COMMUTATIVE HOPF STRUCTURES OVER A LOOP

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ABSTRACT. Let k be an algebraically closed field of characteristic p > 0. For a loop \circlearrowleft , denote its path coalgebra by $k \circlearrowright$. In this paper, all the finite-dimensional commutative Hopf algebras over the sub coalgebras of $k \circlearrowright$ are given. As a direct consequence, all the commutative infinitesimal groups \mathcal{G} with dim_kLie(\mathcal{G}) = 1 are classified.

Keywords Path coalgebra, Unipotent group, Frobenius map2000 MR Subject Classification 16W30, 14L15

1. INTRODUCTION

This paper is concerned with the quiver realization of finite-dimensional cocommutative Hopf algebras. As is well-known, any such algebra can be viewed as the group algebra of a finite algebraic k-group \mathcal{G} . Considerable attention has been received by these algebraic groups.

Quivers are oriented diagrams consisting of vertices and arrows [2]. Due to the well-known theorem of Gabriel given in the early 1970s, these combinatorial stuffs make the abstract elementary algebras and their representations visible. This point of view has since then played a central role and is generally accepted as the starting point in the modern representation theory of associative algebras. Naturally there is a strong desire to apply this handy quiver tool to other algebraic structures. Such idea for Hopf algebras appeared explicitly in [5, 6, 8, 11] and was showed to be very effective in dealing with the structures of finite-dimensional pointed (or dually, basic) Hopf algebras when the characteristic of the base field is 0 [8, 3, 16, 15, 14, 19].

Comparing to the characteristic 0 case, there is hardly any work dealing with the positive characteristic case by using quiver methods, see however a recent work of Cibils, Lauve and Witherspoon [7]. One main difficulty in the positive characteristic case is that general pointed Hopf algebras are not generated by group-likes and skew-primitive elements. While in the characteristic 0 case, the well-known Andruskiewitsch-Schneider Conjecture [1] claims that all finite-dimensional pointed Hopf algebras are indeed generated by their group-likes and skew-primitive elements.

This paper can be considered as our first try to apply quiver methods to the category of pointed Hopf algebras over an algebraically closed field k with characteristic p > 0, especially to finite-dimensional cocommutative Hopf algebras over k or equivalently to the category of finite algebraic kgroups. One can show that the connected component of a finite-dimensional cocommutative Hopf algebra can be embedded into the path coalgebra of a multi-loop quiver (see Corollary 2.2). So as the first step, one can analyze the minimal case, that is, Hopf structures over the path coalgebra of the one-loop quiver. This is exactly what we do in this paper. The main result of the paper is the following theorem (see Theorem 5.1).

Theorem 1.1. Let $n \in \mathbb{N}^+$, any commutative Hopf structure H over $k \oplus_{p^n}$ is isomorphic to a L(n,d) for some d.

See Section 5 for the definition of L(n, d). As a direct consequence of this theorem, all commutative infinitesimal groups \mathcal{G} with $\dim_k \operatorname{Lie}(\mathcal{G}) = 1$ are determined.

The paper is organized as follows. All needed knowledge about path coalgebras is summarized in Section 2. Moreover, the uniserial property of the Hopf structures over $k \circlearrowleft$ is also established in this section. For later use, all endomorphisms of the path coalgebra $k \circlearrowright$ are given in Section 3. As a key step, we need to grasp all possible Hopf structures over $k \circlearrowright_p$ at first and this task is finished in Section 4. In addition, we also show that the property of a Hopf structure over $k \circlearrowright_{p^n}$ is almost determined by that of its restriction to $k \circlearrowright_p$. Combining the work of Farnsteiner-Röhrle-Voigt on unipotent group of complexity 1 [10], the proof of Theorem 1.1 is given in Section 5 at last.

Throughout the paper we will be working over an algebraically closed field k of characteristic p > 0. We freely use the results, notations, and conventions of [18].

2. PATH COALGEBRAS

2.1. Given a quiver $Q = (Q_0, Q_1)$ with Q_0 the set of vertices and Q_1 the set of arrows, denote by kQ the k-space with basis the set of all paths in Q. Over kQ, there is a natural coalgebra structure defined as follows. For $\alpha \in Q_1$, let $s(\alpha)$ and $t(\alpha)$ denote respectively the starting and ending vertex of α . Then comultiplication Δ is given by

$$\Delta(p) = \alpha_l \cdots \alpha_1 \otimes s(\alpha_1) + \sum_{i=1}^{l-1} \alpha_l \cdots \alpha_{i+1} \otimes \alpha_i \cdots \alpha_1 + t(\alpha_l) \otimes \alpha_l \cdots \alpha_1$$

for each path $p = \alpha_l \cdots \alpha_1$ with each $\alpha_i \in Q_1$; and the counit ε is defined to be $\varepsilon(p) = 0$ for $l \ge 1$ and 1 if l = 0 (l = 0 means p is a vertex). This is a coradically graded pointed coalgebra and we also denote it by kQ. Like the path algebras case, the path coalgebras serve as the cofree pointed coalgebras. In fact, Chin and Montgomery showed the following result [4]:

 $\mathbf{2}$

Lemma 2.1. Let C be a pointed cocalgebra, then there exists a unique quiver Q(C) such that C can be embedded into the path coalgebra kQ(C) as a large sub coalgebra.

This unique quiver Q(C) is called the *dual Gabriel quiver* of C. Here "large" means that C contains all group-like elements $Q(C)_0$ and all skew-primitive elements of kQ(C). Note that the skew-primitive elements are indeed corresponding to paths of length 1, i.e., arrows. Now the following conclusion is clear.

Corollary 2.2. Let C be an irreducible cocommutative pointed coalgebra, then its dual Gabriel quiver Q(C) has only one vertex.

