriate control action determined by (5) is always available, provided Assumption 2 holds.

The packet-based control algorithm with the control law in (2) can now be summarized as follows based on Assumptions 1 and 2.

**S1.** At time \( k \), if the packet-based controller receives the delayed state data \( x(k - \tau_{sc,k}) \), then,

**S1a.** Calculates the forward control sequence as in (4);

**S1b.** Packs \( U(k|\tau_{sc,k}) \) and sends it to the actuator in one data packet.

If no data packet is received at time \( k \), then let \( k = k + 1 \) and wait for the next time instant.

**S2.** CAS updates its forward control sequence once a data packet arrives;

**S3.** The control action in (5) is picked out from CAS and applied to the plant.

It is seen from Fig. 2 that with the packet-based control approach the control actions are taken with explicit consideration of the network conditions and therefore the negative effects brought by the inserted communication network to NCSs can be potentially effectively dealt with by using the packet-based control approach and an appropriate controller. In the following section, a stabilized controller design method is hence considered within the Markov jump system framework.

III. STOCHASTIC STABILITY ANALYSIS

In this section, the stochastic stability of the closed-loop system using the aforementioned packet-based control approach is analyzed within the Markov jump system framework.

Let \( X(k) = [x^T(k) \ x^T(k-1) \ldots \ x^T(k-\tilde{\tau})] \). The closed-loop system with the control law in (2) can then be written as

\[
X(k + 1) = \Xi(\tau_k^*)X(k) \quad (6)
\]

where

\[
\Xi(\tau_k^*) = \begin{pmatrix}
A & \cdots & BK(\tau_k^*) & \cdots & 0 \\
I_n & 0 & I_n & \cdots & I_n \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & I_n & 0
\end{pmatrix}
\]

and \( I_n \) is the identity matrix with rank \( n \).

A. The stochastic model of the packet-based control approach for NCSs

In NCSs, it is reasonable to model the round trip delay \( \{\tau_k; k = 0, 1, \ldots\} \) as a homogeneous ergodic Markov chain [15]. In order to take explicit account of the data packet dropout, Markov chain \( \{\tau_k; k = 0, 1, \ldots\} \) in this paper is assumed to take values from \( \mathcal{M} = \{0, 1, 2, \ldots, \infty\} \) where \( \tau_k = 0 \) means no delay in round trip while \( \tau_k \to \infty \) implies a data packet dropout in either the backward or the forward channel. Let the transition probability matrix of \( \{\tau_k; k = 0, 1, \ldots\} \) denoted by \( \Lambda = [\lambda_{ij}] \) where

\[
\lambda_{ij} = P(\tau_{k+1} = j | \tau_k = i), i, j \in \mathcal{M}
\]
$P(\tau_{k+1} = j|\tau_k = i)$ is the probability of $\tau_k$ jumping from state $i$ to $j$, $\lambda_{ij} \geq 0$ and

$$\sum_{j \in \mathcal{M}} \lambda_{ij} = 1, \forall i, j \in \mathcal{M}$$

The initial distribution of $\{\tau_k; k = 0, 1, \ldots\}$ is defined by

$$P(\tau_0 = i) = p_i, i \in \mathcal{M}$$

According to the comparison process, the round trip delay of the control actions that are actually applied to the plant can be determined by the following equation.

$$\tau_{k+1}^* = \begin{cases} \tau_k^* + 1, & \text{if } \tau_{k+1} > \tau_k^*; \\ \tau_k^* - r, & \text{if } \tau_{k+1} - r \leq \tau_k + 1 \leq \tau_k^*. \end{cases} \quad (7)$$

**Lemma I:** $\{\tau_k^*; k = 0, 1, \ldots\}$ is a non-homogeneous Markov chain with state space $\mathcal{M}^* = \{0, 1, 2, \ldots, \bar{\tau}\}$ whose transition probability matrix $\Lambda^*(k) = [\lambda_{ij}^*(k)]$ is defined by

$$\lambda_{ij}^*(k) = \begin{cases} \sum_{l \in \mathcal{M}} \pi_l(k)\lambda_{lj}, & j \leq i; \\ \sum_{l_1 \in \mathcal{M}, l_2 \geq 1} \pi_{l_1}(k)\lambda_{l_1l_2}, & j = i + 1; \\ 0, & \text{otherwise}. \end{cases} \quad (8)$$

where $\pi_j(k) = \sum_{i \in \mathcal{M}} p_i \lambda_{ij}^{(k)}$ and $\lambda_{ij}^{(k)}$ is the $k$-step transition probability of $\tau_k$ from state $i$ to $j$.

