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ON THE CENTRE OF A TRIANGULATED CATEGORY

HENNING KRAUSE[1] AND YU YE[2]

[1] *Fakultät für Mathematik, Universität Bielefeld, 33501 Bielefeld, Germany* (hkrause@math.uni-bielefeld.de)
[2] *Department of Mathematics, University of Science and Technology of China, Hefei 230026, Anhui, People's Republic of China* (yeyu@ustc.edu.cn)

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Abstract We discuss some basic properties of the graded centre of a triangulated category and compute examples arising in representation theory of finite-dimensional algebras.

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1. Introduction

The graded centre of a triangulated category $\mathcal{T}$ with suspension functor Σ is a $\mathbb{Z}$-graded ring. The degree n component consists of all natural transformations from the identity functor Id to Σ^n which commute modulo the sign $(-1)^n$ with Σ. The graded centre is the universal graded commutative ring that acts on $\mathcal{T}$. For instance, the Hochschild cohomology $\mathrm{HH}^*(A)$ of an algebra A acts on the derived category $\boldsymbol{D}(A)$ via a morphism $\mathrm{HH}^*(A) \to Z^*(\boldsymbol{D}(A))$ into the graded centre.

Over the past few years, several authors have studied and used graded centres in various settings: Avramov and Iyengar investigated support varieties of modules over Noetherian rings via central cohomology operations [**3**]. The work of Buchweitz and Flenner uses graded centres for studying the Hochschild cohomology of singular spaces [**5**]. For the related work of Lowen and van den Bergh in the setting of differential graded categories we refer the reader to [**14**]. Blocks of finite groups and their modular representation theory provide the context for recent work of Linckelmann on the graded centre of stable and derived categories [**13**]. Closely related is the study of cohomological support varieties, which depends on the appropriate choice of a graded commutative ring acting on a triangulated category [**4**].

In this paper, we prove some structural results and provide complete descriptions of the graded centre for some small examples. The paper is organized as follows.

In § 2, it is shown that for any abelian category $\mathcal{A}$ with enough projective objects, there is an isomorphism of graded commutative rings $Z^*(\boldsymbol{D}^b(\mathcal{A})) \cong Z^*(\boldsymbol{D}^b(\mathrm{Proj}(\mathcal{A})))$.

Here $\mathrm{Proj}(\mathcal{A})$ denotes the full subcategory of $\mathcal{A}$ consisting of all projective objects and the isomorphism is given by restriction.

In §§ 3 and 4, we deal with derived categories of hereditary categories. Note that, for a hereditary category, the derived category and the bounded derived category have the same graded centre. In § 3, the category $\mathrm{mod}(R)$ of finitely generated modules over a Dedekind domain R is considered. We calculate $Z^*(\boldsymbol{D}(\mathrm{mod}(R)))$ explicitly. As we show, it relates closely to the residue fields of all the maximal ideals of R.

In § 4, we consider the module category of a tame hereditary algebra and the category of coherent sheaves on a weighted projective line of non-negative Euler characteristic. We compute the graded centres of their bounded derived categories. Note that our methods do not apply to wild cases. For a weighted projective line of wild type, we only get a subalgebra of the graded centre.

In §§ 5 and 6, we describe the graded centres of $\boldsymbol{D}^b(\mathrm{mod}(k[x]/x^2))$ and $\underline{\mathrm{mod}}(k[x]/x^n)$ for $n \geqslant 2$, respectively.

2. Morphisms between graded centres

Definition 2.1. Let $\mathcal{T}$ be a triangulated category and Σ the suspension functor of $\mathcal{T}$. We define a $\mathbb{Z}$-graded abelian group $Z^*(\mathcal{T}) = Z^*(\mathcal{T}, \Sigma)$ as follows. For any $n \in \mathbb{Z}$, let $Z^n(\mathcal{T})$ denote the collection of all natural transformations $\eta\colon \mathrm{Id} \to \Sigma^n$ that satisfy $\eta\Sigma = (-1)^n \Sigma\eta$. The composition of natural transformations gives $Z^*(\mathcal{T})$, the structure of a graded commutative ring, and we call it the graded centre of $\mathcal{T}$. *Graded commutative* here means that $\eta\zeta = (-1)^{mn}\zeta\eta$ for all $\eta \in Z^n(\mathcal{T})$ and $\zeta \in Z^m(\mathcal{T})$.

Remarks 2.2.

(i) The definition of the graded centre $Z^*(\mathcal{T})$ makes sense for any graded category, that is, an additive category equipped with an autoequivalence. In particular, the choice of the exact triangles of $\mathcal{T}$ is not relevant for $Z^*(\mathcal{T})$.

(ii) The degree 0 part $Z^0(\mathcal{T})$ is a subring of the usual centre $Z(\mathcal{T})$ of $\mathcal{T}$, which by definition consists of all natural transformations from the identity functor to itself. Note that $Z^0(\mathcal{T}) = Z(\mathcal{T})$ if $\Sigma = \mathrm{Id}$.

(iii) The graded centre $Z^*(\mathcal{T})$ need not be a set in general. However, it will be a set when the category $\mathcal{T}$ is small.