A natural question is when there is a Hopf structure on a path coalgebra. We will see not every quiver can serve as the dual Gabriel quiver of a pointed Hopf algebra and those do are called *Hopf quivers* by Cibils and Rosso [8]. Recall that a *ramification data* r of a group G is a positive central element of the group ring of G: let C be the set of conjugacy classes, $r = \sum_{C \in Cr_C} Cr_C C$ is a formal sum with non-negative integer coefficients.

Definition 2.3. Let G be a group and r a ramification data. The corresponding Hopf quiver Q(G,r) has set of vertices the elements of G and has r_C arrows from x to cx for each $x \in G$ and $c \in C$.

One of the main results in [8] states that there is a graded Hopf algebra structure on the path coalgebra kQ if and only if Q is a Hopf quiver. In this case, kQ_0 is a group algebra and kQ_1 is a kQ_0 -Hopf bimodule. Moreover, the product rule of paths can be displayed as follows.

Let p be a path of length l. An n-thin split of it is a sequence (p_1, \dots, p_n) of vertices and arrows such that the concatenation $p_n \cdots p_1$ is exactly p. These n-thin splits are in one-to-one correspondence with the n-sequences of (n - l) 0's and l 1's. Denote the set of such sequences by D_l^n . Clearly $|D_l^n| = \binom{n}{l}$. For $d = (d_1, \dots, d_n) \in D_l^n$, the corresponding n-thin split is written as $dp = ((dp)_1, \dots, (dp)_n)$, in which $(dp)_i$ is a vertex if $d_i = 0$ and an arrow if $d_i = 1$.

Let $\alpha = a_m \cdots a_1$ and $\beta = b_n \cdots b_1$ be paths of lengths m and n respectively. Let $d \in D_m^{m+n}$ and $\bar{d} \in D_n^{m+n}$ the complement sequence which is obtained from d by replacing each 0 by 1 and each 1 by 0. Define an element in kQ_{m+n} ,

$$(\alpha \cdot \beta)_d = [(d\alpha)_{m+n} \cdot (d\beta)_{m+n}] \cdots [(d\alpha)_1 \cdot (d\beta)_1],$$

where $[(d\alpha)_i.(\bar{d}\beta)_i]$ is understood as the action of kQ_0 -Hopf bimodule on kQ_1 and these terms in different brackets are put together by cotensor product, or equivalently concatenation. In these notations, the formula of the product of α and β is given as follows (see pages 245-246 in [8]):

(2.1)
$$\alpha \cdot \beta = \sum_{d \in D_m^{m+n}} (\alpha \cdot \beta)_d \; .$$

2.2. In this paper, we only consider the very simple Hopf quiver, a loop \circlearrowright . By setting G := e and r := e, one can see that a loop is just the Hopf quiver Q(G, r). For any natural number n, denote the unique path of length n by α_n . Since the group G is trivial now, the Hopf bimodule action is trivial too. Thus the product rule over $k \circlearrowright$ is very simple. That is,

(2.2)
$$\alpha_n \cdot \alpha_m = \begin{pmatrix} m+n \\ n \end{pmatrix} \alpha_{m+n}.$$

This is indeed the familiar Hopf algebra $(k[x])^{\circ}$, the finite dual of k[x]. Sometimes, we denote this Hopf structure still by $k \circlearrowleft$ and one can discriminate the exact meaning by context. Note that this is a graded Hopf algebra with length grading.

For a quiver Q, define $kQ_d := \bigoplus_{i=0}^{d-1} kQ(i)$ where Q(i) is the set of all paths of length i in Q. Clearly, for any $i \ge 0$, $k \bigcirc_{p^i}$ is a sub Hopf algebra of $k \oslash$.

Lemma 2.4. Let H be a finite-dimensional sub Hopf algebra of $k \, \circlearrowright$, then $H \cong k \, \circlearrowright_{p^i}$ for some $i \ge 0$.

Proof. This is follows directly from the known fact that $k[x]/(x^{p^i})$ are all Hopf quotients of k[x].

Van Oystaeyen and Zhang proved the dual Gabriel Theorem for coradically graded pointed Hopf algebras (Theorem 4.5 in [19]):

Lemma 2.5. Let H be a coradically graded pointed Hopf algebra, then its dual Gabriel quiver Q(H) is a Hopf quiver and there is a Hopf embedding

$$H \hookrightarrow kQ(H).$$

Now let $C \subset k \circlearrowleft$ be a finite-dimensional large sub coalgebra of $k \circlearrowright$ and assume there is a Hopf structure H(C) on C.

Proposition 2.6. With notations and the assumption as above, there is a natural number i such that as a coalgebra,

$$C \cong k \circlearrowleft_{p^i}$$
.

Proof. At first, we know that H(C) is a pointed Hopf algebra. Denote its coradical filtration by $\{H(C)_n\}_{n=0}^{\infty}$. Define

$$\operatorname{gr}(H(C)) = H(C)_0 \oplus H(C)_1 / H(C)_0 \oplus H(C)_2 / H(C)_1 \oplus \cdots \cdots$$

as the corresponding coradically graded version. Then gr(H(C)) inherits from H(C) a coradically graded Hopf algebra structure (see e.g. [18]). By Lemma 2.5, $\operatorname{gr}(H(C))$ is a sub Hopf algebra of $k \circlearrowleft$. Thus Lemma 2.4 implies what we want.

Thus, our next aim is to give all possible Hopf structures (not necessarily coradically graded) over the coalgebra $k \circ _{n^i}$.

For any rational number a, denote by [a] the biggest integer which is not bigger than a.

Lemma 2.7. For any positive integers
$$m, n, \binom{m+n}{n} = 0$$
 if and only if

$$\sum_{i \ge 1} \left[\frac{m+n}{p^i}\right] > \sum_{i \ge 1} \left[\frac{m}{p^i}\right] + \sum_{i \ge 1} \left[\frac{n}{p^i}\right].$$
Proof Clear

Proof. Clear.

We call a Hopf algebra is *uniserial* if the set of its sub Hopf algebras forms a totally ordered set under the containing relation.