**Proof:** The comparison rule in (7) implies that the probability event $\{\tau_k^* = i\} \in \sigma(\tau_k, \tau_{k-1}, \ldots, \tau_1, \tau_0)$. Thus it is readily concluded that $\tau_k^*$ is also a Markov chain since $\tau_k$ as a Markov chain evolves independently. It is obvious that $\tau_k^*$ can not be finite and thus its state space is $\mathcal{M}^* = \{0, 1, 2, \ldots, \bar{\tau}\}$. Furthermore, noticing $\{\tau_k^* = i\} = \{\tau_{k-1}^* = i - 1, \tau_k > i - 1 \} \cup \{\tau_{k-1}^* = i, \tau_k = i\}$ we have

1) If $j \leq i$,

$$P(\tau_{k+1}^* = j|\tau_k^* = i) = \sum_{l_1 \in \mathcal{M}, l_2 \geq 1} \pi_{l_1}(k)\lambda_{lj}, \quad (9)$$

2) If $j = i + 1$,

$$P(\tau_{k+1}^* = j|\tau_k^* = i) = \sum_{l_1 \in \mathcal{M}, l_2 \geq 1} \pi_{l_1}(k)\lambda_{l_1l_2}, \quad (10)$$

which completes the proof.

The following well-known result for homogeneous ergodic Markov chains is required for the stochastic stability analysis in this paper.

**Lemma 2 ([17]):** For the homogeneous ergodic Markov chain $\{\tau_k; k = 0, 1, \ldots\}$ with any initial distribution, there exists a limit probability distribution $\pi = \{\pi_i; \pi_i > 0, i \in \mathcal{M}\}$ such that for each $j \in \mathcal{M}$,

$$\sum_{i \in \mathcal{M}} \lambda_{ij}\pi_i = \pi_j, \quad \sum_{i \in \mathcal{M}} \pi_i = 1 \quad (10)$$

and

$$|\pi_i(k) - \pi_i| \leq \eta \xi^k \quad (10)$$

for some $\eta > 0$ and $0 < \xi < 1$.

**Proposition 1:** For $N_1$ that is large enough and some nonzero $\eta^*$ the following inequality holds

$$|\lambda_{ij}^* - \lambda_{ij}^*| \leq \eta^* \xi^k, \quad k > N_1 \quad (11)$$

where $\Lambda^* = [\lambda_{ij}^*]$. 

**Proof:** It can be readily obtained from (8), (10) and (12).

**B. Stochastic stability**

The stochastic stability theorem can now be obtained as follows.

![Packet-Based Controller](image1)

Fig. 2. Packet-based control for networked control systems
**Theorem 1:** The closed-loop system in (6) is stochastically stable if and only if there exists \( P(i) > 0, \forall i \in \mathcal{M}^* \) such that the following \((\tau + 1)\) LMI holds

\[
L(i) = \sum_{j \in \mathcal{M}^*} \lambda_j \Xi^T(j)P(j)\Xi(j) - P(i) < 0, \forall i \in \mathcal{M}^* \quad (13)
\]

**Proof:** Sufficiency. For the closed-loop system in (6), consider the following quadratic function

\[
V(X(k), k) = X^T(k)P(\tau^*_k)X(k)
\]

(14)

We have

\[
E\{\Delta V(X(k), k)\} = E\{X^T(k + 1)P(\tau^*_k)X(k + 1) - X^T(k)P(\tau^*_k)X(k)\}
\]

(13) yields

\[
X^T(k)[\sum_{j \in \mathcal{M}^*} \lambda_j \Xi^T(j)P(j)\Xi(j) - P(i)]X(k)
\]

\[
\leq -\lambda_{\text{min}}(-L(i))X^T(k)X(k)
\]

\[
\leq -\beta ||X(k)||^2
\]

where \( \beta = \inf\{\lambda_{\text{min}}(-L(i)): i \in \mathcal{M}^*\} > 0 \). Thus for \( k > N_1 \),

\[
E\{\Delta V(X(k), k)\} \leq -\beta ||X(k)||^2
\]

(15)

where \( \beta^* = \beta - \alpha^* \xi^{N_2 + 1} > 0 \). Summing from \( N_2 \) to \( N > N_2 \) we obtain

\[
E\{\sum_{k = N_2}^{N} ||X(k)||^2\} \leq \frac{1}{\beta^*}E\{V(X(N_2), N_2) - V(X(N + 1), N + 1)\}
\]

(16)

which implies that

\[
E\{\sum_{k = 0}^{N} ||X(k)||^2\} \leq \frac{1}{\beta^*}E\{V(X(N_2), N_2)\} + E\{\sum_{k = 0}^{N_2 - 1} ||X(k)||^2\}
\]

which proves the stochastic stability of the closed-loop system in (6).