(iv) For any object M in $\mathcal{T}$ we define the graded ring $\mathrm{Ext}^*_{\mathcal{T}}(M, M)$ by setting

$$\mathrm{Ext}^n_{\mathcal{T}}(M, M) = \mathrm{Hom}_{\mathcal{T}}(M, \Sigma^n M)$$

for any integer n. By definition there is a canonical graded ring morphism

$$Z^*(\mathcal{T}) \to \mathrm{Ext}^*_{\mathcal{T}}(M, M)$$

mapping a natural transformation $\eta\colon \mathrm{Id} \to \Sigma^n$ to the morphism $\eta_M\colon M \to \Sigma^n M$.

Following Rouquier [**15**], we set $\langle M\rangle_1$ to be the full additive subcategory of $\mathcal{T}$ which contains M and is closed under finite direct sums, summands and the action of Σ, and for $i \geqslant 2$ we define inductively $\langle M\rangle_i$ as the full additive subcategory of $\mathcal{T}$ consisting of all objects isomorphic to direct summands of objects Z for which there exists an exact triangle $X \to Y \to Z \to \Sigma X$ with $X \in \langle M\rangle_1$ and $Y \in \langle M\rangle_{n-1}$.

Now suppose that M is an object of $\mathcal{T}$ with $\mathcal{T} = \langle M\rangle_{d+1}$ for some positive integer d. We set $\mathcal{N}$ to be the kernel of the canonical morphism $Z^*(\mathcal{T}) \to \mathrm{Ext}^*_{\mathcal{T}}(M,M)$. It can be shown in this case that $\mathcal{N}$ is a nilpotent ideal satisfying $\mathcal{N}^{2^d} = 0$; see [**13**] for a proof. In particular, $Z^*(\mathcal{T})$ is modulo nilpotent elements a set.

Let $F\colon \mathcal{S} \to \mathcal{T}$ be an exact functor between triangulated categories. An obvious question to ask is when the functor F induces morphisms between $Z^*(\mathcal{S})$ and $Z^*(\mathcal{T})$. Recently, Linckelmann gave an affirmative answer to this question in the case in which there exists a functor $G\colon \mathcal{T} \to \mathcal{S}$ that is simultaneously left and right adjoint to F and satisfies some further compatibility conditions [**13**]. The answer for general F seems to be unknown. The following proposition shows that in some very specific situation, for instance when F is fully faithful, we do obtain some morphisms between the graded centres.

Proposition 2.3. *Let $\mathcal{T}$ be a triangulated category and let $\mathcal{S}$ be a full triangulated subcategory.*

(i) *The inclusion functor $i\colon \mathcal{S} \to \mathcal{T}$ induces a morphism of graded rings*

$$i^*\colon Z^*(\mathcal{T}) \to Z^*(\mathcal{S}),$$

where $i^*(\eta)_X = \eta_X$ for any $\eta \in Z^*(\mathcal{T})$ and $X \in \mathcal{S}$.

(ii) *The canonical functor $\pi\colon \mathcal{T} \to \mathcal{T}/\mathcal{S}$ induces a morphism of graded rings*

$$\pi_*\colon Z^*(\mathcal{T}) \to Z^*(\mathcal{T}/\mathcal{S}),$$

where $\pi_*(\eta)_X = \pi(\eta_X)$ for any $\eta \in Z^*(\mathcal{T})$ and $X \in \mathcal{T}/\mathcal{S}$.

Proof. The proof is routine. To check that π_* is well defined, one uses the fact that, for any commutative diagram in $\mathcal{T}$,

$$\begin{tikzcd} X \arrow[d, "\alpha"] & Z \arrow[l, dashed, "s"'] \arrow[r, "f"] \arrow[d, "\beta"] & Y \arrow[d, "\gamma"] \\ X' & Z' \arrow[l, dashed, "s'"'] \arrow[r, "f'"] & Y' \end{tikzcd}$$

with $\mathrm{Cone}(s)$ and $\mathrm{Cone}(s')$ in $\mathcal{S}$, we have $\gamma \circ (f/s) = (f'/s') \circ \alpha$ in the quotient category $\mathcal{T}/\mathcal{S}$, where we use $\dashrightarrow$ to denote the morphisms whose cones are in $\mathcal{S}$. $\square$

Until now, little seems to have been known about the properties of the above morphisms. For example, the question of when i^* and π_* are surjective or injective is of special interest to us. Also, one might study the induced morphism of graded rings

$$(i^*, \pi_*)\colon Z^*(\mathcal{T}) \to Z^*(\mathcal{S}) \times Z^*(\mathcal{T}/\mathcal{S}).$$

Example 2.4. Let $\mathcal{S} \amalg \mathcal{T}$ denote the direct product of two triangulated categories, $\mathcal{S}$ and $\mathcal{T}$. We view $\mathcal{S}$ as a thick subcategory of $\mathcal{S} \amalg \mathcal{T}$ and the corresponding quotient is equivalent to $\mathcal{T}$. Then we have $Z^*(\mathcal{T} \amalg \mathcal{S}) \cong Z^*(\mathcal{T}) \times Z^*(\mathcal{S})$ via the morphism (i^*, π_*).