Convention. Let C and D be two coalgebras and assume that C is a sub coalgebra of D. If there is a Hopf structure H(D) over D, then we use the notion H(C) to denote the restriction, if applicable, of the structure of H(D) to C.

Proposition 2.8. Let n be a positive natural number and assume that there is a Hopf structure $H(k \oplus_{p^n})$ over $k \oplus_{p^n}$. Then $H(k \oplus_{p^n})$ is a uniserial Hopf algebra with the composition series

$$k \subset H(k \oslash_{p^1}) \subset \cdots \subset H(k \oslash_{p^{i-1}}) \subset H(k \oslash_{p^i}) \subset \cdots \subset H(k \oslash_{p^n}).$$

Proof. By Proposition 2.6, it is enough to show that $H(k \circ_{p^i})$ for $i \leq n$ are sub Hopf algebras. Thus, it is enough to show that they are closed under the multiplication. But this is the direct consequence of the product rule (2.1) and Lemma 2.7.

3. Endomorphisms of $k \circlearrowleft$

For later use, we characterize all the possible endomorphisms of the path coalgebra $k \circlearrowleft$ in this section.

Theorem 3.1. (i) Let $f : k \circlearrowleft \to k \circlearrowright$ be a coalgebra map, then there are $\{\lambda_i \in k | i \in \mathbb{N}^+\}$ such that

$$f(\alpha_n) = \sum_{r=1}^n (\sum_{n_1 + \dots + n_r = n} \lambda_{n_1} \cdots \lambda_{n_r}) \alpha_r$$

for any n.

(ii) All coalgebra endomorphisms of $k \bigcirc$ are precisely given in this way.

Proof. (i) Let's find such λ'_i 's. Since f is a coalgebra map, f(1) is a grouplike element and $f(\alpha_1)$ is a primitive element. Thus f(1) = 1 and there is a $\lambda_1 \in k$ such that $f(\alpha_1) = \lambda_1 \alpha_1$ since $k\alpha_1$ are all primitive elements. Suppose we have found $\{\lambda_1, \ldots, \lambda_n\}$ and let's find λ_{n+1} . By f is a coalgebra map,

$$\Delta(f(\alpha_{n+1}) - \sum_{r=2}^{n+1} (\sum_{n_1 + \dots + n_r = n+1} \lambda_{n_1} \cdots \lambda_{n_r}) \alpha_r)$$

$$= f(\alpha_{n+1}) \otimes 1 + 1 \otimes f(\alpha_{n+1}) + \sum_{i=1}^{n} f(\alpha_i) \otimes f(\alpha_{n+1-i}) - \sum_{r=2}^{n+1} (\sum_{n_1 + \dots + n_r = n+1}^{n} \lambda_{n_1} \dots \lambda_{n_r}) \sum_{s+t=r}^{n} \alpha_s \otimes \alpha_t)$$

$$= f(\alpha_{n+1}) \otimes 1 + 1 \otimes f(\alpha_{n+1}) + \sum_{i=1}^{n} (\sum_{s=1}^{i} (\sum_{n_1 + \dots + n_s = i}^{n} \lambda_{n_1} \dots \lambda_{n_s}) \alpha_s$$

$$\otimes \sum_{t=1}^{n+1-i} (\sum_{m_1 + \dots + m_t = n+1-i}^{n+1-i} \lambda_{m_1} \dots \lambda_{m_t}) \alpha_t) - \sum_{s+t=r}^{n+1} (\sum_{n_1 + \dots + n_r = n+1}^{n+1-i} \lambda_{n_1} \dots \lambda_{n_r}) \sum_{s+t=r}^{n+1} \alpha_s \otimes \alpha_t).$$

Replace $\lambda_{m_1} \cdots \lambda_{m_t}$ by $\lambda_{n_{s+1}} \cdots \lambda_{n_{s+t}}$ and set r = s + t, one can find that

$$\sum_{i=1}^{n} \left(\sum_{s=1}^{i} \left(\sum_{n_1+\cdots n_s=i} \lambda_{n_1}\cdots \lambda_{n_s}\right) \alpha_s \otimes \sum_{t=1}^{n+1-i} \left(\sum_{m_1+\cdots m_t=n+1-i} \lambda_{m_1}\cdots \lambda_{m_t}\right) \alpha_t\right)$$

$$=\sum_{r=2}^{n+1} (\sum_{n_1+\cdots n_r=n+1} \lambda_{n_1} \cdots \lambda_{n_r}) \sum_{s+t=r, s\neq 0\neq t} \alpha_s \otimes \alpha_t).$$
(*)

Thus let $y := f(\alpha_{n+1}) - \sum_{r=2}^{n+1} (\sum_{n_1 + \cdots + n_r = n+1} \lambda_{n_1} \cdots \lambda_{n_r}) \alpha_r$, then $\Delta(y) = y \otimes 1 + 1 \otimes y$. Thus $f(\alpha_{n+1}) - \sum_{r=2}^{n+1} (\sum_{n_1 + \cdots + n_r = n+1} \lambda_{n_1} \cdots \lambda_{n_r}) \alpha_r$ is a primitive element and so there is a λ_{n+1} such that $y = \lambda_{n+1} \alpha_1$. Equivalently,

$$f(\alpha_{n+1}) = \sum_{r=1}^{n+1} (\sum_{n_1 + \dots + n_r = n+1} \lambda_{n_1} \cdots \lambda_{n_r}) \alpha_r.$$

(ii) By (i), it is enough to show that for any $\{\lambda_i \in k | i \in \mathbb{N}^+\}$ and the linear map f defined by $f(\alpha_n) = \sum_{r=1}^n (\sum_{n_1 + \cdots + n_r = n} \lambda_{n_1} \cdots \lambda_{n_r}) \alpha_r$ is indeed

a coalgebra map. In fact,

$$(f \otimes f)\Delta(\alpha_n) = f(\alpha_n) \otimes 1 + 1 \otimes f(\alpha_n) + \sum_{i=1}^{n-1} f(\alpha_i) \otimes f(\alpha_{n-i})$$