Necessity. Suppose the closed-loop system in (6) is stochastically stable, that is,

\[
E\{\sum_{k = 0}^{\infty} ||X(k)||^2|X_0, \tau^*_n \} < X_0^T W X_0
\]

(17)

Define

\[
X^T(n)P(N - n, \tau^*_n)X(n) = \sum_{k = N}^{N} X^T(k)Q(\tau^*_k)X(k)|X_n, \tau^*_n
\]

(18)

with \( Q(\tau^*_n) > 0 \). It is noticed that \( X^T(n)P(N - n, \tau^*_n)X(n) \) is upper bounded from (17) and monotonically non-decreasing as \( N \) increases since \( Q(\tau^*_n) > 0 \). Therefore its limit exists which is denoted by

\[
X^T(n)P(i)X(n) = \lim_{N \to \infty} X^T(n)P(N - n, \tau^*_n)X(n)
\]

(19)

Since (19) is valid for any \( X(n) \), we obtain

\[
P(i) = \lim_{N \to \infty} P(N - n, \tau^*_n = i) > 0
\]

(20)

Now consider

\[
E\{X^T(n)P(N + 1 - n, \tau^*_n)X(n) - X^T(n)P(N - n, \tau^*_n)X(n + 1)|X_n, \tau^*_n = i\}
\]

\[
= X^T(n)P(n - i) - \sum_{j \in \mathcal{M}^*} \lambda_j^*(n + 1)\Xi^T(j)P(N - n - 1, j)\Xi(j)|X(n)
\]

(21)

\[
\leq X^T(n)Q(i)X(n)
\]

(22)

Letting \( N \to \infty \) and then \( n \to \infty \) we obtain

\[
P(i) - \sum_{j \in \mathcal{M}^*} \lambda_j^*(i)\Xi^T(j)P(i)\Xi(j) > 0
\]

which completes the proof.  

\[\square\]

**IV. A Numerical Example**

Consider the example in [15] where the system matrices are

\[
A = \begin{pmatrix}
1.0000 & 0.1000 & -0.0166 & -0.0005 \\
0 & 1.0000 & -0.3374 & -0.0166 \\
0 & 0 & 1.0996 & 0.1033 \\
0 & 0 & 2.0247 & 1.0996
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
0.0045 \\
0.0896 \\
-0.0068 \\
-0.1377
\end{pmatrix}
\]

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This system is open-loop unstable with the eigenvalues at 1, 1, 1.5569 and 0.6423. Let \( \tau_k \in M = \{0, 1, 2, 3, 4, \infty\} \) with the following transition probability matrix,

\[
\Lambda = \begin{pmatrix}
0.1 & 0.2 & 0.2 & 0.3 & 0.2 & 0 \\
0.2 & 0.2 & 0.2 & 0.1 & 0.3 & 0 \\
0.24 & 0.06 & 0.48 & 0.12 & 0.1 & 0 \\
0.15 & 0.25 & 0.3 & 0.15 & 0.1 & 0.05 \\
0.3 & 0.3 & 0.2 & 0.1 & 0.1 & 0 \\
0.3 & 0.3 & 0.15 & 0.15 & 0.1 & 0 \\
\end{pmatrix}.
\]

The limit distribution of the above ergodic Markov chain can be simply obtained as in Lemma 2,

\[
\pi = (0.1982 \ 0.1814 \ 0.3000 \ 0.1738 \ 0.1198 \ 0.0268).
\]

\( \Lambda^* \) in Proposition 1 can then be calculated by (12) as

\[
\Lambda^* = \begin{pmatrix}
0.1982 & 0.8018 & 0 & 0 & 0 \\
0.2224 & 0.1767 & 0.6008 & 0 & 0 \\
0.2290 & 0.1699 & 0.3612 & 0.2398 & 0 \\
0.2186 & 0.2729 & 0.2501 & 0.1313 & 0.1271 \\
0.3000 & 0.3000 & 0.1909 & 0.1091 & 0.1000 \\
\end{pmatrix}.
\]

The packet-based controller is designed as follows, with which the closed-loop system can be shown to be stable by Theorem 1,

\[
K(0) = (0.5292 \ 0.6489 \ 22.4115 \ 2.8205) \\
K(1) = (0.3792 \ 0.8912 \ 20.2425 \ 5.3681) \\
K(2) = (0.0499 \ 0.4266 \ 15.6574 \ 5.7322) \\
K(3) = (-0.4400 \ -0.3003 \ 9.2976 \ 5.0540) \\
K(4) = (-0.8400 \ -1.3422 \ 2.7723 \ 2.9173).
\]

The state trajectories of the closed-loop system under the packet-based controller are shown in Fig. 3 with the initial states \( x(-3) = x(-2) = x(-1) = x(0) = [0 \ 0.1 \ 0 - 0.1]^T \), which illustrates the stochastic stability of the closed-loop system.

V. Conclusion

A packet-based control approach is proposed for NCSs to deal with network-induced delay, data packet dropout and data packet disorder simultaneously. The sufficient and necessary condition for the stochastic stability of the closed-loop system is obtained by modeling the communication constraints as a homogeneous ergodic Markov chain, which is then validated by a numerical example.

References