For the rest of this section we focus on homotopy categories and derived categories. Firstly, we introduce some basic notation and conventions. Let $\mathcal{A}$ be any additive category. We denote by $\boldsymbol{C}(\mathcal{A})$ the category of chain complexes in $\mathcal{A}$. Recall that a chain complex in $\mathcal{A}$ is a sequence of morphisms in $\mathcal{A}$:

$$X = (\,\cdots \longrightarrow X_n \xrightarrow{d_n^X} X_{n-1} \longrightarrow \cdots\,)$$

with $d_n^X d_{n+1}^X = 0$ for all $n \in \mathbb{Z}$. A morphism of complexes is a chain map $f\colon X \to Y$ consisting of a family of morphisms $f_n\colon X_n \to Y_n$ in $\mathcal{A}$ with $n \in \mathbb{Z}$ such that $f_n \circ d_{n+1}^X = d_{n+1}^Y \circ f_{n+1}$ for all n, that is, the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & X_{n+1} \arrow[r, "d_{n+1}^X"] \arrow[d, "f_{n+1}"'] & X_n \arrow[r, "d_n^X"] \arrow[d, "f_n"'] & X_{n-1} \arrow[r] \arrow[d, "f_{n-1}"'] & \cdots \\
\cdots \arrow[r] & Y_{n+1} \arrow[r, "d_{n+1}^Y"] & Y_n \arrow[r, "d_n^Y"] & Y_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

commutes. We denote by $\boldsymbol{C}^+(\mathcal{A})$ the full subcategory of $\boldsymbol{C}(\mathcal{A})$ that consists of all bounded below complexes, that is, the complexes X with $X_n = 0$ for $n \ll 0$. Similarly, we denote by $\boldsymbol{C}^-(\mathcal{A})$ and $\boldsymbol{C}^b(\mathcal{A})$ the full subcategory of bounded above complexes and complexes bounded in both directions, respectively.

If, moreover, $\mathcal{A}$ is abelian, then for any integer n the nth homology group $H_n(X)$ is by definition $\operatorname{Ker}(d_n^X)/\operatorname{Im}(d_{n+1}^X)$, and any morphism f of complexes induces morphisms of homology groups $H_n(f)\colon H_n(X) \to H_n(Y)$ for all $n \in \mathbb{Z}$.

The homotopy category $\boldsymbol{K}(\mathcal{A})$ has the same objects as $\boldsymbol{C}(\mathcal{A})$. The morphisms are the equivalence classes of the morphisms in $\boldsymbol{C}(\mathcal{A})$ modulo the null-homotopic morphisms, that is, those with components of the form

$$d_{n+1}^Y \circ h_n + h_{n-1} \circ d_n^X$$

for some family of morphisms $h_n\colon X_n \to Y_{n+1}$ in $\mathcal{A}$, $n \in \mathbb{Z}$.

The suspension functor (or shift functor) Σ of $\boldsymbol{C}(\mathcal{A})$ is defined by $(\Sigma X)_n = X_{n-1}$, $d_n^{\Sigma X} = -d_{n-1}^X$ on the objects and by $(\Sigma f)_n = f_{n-1}$ on any morphism f. Clearly, Σ is not only an autoequivalence but also an automorphism of $\boldsymbol{C}(\mathcal{A})$. Moreover, Σ also induces an automorphism of $\boldsymbol{K}(\mathcal{A})$ and $\boldsymbol{K}(\mathcal{A})$ admits a triangulated structure with suspension functor Σ.

Let $\boldsymbol{D}(\mathcal{A})$ denote the derived category of $\mathcal{A}$, i.e. the localization of $\boldsymbol{K}(\mathcal{A})$ with respect to the quasi-isomorphisms. Note that $\boldsymbol{D}(\mathcal{A})$ is again a triangulated category with suspension functor Σ. One defines $\boldsymbol{K}^*(\mathcal{A})$ and $\boldsymbol{D}^*(\mathcal{A})$ with $* \in \{+, b, -\}$ in a similar way.

Now let $\mathcal{A}$ be an abelian category with enough projective objects, and let $\mathcal{P}$ be the full subcategory consisting of all projective objects. We denote by $\boldsymbol{K}^{+,b}(\mathcal{P})$ the thick subcategory of $\boldsymbol{K}^+(\mathcal{P})$ that consists of bounded below complexes X with $H_n(X) = 0$ for almost all n. Clearly, we have $\boldsymbol{K}^b(\mathcal{P}) \subseteq \boldsymbol{K}^{+,b}(\mathcal{P})$. In some cases, objects in $\boldsymbol{K}^b(\mathcal{P})$ are also called perfect complexes.