= $f(\alpha_n) \otimes 1 + 1 \otimes f(\alpha_n) + \sum_{i=1}^{n-1} (\sum_{s=1}^{i} (\sum_{n_1 + \dots + n_s = i} \lambda_{n_1} \cdots \lambda_{n_s}) \alpha_s$
 $\otimes \sum_{t=1}^{n-i} (\sum_{m_1 + \dots + m_t = n-i} \lambda_{m_1} \cdots \lambda_{m_t}) \alpha_t).$

While

$$\Delta(f(\alpha_n)) = \Delta(\sum_{r=1}^n (\sum_{n_1 + \dots + n_r = n} \lambda_{n_1} \cdots \lambda_{n_r}) \alpha_r)$$

=
$$\sum_{r=1}^n (\sum_{n_1 + \dots + n_r = n} \lambda_{n_1} \cdots \lambda_{n_r}) \sum_{s+t=r} (\alpha_s \otimes \alpha_t)$$

=
$$f(\alpha_n) \otimes 1 + 1 \otimes f(\alpha_n) + \sum_{r=2}^n (\sum_{n_1 + \dots + n_r = n} \lambda_{n_1} \cdots \lambda_{n_r})$$

$$\times \sum_{s+t=r, s \neq 0 \neq t} (\alpha_s \otimes \alpha_t).$$

Then equation (\star) implies that $\Delta(f(\alpha_n)) = (f \otimes f) \Delta(\alpha_n)$.

By the proof, we know that if $f : k \circlearrowleft \to k \circlearrowright$ is a coalgebra map, then $f(\alpha_1) = \lambda_1 \alpha_1$ for some $\lambda_1 \in k$. The next result is to provide a criterion to determine when f is indeed an automorphism.

Proposition 3.2. With notions as the above, f is an automorphism if and only if $\lambda_1 \neq 0$.

Proof. By Theorem 3.1, $f(k \circ n) \subseteq k \circ n$ for any $n \in \mathbb{N}^+$ and thus $f|_{k \circ n}$ is a coalgebra endomorphism of $k \circ n$. By $\lambda_1 \neq 0$, $f|_{k \circ 2}$ is injective and so $f|_{k \circ n}$ is injective by Heynaman-Radford's result [12]. Since $\dim_k k \circ n < \infty$, $f|_{k \circ n}$ is bijective. This indeed implies that f is an automorphism of $k \circ N$. The converse is obvious since one always has $f(\alpha_1) = \lambda_1 \alpha_1$.

Corollary 3.3. For any natural numbers m > n > 0 and assume that f is an automorphism of the coalgebra $k \circ n$, then f can be extended to be automorphisms of the coalgebra $k \circ n$.

Proof. By the proof of Theorem 3.1, there are $\{\lambda_1, \ldots, \lambda_{n-1}\}$ such that

$$f(\alpha_i) = \sum_{r=1}^{i} (\sum_{n_1 + \dots + n_r = n} \lambda_{n_1} \cdots \lambda_{n_r}) \alpha_r$$

for $1 \leq i \leq n-1$. By setting $\lambda_j = 0$ for all $j \geq n$ and by Theorem 3.1, if we define a map $F: k \circlearrowleft \to k \circlearrowright$ through

$$F(\alpha_l) = \sum_{r=1}^{l} (\sum_{n_1 + \dots + n_r = l} \lambda_{n_1} \cdots \lambda_{n_r}) \alpha_r$$

for any natural number l, then F is a coalgebra endomorphism of $k \circlearrowleft$. Clearly, $F|_{k \circlearrowright n} = f$. Owning to Proposition 3.2, F is an automorphism. Theorem 3.1 deduces that $F(k \circlearrowright m) \subseteq k \circlearrowright m$ and thus $F|_{k \circlearrowright m}$ is the extension of f to $k \circlearrowright m$.

4. HOPF STRUCTURES ON $k \bigcirc_p$

The following result seems well-known and we write its proof out for completeness.

Lemma 4.1. Let H be a Hopf structure over $k riangle_{p^1}$, then as a Hopf algebra H is isomorphic to either $(k\mathbb{Z}_p)^*$, dual of the group algebra $k\mathbb{Z}_p$, or $k[x]/(x^p)$.

Proof. At first, it is not hard to see that H is generated by $\alpha_{p^0} = \alpha_1$. Consider the element α_1^p . For it, we have

$$\Delta(\alpha_1^p) = \Delta(\alpha_1)^p = (1 \otimes \alpha_1 + \alpha_1 \otimes 1)^p = 1 \otimes \alpha_1^p + \alpha_1^p \otimes 1.$$

Thus α_1^p is a primitive element. Since the space spanned by α_1 are all primitive elements in the coalgebra $k \circ _{p^1}$, there is a $\lambda \in k$ such that

$$\alpha_1^p = \lambda \alpha_1.$$

If $\lambda = 0$, then $H \cong k[x]/(x^p)$. If $\lambda \neq 0$, take λ' to be a solution of the equation $\lambda x^p - x = 0$. Then

$$(\lambda'\alpha_1)^p = \lambda\lambda'^p\alpha_1^p = \lambda'\alpha_1.$$

In one word, if $\lambda \neq 0$, we can always assume that $\lambda = 1$ and thus $H \cong (k\mathbb{Z}_p)^*$.

We find that the property of $H(k \circ p^n)$ is largely determined by that of $H(k \circ p^1)$.