It is known that the composition of functors

$$\begin{tikzcd} \boldsymbol{K}^{+,b}(\mathcal{P}) \arrow[rr, dashed, "\cong"] \arrow[d, hook] & & \boldsymbol{D}^b(\mathcal{A}) \arrow[d, hook] \\ \boldsymbol{K}^+(\mathcal{P}) \arrow[r] \arrow[rr, bend left, "\cong"] & \boldsymbol{K}^+(\mathcal{A}) \arrow[r] & \boldsymbol{D}^+(\mathcal{A}) \end{tikzcd}$$

induces equivalences $\boldsymbol{K}^+(\mathcal{P}) \xrightarrow{\sim} \boldsymbol{D}^+(\mathcal{A})$ and $\boldsymbol{K}^{+,b}(\mathcal{P}) \xrightarrow{\sim} \boldsymbol{D}^b(\mathcal{A})$ of triangulated categories. The quotient category $\boldsymbol{D}_{\mathrm{sg}}(\mathcal{A}) = \boldsymbol{D}^b(\mathcal{A})/\boldsymbol{K}^b(\mathcal{P})$ is called the triangulated category of singularities of $\mathcal{A}$ because it is an invariant of the singularities provided that $\mathcal{A}$ is the category of sheaves on some variety. We know that $\boldsymbol{D}_{\mathrm{sg}}(\mathcal{A}) = 0$ if and only if all objects of $\mathcal{A}$ have finite homological dimension. When $\mathcal{A} = \mathrm{Mod}(A)$ for some self-injective ring A, $\boldsymbol{D}_{\mathrm{sg}}(\mathcal{A})$ is equivalent to the stable module category $\underline{\mathrm{Mod}}(A)$ of A.

We are now in a position to state the main result of this section.

Theorem 2.5. *Let $\mathcal{A}$ be an abelian category with enough projective objects and let $\mathcal{P}$ be the full subcategory consisting of all projective objects. Then the embedding $\boldsymbol{K}^b(\mathcal{P}) \to \boldsymbol{K}^{+,b}(\mathcal{P})$ induces an isomorphism $Z^*(\boldsymbol{K}^b(\mathcal{P})) \xrightarrow{\sim} Z^*(\boldsymbol{D}^b(\mathcal{A}))$ of graded commutative rings.*

To prove the theorem, we need some preparations.

For each $n \in \mathbb{Z}$ the nth truncation functor $\iota^n \colon \boldsymbol{C}(\mathcal{A}) \to \boldsymbol{C}(\mathcal{A})$ is defined for a complex X by $(\iota^n X)_i = X_i$ for $i \leqslant n$ and 0 for $i > n$, and $d_i^{\iota^n X} = d_i^X$ for $i \leqslant n$ and 0 for $i > n$. Clearly, ι^n sends $\boldsymbol{C}(\mathcal{A})$ to $\boldsymbol{C}^-(\mathcal{A})$ and $\boldsymbol{C}^+(\mathcal{A})$ to $\boldsymbol{C}^b(\mathcal{A})$. Note that we have a natural morphism $i^n \colon \iota^n X \to X$, and sometimes we use i_X^n to emphasize X. We have $(i^n)_s = \mathrm{id}$ for $s \leqslant n$ and 0 for $s > n$. The following lemma is crucial in the proof of the main theorem.

Lemma 2.6. *Let $X \in \boldsymbol{C}(\mathcal{P})$, $Y \in \boldsymbol{C}(\mathcal{A})$ and $f \colon X \to Y$ be a chain map with $H_n(Y) = 0$ for $n > 0$. Then f is null-homotopic if and only if the composition $f \circ i^n \colon \iota^n X \to X \to Y$ is null-homotopic for some $n \geqslant 0$.*

Proof. One direction is clear, since the null-homotopic morphisms form an ideal. Conversely, suppose that $f \circ i^n$ is null-homotopic for some $n \geqslant 0$. To show that f is also null-homotopic, it suffices to find a family $\{h_n \colon X_n \to Y_{n+1} \mid n \in \mathbb{Z}\}$, such that $f_n = d_{n+1}^Y h_n + h_{n-1} d_n^X$ holds for all n. By applying the shift functor, one can assume without loss of generality that $f \circ i^0$ is null-homotopic. Thus, there exists a family of morphisms in $\mathcal{A}$, say $\{h_n \colon X_n \to Y_{n+1}, n \leqslant 0\}$, such that $f_n = d_{n+1}^Y h_n + h_{n-1} d_n^X$ for all $n \leqslant 0$.

Since f is a chain map, we have

$$d_1^Y f_1 = f_0 d_1^X = d_1^Y h_0 d_1^X + h_{-1} d_0^X d_1^X,$$

and hence $d_1^Y(f_1 - h_0 d_1^X) = 0$, which implies that

$$\mathrm{Im}(f_1 - h_0 d_1^X) \subseteq \mathrm{Ker}(d_1^Y) = \mathrm{Im}(d_2^Y);$$

the last equality holds because $H_1(Y) = 0$. Now, the fact that X_1 is projective implies that $f_1 - h_0 d_1^X$ factors through d_2^Y, i.e. there exists $h_1 \colon X_1 \to Y_2$ such that $f_1 = d_2^Y h_1 + h_0 d_1^X$; thus, we get the required h_1. Now repeat the argument and the lemma follows. $\square$

Proposition 2.7. *Let $t \in \mathbb{Z}$ and let $\eta\colon \mathrm{Id} \to \Sigma^t$ be a natural transformation for the category $\boldsymbol{K}^b(\mathcal{P})$. Then η extends uniquely to a natural transformation $\tilde{\eta}\colon \mathrm{Id} \to \Sigma^t$ for the category $\boldsymbol{K}^{+,b}(\mathcal{P})$.*

Proof. First we will construct a morphism $\tilde{\eta}_X \colon X \to \Sigma^t X$ for any $X \in \boldsymbol{K}^{+,b}(\mathcal{P})$. The idea is to use certain approximations.

Since η is a natural transformation for $\boldsymbol{K}^b(\mathcal{P})$, we have for each n a morphism $\bar{\zeta}^n = \eta_{\iota^n X} \colon \iota^n X \to \Sigma^t \iota^n X$. Now we fix a chain map $\zeta^0 \colon \iota^0 X \to \Sigma^t \iota^0 X$ that is a representative of $\bar{\zeta}^0$. We can construct inductively the representatives ζ^n of $\bar{\zeta}^n$ for all $n \geqslant 0$, such that $\zeta_i^{n+1} = \zeta_i^n$ for all $n \geqslant 0$ and $i \leqslant n$.

In fact, suppose that ζ^n has been constructed, and let ξ be any representative of $\bar{\zeta}^{n+1} X$. Consider the morphism $j \colon \iota^n X \to \iota^{n+1} X$ which is given by $j_m = \mathrm{id}_{X_m}$ for all $m \leqslant n$ and 0 otherwise. Since η is a natural transformation, the diagram

$$\begin{tikzcd}
\iota^n X \arrow[r, "j"] \arrow[d, "\zeta^n"'] & \iota^{n+1} X \arrow[d, "\xi"] \\
\Sigma^t \iota^n X \arrow[r, "\Sigma^t j"] & \Sigma^t \iota^{n+1} X
\end{tikzcd}$$

commutes in the category $\boldsymbol{K}^b(\mathcal{P})$, i.e. $\delta := \xi \circ j - \Sigma^t j \circ \zeta^n$ is null-homotopic. Explicitly, $\delta_i = \xi_i - \eta_i$ for $i \leqslant n$ and $\delta_i = 0$ for $i \geqslant n+1$.

Now there exists a family of morphisms $\{h_i \colon (\iota^n X)_i \to (\Sigma^t \iota^{n+1} X)_{i+1} \mid i \in \mathbb{Z}\}$ with $h_i = 0$ for $i > n$, such that

$$\delta_i = d_{i+1}^{\Sigma^t \iota^{n+1} X} \circ h_i + h_{i-1} \circ d_i^{\iota^n X}.$$

The family $\{h_i\}$ can be viewed as a family of morphisms $\{h_i \colon (\iota^{n+1} X)_i \to (\Sigma^t \iota^{n+1} X)_{i+1} \mid i \in \mathbb{Z}\}$; thus, it gives a null-homotopic morphism $\delta' \colon \iota^{n+1} X \to \Sigma^t \iota^{n+1} X$, which satisfies $\delta'_i = \delta_i$ for all $i \leqslant n$. We are done by setting $\zeta^{n+1} = \xi - \delta'$.

Now we define $\tilde{\eta}_X$ by $(\tilde{\eta}_X)_n = \zeta_n^0$ for $n \leqslant 0$ and $(\tilde{\eta}_X)_n = \zeta_n^n$ for $n > 0$. We claim that $\tilde{\eta}$ is a natural transformation from Id to Σ^t for the category $\boldsymbol{K}^{+,b}(\mathcal{P})$.

Note that, by construction, $\tilde{\eta}_X$ satisfies the following condition: for any $n \geqslant 0$, there exists a representative ζ_X^n for $\eta_{\iota^n X}$, which is given by $(\zeta_X^n)_i = (\tilde{\eta}_X)_i$ for $i \leqslant n$, and $(\zeta_X^n)_i = 0$ for $i \geqslant n+1$. In other words, $\tilde{\eta}_X \circ i_X^n = \Sigma^t i_X^n \circ \eta_{\iota^n X}$ as chain maps for all $n \geqslant 0$, where i_X^n denotes the natural morphism from $\iota^n X$ to X as before.

Now let $X, Y \in \boldsymbol{K}^{+,b}(\mathcal{P})$ and let $f \colon X \to Y$ be any chain map. Assume that $\tilde{\eta}_X : X \to \Sigma^t X$ and $\tilde{\eta}_Y : Y \to \Sigma^t Y$ are arbitrary chain maps with the property

$$\tilde{\eta}_X \circ i_X^n = \Sigma^t i_X^n \circ \eta_{\iota^n X} \quad \text{and} \quad \tilde{\eta}_Y \circ i_Y^n = \Sigma^t i_Y^n \circ \eta_{\iota^n Y} \quad \text{for } n \geqslant 0.$$