Proposition 4.2. Let *n* be a positive integer and assume that there is a commutative Hopf structure $H(k \oplus_{p^n})$ over $k \oplus_{p^n}$. If $H(k \oplus_{p^1}) \cong (k\mathbb{Z}_p)^*$, then

$$H(k \circ p^n) \cong (k\mathbb{Z}_{p^n})^*.$$

Proof. Claim. Up to a Hopf isomorphism, $\alpha_l^p = \alpha_l$ for $0 < l < p^n$. We prove this fact by using induction on l. If l = 1, this is just assumption. Assume that $\alpha_l^p = \alpha_l$ for $l \leq m - 1$, let's prove that $\alpha_m^p = \alpha_m$. By the

definition of path coalgebra and the assumption of commutativity, we always have

$$\Delta(\alpha_m^p) = (1 \otimes \alpha_m + \alpha_m \otimes 1 + \sum_{0 < l < m} \alpha_l \otimes \alpha_{m-l})^p$$
$$= 1 \otimes \alpha_m^p + \alpha_m^p \otimes 1 + \sum_{0 < l < m} \alpha_l^p \otimes \alpha_{m-l}^p.$$

The inductive assumption implies that $\alpha_l^p = \alpha_l$ for l < m. Thus

$$\Delta(\alpha_m^p) = 1 \otimes \alpha_m^p + \alpha_m^p \otimes 1 + \sum_{0 < l < m} \alpha_l \otimes \alpha_{m-l}$$

and so

$$\Delta(\alpha_m^p - \alpha_m) = (\alpha_m^p - \alpha_m) \otimes 1 + 1 \otimes (\alpha_m^p - \alpha_m).$$

Therefore, there is $\lambda \in k$ such that $\alpha_m^p - \alpha_m = \lambda \alpha_1$. If $\lambda = 0$, done. If $\lambda \neq 0$, take ξ to be a solution of the equation $x^p - x + \lambda = 0$ and let $\alpha'_m := \alpha_m + \xi \alpha_1$. Clearly, the map

$$f: k \oslash_{m+1} \to k \oslash_{m+1}, \ \alpha_i \mapsto \alpha_i \text{ for } i \neq m; \ \alpha_m \mapsto \alpha'_m$$

is an automorphism of $k \circ _{m+1}$. Corollary 3.3 implies f can be extended to be an automorphism of $k \circ _{p^n}$. Since this automorphism is equivalent to choose a new basis of $k \circ _{p^n}$, f is an automorphism of Hopf algebras of $H(k \circ _{p^n})$. Now

$$(\alpha'_{p^m})^p = (\alpha_{p^m} + \xi \alpha_1)^p = \alpha_{p^m} + \lambda \alpha_1 + \xi^p \alpha_1 = \alpha_{p^m} + \xi \alpha_1 = \alpha'_{p^m}.$$

The claim is proved.

Construct the element

$$t := \prod_{0 < l < p^n} (1 - \alpha_l^{p-1}).$$

Thus the claim implies for any $0 < m < p^n$,

$$\alpha_m t = 0 = \varepsilon(\alpha_m)t.$$

This means that $t \in \int_{H}$, the set of integrals. Since $\varepsilon(t) = 1 \neq 0$, $H(k \circ p^{n})$ is a simisimple Hopf algebra (Theorem 2.2.1 in [18]). Thus $H(k \circ p^{n}) \cong (kG)^{*}$ for some finite abelian group. Since $H(k \circ p^{n})$ is cogenerated by α_{1} , kG is generated by one element. Thus $G \cong \mathbb{Z}_{p^{n}}$.

Remark 4.3. We would like to thank Professor A. Masuoka for pointing out to us that the above proposition can be deduced from Chapter IV, Section 3, 3.4 of [9] or Theorem 0.1 in [17].

Recall an affine algebraic group \mathcal{G} is *finite* if its coordinate ring $\mathcal{O}(\mathcal{G})$ is a finite-dimensional Hopf algebra. A finite algebraic group \mathcal{G} is called *infinitesimal* if $\mathcal{O}(\mathcal{G})$ is a local algebra. And, we call a finite algebraic group

 \mathcal{G} unipotent if its distribution algebra $\mathcal{H}(\mathcal{G}) := (\mathcal{O}(\mathcal{G}))^*$ is a local algebra. There is an equivalence between the category of finite algebraic groups and the category of finite-dimensional cocommutative Hopf algebras. Explicitly, sending finite algebraic group \mathcal{G} to $\mathcal{H}(\mathcal{G})$ gives us the equivalence. For more knowledge about affine algebraic groups, see [9, 20].

Now assume that there is a commutative Hopf structure $H(k \circ p^n)$ over $k \circ p^n$, then there is a finite algebraic group \mathcal{G}_{p^n} such that

$$\mathcal{H}(\mathcal{G}_{p^n}) = H(k \circlearrowleft_{p^n}).$$

Proposition 4.4. Keep the above notations. If $H(k \circ p) \cong k[x]/(x^p)$, then \mathcal{G}_{p^n} is an infinitesimal unipotent group.

Proof. Owning to the fact that $(k \circ p^n)^* \cong k[x]/(x^{p^n})$ as algebras, \mathcal{G}_{p^n} is infinitesimal. So in order to show \mathcal{G}_{p^n} is unipotent, it is enough to show that $H(k \circ p^n)$ is a local algebra.

By Proposition 2.8, $H(k \circ p^n)$ is uniserial with the composition series

$$k \subset H(k \circ _{p^1}) \subset \cdots \subset H(k \circ _{p^{i-1}}) \subset H(k \circ _{p^i}) \subset \cdots \subset H(k \circ _{p^n}).$$

Lemma 4.1 implies that either $H(k \circ p_i)/H(k \circ p_{i-1})^+ H(k \circ p_i) \cong (k\mathbb{Z}_p)^*$ or $H(k \circ p_i)/H(k \circ p_{i-1})^+ H(k \circ p_i) \cong k[x]/(x^p)$ for any $1 \leq i \leq n$. Here for a Hopf algebra H, H^+ stands for the kernel of $\varepsilon : H \to k$. To show $H(k \circ p_i)$ is local, it is enough to show $H(k \circ p_i)/H(k \circ p_{i-1})^+ H(k \circ p_i) \cong k[x]/(x^p)$ for all $1 \leq i \leq n$ (In fact, if so then all non-trivial paths will be nilpotent).