We will show that $\tilde{\eta}_Y \circ f = \Sigma^t f \circ \tilde{\eta}_X$ in $\boldsymbol{K}^{+,b}(\mathcal{P})$. Note that in the cube below the other five faces are commutative by the construction of $\tilde{\eta}_X$, $\tilde{\eta}_Y$, ι_X^n and ι_Y^n:

$$\begin{tikzcd}
 & X \arrow[rr, "\tilde{\eta}_X"] \arrow[dd, "f" near end] & & \Sigma^t X \arrow[dd, "\Sigma^t f"] \\
\iota^n X \arrow[ur, "i_X^n"] \arrow[rr, "\eta_{\iota^n X}" near start] \arrow[dd] & & \Sigma^t \iota^n X \arrow[ur] \arrow[dd] & \\
 & Y \arrow[rr, "\tilde{\eta}_Y" near end] & & \Sigma^t Y \\
\iota^n Y \arrow[ur, "i_Y^n"] \arrow[rr, "\eta_{\iota^n Y}"] & & \Sigma^t \iota^n Y \arrow[ur] &
\end{tikzcd}$$

By Lemma 2.6 it suffices to show that $\tilde{\eta}_Y \circ f \circ i_X^n = \Sigma^t f \circ \tilde{\eta}_X \circ i_X^n$ for some sufficiently large n. This is equivalent to showing that $\tilde{\eta}_Y \circ i_Y^n \circ \iota^n f = \Sigma^t f \circ \Sigma^t i_X^n \circ \eta_{\iota^n X}$. The left-hand side is $\Sigma^t i_Y^n \circ \eta_{\iota^n Y} \circ \iota^n f$, and since $\eta_{\iota^n Y} \circ \iota^n f = \Sigma^t \iota^n f \circ \eta_{\iota^n X}$ and $\Sigma^t f \circ \Sigma^t i_X^n = \Sigma^t i_Y^n \circ \Sigma^t \iota^n f$, the equality holds.

Thus, by fixing such $\tilde{\eta}_X$ for each X, we can extend η to the category $\boldsymbol{K}^{+,b}(\mathcal{P})$. For the uniqueness, we need only to take $f = \mathrm{id}_X$ in the above argument. This completes the proof. □

Corollary 2.8. *Let $\mathcal{A}$ and $\mathcal{P}$ be as before. Then $Z^t(\boldsymbol{K}^b(\mathcal{P})) = 0$ for all $t < 0$, and therefore $Z^*(\boldsymbol{K}^b(\mathcal{P}))$ and $Z^*(\boldsymbol{D}^b(\mathcal{A}))$ are positively graded.*

Proof. Suppose that η is a natural transformation from $\mathrm{Id}_{\boldsymbol{K}^b(\mathcal{P})}$ to $\Sigma^t_{\boldsymbol{K}^b(\mathcal{P})}$ for some $t < 0$. We prove that $\eta_X = 0$ by using induction on the length of the support of X, where the support of X means the interval $[i, j]$, such that i and j are respectively the minimal and maximal integer m with $X_m \neq 0$. Without loss of generality, we may assume that $i = 0$ and we use induction on j.

In the case $j = 0$, clearly $\mathrm{Hom}_{\boldsymbol{K}^b(\mathcal{P})}(X, \Sigma^t X) = 0$ for $t < 0$. Suppose $\eta_X = 0$ for all $j \leqslant m$ and suppose $X = (\cdots \to 0 \to X_{m+1} \to \cdots \to X_0 \to 0 \to \cdots)$. By the same argument as in the proof of Proposition 2.7, there is a representative ζ of η_X, such that $\zeta_i = 0$ for all $i \leqslant m$, and now the assumption $t < 0$ implies that $(\Sigma^t X)_{m+1} = 0$, which forces that $\zeta_{m+1} = 0$, thus $\zeta = 0$ and hence $\eta_X = 0$. □

With the above preparations, we can now prove the main theorem.

Proof of Theorem 2.5. Fix $\eta \in Z^t(\boldsymbol{K}^b(\mathcal{P}))$. By Proposition 2.7, η extends uniquely to a natural transformation $\tilde{\eta}\colon \mathrm{Id} \to \Sigma^t$ for the category $\boldsymbol{K}^{+,b}(\mathcal{P})$, and clearly $i^*(\tilde{\eta}) = \eta$, where i^* is induced by the embedding $i\colon \boldsymbol{K}^b(\mathcal{P}) \to \boldsymbol{K}^{+,b}(\mathcal{P})$. By the same argument as in the last part of the proof of Proposition 2.7, one can show that $\tilde{\eta}\Sigma = (-1)^n \Sigma\tilde{\eta}$, which implies that $\tilde{\eta} \in Z^n(\boldsymbol{K}^{+,b}(\mathcal{P}))$. This proves the surjectivity of i^*. The injectivity of i^* follows from the uniqueness of the extension. □

Remark 2.9. Suppose there are enough injective objects in $\mathcal{A}$ and denote by $\mathcal{I}$ the full subcategory of injectives. Then we have $\boldsymbol{D}^b(\mathcal{A}) \cong \boldsymbol{K}^{-,b}(\mathcal{I})$ and the dual version of the theorem says that there is an isomorphism of graded centres $Z^*(\boldsymbol{K}^b(\mathcal{I})) \cong Z^*(\boldsymbol{D}^b(\mathcal{A}))$.

3. Finitely generated modules over Dedekind domains

The following two sections are devoted to studying the graded centre of the derived category of some hereditary categories. We look at some basic examples and use explicit calculations. First we discuss the derived category of the category of finitely generated modules $\mathrm{mod}(R)$ for any Dedekind domain R. We start with some preparation.

Let R be an arbitrary unitary ring and denote by $Z(R)$ the centre of R. Let $z \in Z(R)$ and $M \in \mathrm{mod}(R)$. Then we have a morphism $l_z \in \mathrm{Hom}_R(M, M)$, which is given by $l_z(m) = z \cdot m$. This is indeed a morphism of modules since z is in the centre of R. Moreover, l_z induces a natural transformation from the identity functor to itself for $\mathrm{mod}(R)$ as well as for $\boldsymbol{D}^b(\mathrm{mod}(R))$.

Now let $\mathcal{H}$ be a hereditary abelian category, that is, $\mathrm{Ext}^i_{\mathcal{H}}(M, N) = 0$ for any $M, N \in \mathcal{H}$ and $i \geqslant 2$. Consider the derived category of $\mathcal{H}$ and observe that any object $X \in D(\mathcal{H})$ is isomorphic to $\bigoplus_{i\in\mathbb{Z}} \Sigma^i(H_i(X))$. Here, Σ is the shift functor and $H_i(X)$ is viewed as a stalk complex concentrated in degree 0. For a simple proof of this, see [**10**, § 1]. We have the following easy lemmas.

Lemma 3.1. *Left multiplication induces an injective ring homomorphism $Z(R) \to Z^0(\boldsymbol{D}^b(\mathrm{mod}(R)))$. Moreover, if R is hereditary, then this is an isomorphism.*

Proof. For a proof, we just use the fact that left multiplication gives an isomorphism from $Z(R)$ to the usual centre of $\mathrm{mod}(R)$, i.e. the ring of natural transformations from the identity functor to itself, and that $\mathrm{mod}(R)$ is a full subcategory of $\boldsymbol{D}^b(\mathrm{mod}(R))$.

Moreover, if R is hereditary, then all objects of $\boldsymbol{D}^b(\mathrm{mod}(R))$ are of the form $\bigoplus_{i\in\mathbb{Z}} \Sigma^i M_i$ with $M_i \in \mathrm{mod}(R)$ viewed as a stalk complex concentrated in degree 0. Now the lemma follows easily. □

Remark 3.2. Note that the morphism in the lemma need not be an isomorphism (see [**11**] or § 5).

Lemma 3.3. *Let $\mathcal{H}$ be an arbitrary hereditary category. Then $Z^*(\boldsymbol{D}^b(\mathcal{H}))$ is concentrated in degrees 0 and 1. Moreover, the inclusions $\boldsymbol{D}^b(\mathcal{H}) \subseteq \boldsymbol{D}^-(\mathcal{H}) \subseteq \boldsymbol{D}(\mathcal{H})$ induce isomorphisms of graded centres $Z^*(\boldsymbol{D}^b(\mathcal{H})) \cong Z^*(\boldsymbol{D}^-(\mathcal{H})) \cong Z^*(\boldsymbol{D}(\mathcal{H}))$.*

Proof. We have $\mathrm{Hom}_{\boldsymbol{D}(\mathcal{H})}(M, \Sigma^m M) = \mathrm{Ext}^m_{\mathcal{H}}(M, M) = 0$ for all $M \in \mathcal{H}$ and $m \geqslant 2$, since $\mathcal{H}$ is hereditary. Thus, there is no non-trivial natural transformations from Id to Σ^m for the category $\boldsymbol{D}^b(\mathcal{H})$ for $m \geqslant 2$, and the first part of the lemma follows. The last assertion follows from the fact that any element in the graded centre $Z^*(\boldsymbol{D}(\mathcal{H}))$ is uniquely determined by the restriction to the stalk complexes. The minor difference between the two derived categories is that any object in $\boldsymbol{D}(\mathcal{H})$ is an infinite direct sum of stalk complexes while objects in $\boldsymbol{D}^b(\mathcal{H})$ can always be written as finite direct sums. Similarly, we have $Z^*(\boldsymbol{D}^+(\mathcal{H})) \cong Z^*(\boldsymbol{D}(\mathcal{H}))$. □

Due to the lemma, to study the graded centre of the derived categories of hereditary abelian categories, we need only to consider the bounded ones.