Otherwise, there is an *i* such that $H(k \circ p_i)/H(k \circ p_{i-1})^+H(k \circ p_i) \cong (k\mathbb{Z}_p)^*$. Take such *i* as small as possible. By assumption, $i \ge 2$. Thus $H(k \circ p_i)/H(k \circ p_{i-1})^+H(k \circ p_i) \cong (k\mathbb{Z}_p)^*$ implies that

$$\alpha_{p^{i-1}}^p \equiv \alpha_{p^{i-1}} \mod k \circ _{p^{i-1}}.$$

And thus $\alpha_{p^{i-1}}^{p^i} \equiv \alpha_{p^{i-1}} \mod k \oslash_{p^{i-1}}$. Therefore there is an element $a \in k \oslash_{p^{i-1}}$ such that $\alpha_{p^{i-1}}^{p^{i-1}} = \alpha_{p^{i-1}} + a$. Since *i* is as small as possible, $H(k \oslash_{p^{i-1}})$ is local and all non-trivial paths in $k \oslash_{p^{i-1}}$ are nilpotent. More precisely, let α be a non-trivial path living in $k \oslash_{p^{i-1}}$, then $\alpha^{p^{i-1}} = 0$. Thus

$$\begin{aligned} \Delta(\alpha_{p^{i-1}}^{p^{i-1}}) &= (1 \otimes \alpha_{p^{i-1}} + \alpha_{p^{i-1}} \otimes 1 + \sum_{0 < l < p^{i-1}} \alpha_l \otimes \alpha_{p^{i-1}-l})^{p^{i-1}} \\ &= 1 \otimes \alpha_{p^{i-1}}^{p^{i-1}} + \alpha_{p^{i-1}}^{p^{i-1}} \otimes 1 + \sum_{0 < l < p^{i-1}} \alpha_l^{p^{i-1}} \otimes \alpha_{p^{i-1}-l}^{p^{i-1}} \\ &= 1 \otimes \alpha_{p^{i-1}}^{p^{i-1}} + \alpha_{p^{i-1}}^{p^{i-1}} \otimes 1. \end{aligned}$$

This implies that $\alpha_{p^{i-1}}^{p^{i-1}} = \alpha_{p^{i-1}} + a$ is a primitive element and so there is a $\lambda \in k$ such that $\alpha_{p^{i-1}} + a = \lambda \alpha_1$. Therefore, $\alpha_{p^{i-1}} \in k \bigcirc_{p^{i-1}}$ which is impossible.

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Combining Lemma 4.1, Propositions 4.2, 4.4 and 2.8, we get

Corollary 4.5. Let n be a positive integer and assume that there is a commutative Hopf structure $H(k \circ p^n)$ over $k \circ p^n$. Then either $H(k \circ p^n) \cong (k\mathbb{Z}_{p^n})^*$ or $H(k \circ p^n)$ is the distribution algebra of a uniserial infinitesimal unipotent commutative k-group.

5. CLASSIFICATION AND APPLICATION

Fix a positive integer n and consider the coalgebra $k \circ p^n$. Assume that there is a Hopf structure on $k \circ p^n$. Since its coradically graded version is generated by $\{\alpha_{p^i} | 1 \le i \le n-1\}$, it is also generated by $\{\alpha_{p^i} | 1 \le i \le n-1\}$. So in order to give the Hopf structure, it is enough to characterize the relations between $\{\alpha_{p^i} | 1 \le i \le n-1\}$.

For any $0 \le d \le n$, the Hopf algebra L(n,d) (it is indeed a Hopf algebra by the following theorem) is defined to be the Hopf algebra over $k \circ p^n$ with relations:

(5.1) $\alpha_{p^{i}}\alpha_{p^{j}} = \alpha_{p^{j}}\alpha_{p^{i}}, \text{ for } 0 \le i, j \le n-1;$

(5.2)
$$\alpha_{n^i}^p = 0, \text{ for } i < d;$$

(5.3)
$$\alpha_{p^i}^p = \alpha_{p^{i-d}}, \text{ for } i \ge d.$$

The main result of this section is the following.

Theorem 5.1. L(n,d) is a Hopf algebra and any commutative Hopf structure $H(k \oplus_{p^n})$ over $k \oplus_{p^n}$ is isomorphic to an L(n,d) for some d.

One of the main ingredients of the proof is the classification result given in [10]. Let's recall it. By $\mathcal{W} : \mathbb{M}_k \to \mathbb{M}_{\mathbb{Z}}$ we denote the affine commutative group scheme of *Witt vectors*. For any positive natural number m let $\mathcal{W}_m : \mathbb{M}_k \to \mathbb{M}_{\mathbb{Z}}$ be the affine commutative group scheme of *Witt vectors* of length m. Denote the Frobenius map and Verschiebung of \mathcal{W}_m by \mathcal{F} and \mathcal{V} respectively. For any finite commutative algebraic group \mathcal{G} , its Cartier dual is denoted by $\mathcal{D}(\mathcal{G})$. For details, see [9]. An infinitesimal unipotent commutative group \mathcal{U} is called \mathcal{V} -uniserial if Coker $\mathcal{V} \cong \text{Spec}_k(k[x]/(x^p))$. Likewise, a unipotent infinitesimal group \mathcal{U} is called \mathcal{F} -uniserial if Ker $\mathcal{F} \cong$ $\text{Spec}_k(k[x]/(x^p))$. Note that \mathcal{G} is \mathcal{V} -uniserial or \mathcal{F} -uniserial is equivalent to its distribution algebra $\mathcal{H}(\mathcal{G})$ is uniserial (see Lemma 2.5 in [10]).

Let $d, j, n \in \mathbb{N}$ and for $n \geq 1$, $d \geq 1$, we denote by $\mathcal{U}_{n,d}$ the kernel of the endomorphism $\mathcal{V}^d - \mathcal{F} : \mathcal{W}_m \to \mathcal{W}_m$ with m = n(d+1). Denote by $\mathcal{U}_{n,d}^j$ the intersection of $\mathcal{U}_{n,d}$ with the kernel of the endomorphism $\mathcal{V}^{(n-1)(d+1)+j}$: $\mathcal{W}_m \to \mathcal{W}_m$ for $1 \leq j \leq d$. The following is the main result of [10] (Theorem 1.2 in [10]). **Lemma 5.2.** The following gives a complete list of representatives of isomorphism classes of non-trivial uniserial infinitesimal unipotent commutative k-groups:

(i) $(\mathcal{W}_d)_1$; (ii) $\mathcal{U}_{n,d}$; (iii) $\mathcal{U}_{n,d}^j$;

(iv) $\mathcal{D}((\mathcal{W}_d)_1)$; (v) $\mathcal{D}(\mathcal{U}_{n,d})$; (vi) $\mathcal{D}(\mathcal{U}_{n,d}^j)$.

Moreover, the groups labeled (i)-(iii) are \mathcal{V} -uniserial and those in (iv)-(vi) are \mathcal{F} -uniserial.

Proof of the Theorem 5.1. At first, since $H(k \circ_{p^n})$ is commutative, there is a k-group \mathcal{G} such that $\mathcal{G} = \operatorname{Spec}_k(H(k \circ_{p^n}))$. So the Frobenius map \mathcal{F} and Verschiebung \mathcal{V} can be defined for $H(k \circ_{p^n})$ too. Let's see what they are. In order to explain our understanding, there is no harm to assume that both \mathcal{F} and \mathcal{V} are Hopf endomorphisms of $H(k \circ_{p^n})$ for simplicity since the path coalgebra can clearly be defined over \mathbb{Z} . By the definition of Frobenius map, we know that

$$\mathcal{F}: \ H(k \circlearrowleft_{p^n}) \to H(k \circlearrowright_{p^n}), \ \ \alpha_{p^i} \mapsto \alpha_{p^i}^p, \ \ \text{for} \ 0 < i < n.$$

Note that \mathcal{V} is just the dual map of Frobenius map of $\mathcal{D}(\mathcal{G})$. Since as an algebra we have $(k \circ p^n)^* \cong k[x]/(x^{p^n})$, the Frobenius map for $(k \circ p^n)^*$ is given by $x \mapsto x^p$. Note also that $\{\alpha_i | 0 \leq i \leq p^n - 1\}$ are the dual basis of $\{x^i | 0 \leq i \leq p^n - 1\}$. Thus \mathcal{V} is given by

$$\mathcal{V}: \ H(k \bigcirc_{p^n}) \to H(k \bigcirc_{p^n}), \ \alpha_{p^i} \mapsto \alpha_{p^{i-1}}, \ \text{ for } 0 < i < n.$$

Thus if $\operatorname{Spec}_k(H(k \circ p^n))$ is unipotent, then it is a \mathcal{V} -uniserial group.

According to Corollary 4.5, either $H(k \circ p^n) \cong (k\mathbb{Z}_{p^n})^*$ or $H(k \circ p^n)$ is the distribution algebra of a uniserial infinitesimal unipotent commutative k-group. If $H(k \circ p^n) \cong (k\mathbb{Z}_{p^n})^*$, then $H(k \circ p^n) \cong L(n,0)$. Otherwise, $H(k \circ p^n)$ is a local algebra which implies that $\operatorname{Spec}_k(H(k \circ p^n))$ is infinitesimal and thus a unipotent group. By the discussion above, $\operatorname{Spec}_k(H(k \circ p^n))$ is an infinitesimal unipotent \mathcal{V} -uniserial group. By Lemma 5.2, we have $\operatorname{Spec}_k(H(k \circ p^n)) \cong (\mathcal{W}_d)_1$ or $\operatorname{Spec}_k(H(k \circ p^n)) \cong \mathcal{U}_{m,d}$ or $\operatorname{Spec}_k(H(k \circ p^n))$ $)) \cong \mathcal{U}_{m,d}^j$ for some m, d, j. The first case implies that $H(k \circ p^n) \cong L(n, n)$. Let us analyze the last two cases. Recall that the coordinate ring of \mathcal{W}_n is $k[x_1, \ldots, x_n]$. If (d+1)|n (that is, we consider the second case), we have a Hopf epimorphism

$$\pi: k[x_1,\ldots,x_n] \twoheadrightarrow H(k \bigcirc_{p^n})$$

and the following commutative diagram

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By $\operatorname{Spec}_k(H(k \circ p^n)) \cong \mathcal{U}_{\frac{n}{d+1},d}$, $\dim_k \mathcal{O}(\mathcal{U}_{\frac{n}{d+1},d}) = p^n$. Therefore the above commutative diagram and the definitions of \mathcal{F}, \mathcal{V} for $k \circ p^n$ imply that $H(k \circ p^n)$ satisfies equations (5.1)-(5.3) automatically. By comparing the dimension, equations (5.1)-(5.3) are all the relations for $H(k \circ p^n)$. Thus $H(k \circ p^n) \cong L(n,d)$ with (d+1)|n. For the last case (that is, $\operatorname{Spec}_k(H(k \circ p^n)) \cong \mathcal{U}_{m,d}^j)$, the analysis is almost the same as the second case and the only point we need to say is that the condition "intersection with kernel of the endomorphism $\mathcal{V}^{(n-1)(d+1)+j}$ ", appearing in the definition of $\mathcal{U}_{m,d}^j$, is equivalent to the condition $(d+1) \nmid n$.

Of course, L(n, d) are all Hopf algebras now. In fact, the above discussions show that we have

$$L(n,0) \cong (k\mathbb{Z}_{p^n})^*, \ L(n,n) \cong k[x]/(x^{p^n}),$$

and $L(n,d) \cong \mathcal{O}(\mathcal{U}_{\frac{n}{d+1},d})$ in case (d+1)|n. If $(d+1) \nmid n$, then n = m(d+1)+jfor some m, j with 0 < j < d+1. The above discussions indicate that $L(n,d) \cong \mathcal{O}(\mathcal{U}_{m+1,d}^j)$. \Box

Corollary 5.3. Up to Hopf isomorphisms there are exactly n + 1 classes of non-isomorphic commutative Hopf structures on the coalgebra $k riangle_{p^n}$ for any natural number n.

As another direct consequence of this theorem, the commutative infinitesimal groups with 1-dimensional Lie algebras can be classified now.

Corollary 5.4. Let \mathcal{G} be a commutative infinitesimal group. If $\dim_k Lie(\mathcal{G}) = 1$, then $\mathcal{H}(\mathcal{G}) \cong L(n,d)$ for some n, d.

Proof. By dim_kLie(\mathcal{G}) = 1, the set of primitive elements of $\mathcal{H}(\mathcal{G})$ is 1dimensional. Note that $\mathcal{H}(\mathcal{G})$ is always pointed, $\mathcal{H}(\mathcal{G})$ can be embedded into the path coalgebra $k \circlearrowleft$ (Lemma 2.1). Thus there is a natural number nsuch that $\mathcal{H}(\mathcal{G})$ is a Hopf structure over $k \circlearrowright_{p^n}$ by Proposition 2.6. Thanks to Theorem 5.1, $\mathcal{H}(\mathcal{G}) \cong L(n, d)$ for some d.

Remark 5.5. (1) It is known that if we take $k := \mathbb{F}_p$ then the multiplication of the Witt vector group scheme indeed corresponds to the additive of the padic numbers. Theorem 5.1 gives us some hint that sometimes it is possible to explain the addition of the p-adic numbers through the comultiplication of path coalgebras.

(2) For any L(n,d), there is still one thing which is not clear to us. That is, we don't know how to give the expression of each path through generators although we can give in some special cases (see the example below).

(3) Not all Hopf structures over $k \, \bigcirc_{p^n}$ for some $n \ge 2$ are always commutative. In fact, set p = 2 and consider the associative algebra H generated by x, y with relations

$$xy - yx = x$$
, $x^2 = y^2 = 0$.

Define the comultiplication Δ , counit ε and the antipode through

$$\begin{split} \Delta(x) &= 1 \otimes x + x \otimes 1, \ \ \Delta(y) = 1 \otimes y + y \otimes 1 + x \otimes x \\ \varepsilon(x) &= \varepsilon(y) = 0, \ \ S(x) = -x, \ \ S(y) = -y. \end{split}$$

It is straightforward to show that H is indeed a Hopf algebra over the path coalgebra $k \circ _{p^2}$. Clearly, it is not commutative.

Example 5.6. For the Hopf algebra L(2, 1), one can see that up to a coalgebra automorphism

$$\alpha_{sp+t} = \frac{1}{s!t!} \alpha_p^s \alpha_1^t$$
, for $0 \le s \le p-1$, $0 \le t \le p-1$.

In fact, we can prove this by using induction on the lengths of pathes. If the length is 1, it is clear. Now assume it is true for the pathes with lengths not more than sp + t. Now we consider the case sp + t + 1. To show this case, begin with an observation at first. For any element p in k cito, one always have

$$\Delta(p) = \sum_{i=0}^{n} \alpha_i \otimes p_{(i)}$$

where $p_{(i)}$ are uniquely determined since $1, \alpha_1, \alpha_2, \ldots$ is a basis of $k \circlearrowleft$. For two elements p, q, the basic observation is, up to a coalgebra automorphism,

$$p = q$$
 if and only if $p_{(1)} = q_{(1)}$. (\star)

Now we consider the case sp + t + 1. If 0 < t < p - 1, we just need to show that $\alpha_{sp+t+1} = \frac{1}{s!(t+1)!} \alpha_p^s \alpha_1^{t+1}$. By (\star), it is enough to show that $\alpha_{sp+t} = (\frac{1}{s!(t+1)!} \alpha_p^s \alpha_1^{t+1})_{(1)}$. Note that by assumption $\frac{1}{s!(t+1)!} \alpha_p^s \alpha_1^{t+1} = \frac{1}{t+1} \alpha_{sp+t} \alpha_1$ and direct computation shows that $(\alpha_{sp+t}\alpha_1)_{(1)} = (t+1)\alpha_{sp+t}$.

If t = 0, we need show that $\alpha_{sp+1} = \frac{1}{s!} \alpha_p^s \alpha_1 = \alpha_{sp} \alpha_1$ by assumption. Clearly, $(\alpha_{sp+1})_{(1)} = \alpha_{sp}$ and $(\alpha_{sp}\alpha_1)_{(1)} = \alpha_{(s-1)p+(p-1)}\alpha_1 + \alpha_{sp}$. Note that in L(2,1), $\alpha_1^p = 0$ and so $\alpha_{(s-1)p+(p-1)}\alpha_1 = 0$. By (\star) again, $\alpha_{sp+1} = \frac{1}{s!} \alpha_p^s \alpha_1$.

If t = p - 1, the equality that we need check is $\alpha_{(s+1)p} = \frac{1}{(s+1)!} \alpha_p^{s+1}$. Also, computations show that $(\alpha_p^{s+1})_{(1)} = (s+1)\alpha_p^s \alpha_{p-1} = (s+1)\frac{1}{(p-1)!}\alpha_p^s \alpha_1^{p-1}$ and so $(\frac{1}{(s+1)!}\alpha_p^{s+1})_{(1)} = \frac{1}{s!(p-1)!}\alpha_p^s \alpha_1^{p-1}$. Meanwhile, $(\alpha_{(s+1)p})_{(1)} = \alpha_{sp+(p-1)} = \frac{1}{s!(p-1)!}\alpha_p^s \alpha_1^{p-1}$ by assumption. Using (\star) again, $\alpha_{(s+1)p} = \frac{1}{(s+1)!}\alpha_p^{s+1}$.

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