Now suppose that $\mathcal{H} = \mathcal{H}_1 \vee \mathcal{H}_2$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are full additive subcategories of $\mathcal{H}$, and we use $\vee$ to indicate that any object of $\mathcal{H}$ is a direct sum of an object of $\mathcal{H}_1$

and an object of $\mathcal{H}_2$, and $\mathrm{Hom}_{\mathcal{H}}(M_2, M_1) = \mathrm{Ext}^1_{\mathcal{H}}(M_1, M_2) = 0$ for all $M_1 \in \mathcal{H}_1$ and $M_2 \in \mathcal{H}_2$. We set $\Sigma^*\mathcal{H}_1$ to be the minimal additive subcategory of $\boldsymbol{D}^b(\mathcal{H})$ that contains $\mathcal{H}_1$ and is closed under Σ: in other words, the subcategory consisting of all complexes with homologies contained in $\mathcal{H}_1$. Note that $\Sigma^*\mathcal{H}_1$ is not a triangulated subcategory in general. This will happen if $\mathcal{H}_1$ is a subcategory of $\mathcal{H}$ that is closed under extensions, kernels and cokernels. In this case, $\mathcal{H}_1$ is also a hereditary abelian category and $\Sigma^*\mathcal{H}_1 \cong \boldsymbol{D}^b(\mathcal{H}_1)$. For a proof of this, one again uses the fact that any object in $\boldsymbol{D}^b(\mathcal{H})$ is a direct sum of stalk complexes. Since Σ is an autoequivalence of $\Sigma^*\mathcal{H}_1$, we can also define the graded centre of $\Sigma^*\mathcal{H}_1$ with respect to Σ, and denote it by $Z^*(\Sigma^*\mathcal{H}_1)$.

Proposition 3.4. *Let $\mathcal{H} = \mathcal{H}_1 \vee \mathcal{H}_2$ be a hereditary abelian category. Then the restriction map induces an isomorphism of abelian groups*

$$Z^1(\boldsymbol{D}^b(\mathcal{H})) \cong Z^1(\Sigma^*\mathcal{H}_1) \times Z^1(\Sigma^*\mathcal{H}_2).$$

Proof. We produce an inverse map. First observe that any object in $\boldsymbol{D}^b(\mathcal{H})$ can be written uniquely as $X_1 \oplus X_2$ with $X_1 \in \Sigma^*\mathcal{H}_1$ and $X_2 \in \Sigma^*\mathcal{H}_2$. Let $\eta_1\colon \mathrm{Id}_{\Sigma^*\mathcal{H}_1} \to \Sigma_{\Sigma^*\mathcal{H}_1}$ and $\eta_2\colon \mathrm{Id}_{\Sigma^*\mathcal{H}_2} \to \Sigma_{\Sigma^*\mathcal{H}_2}$ be natural transformations. Then we define $\eta\colon \mathrm{Id}_{\boldsymbol{D}^b(\mathcal{H})} \to \Sigma_{\boldsymbol{D}^b(\mathcal{H})}$ by setting $\eta_{X_1 \oplus X_2}$ to be the map $(\eta_1)_{X_1} \oplus (\eta_2)_{X_2}$. We will show that η is indeed a natural transformation. To this end we need to check that, for any morphism $f\colon X \to Y$ in $\boldsymbol{D}^b(\mathcal{H})$, we have $\Sigma f \circ \eta_X = \eta_Y \circ f$.

Since any object of $\boldsymbol{D}^b(\mathcal{H})$ can be uniquely written as $\bigoplus_{i\in\mathbb{Z}} \Sigma^i(M_1^i \oplus M_2^i)$ with $M_1^i \in \mathcal{H}_1$ and $M_2^i \in \mathcal{H}_2$, we need only to check the above compatibility for the morphisms of the form $f\colon \Sigma^i M_1 \to \Sigma^j M_2$ and $g\colon \Sigma^i M_2 \to \Sigma^j M_1$ with $M_1 \in \mathcal{H}_1$ and $M_2 \in \mathcal{H}_2$. We claim that $\Sigma f \circ \eta_{M_1} = \eta_{M_2} \circ f = 0$ and $\Sigma g \circ \eta_{M_2} = \eta_{M_1} \circ g = 0$. In fact, since $\mathcal{H}$ is hereditary, both sides will vanish unless $j = i + 1$ or $j = i$. If $j = i + 1$, the equalities hold since $\mathrm{Ext}^2_{\mathcal{H}}(M, N) = 0$ for all $M, N \in \mathcal{H}$. Otherwise, if $j = i$, we have $g = 0$ and $\mathrm{Ext}^1_{\mathcal{H}}(M_1, M_2) = 0$. Now the assertion follows easily, and this completes the proof. □

Now we can begin the study of $Z^*(\boldsymbol{D}^b(\mathrm{mod}(R)))$ for a Dedekind domain R. A Dedekind domain is an integral domain such that each ideal can be written as a finite product of prime ideals or, equivalently, a Noetherian integrally closed domain with Krull dimension at most one. This name was given to such rings in honour of Dedekind, who was one of the first to study such rings in the 1870s. The rings of algebraic integers of number fields provide an important class of Dedekind domains, which play a crucial role in algebraic number theory.

The assumption on the Krull dimension implies that each non-zero prime ideal of R is maximal, and that the category $\mathcal{H} = \mathrm{mod}(R)$ is hereditary and any object M of $\mathrm{mod}(R)$ is a direct sum of a torsion-free module and a torsion module. Any finitely generated torsion-free module is projective, and any finitely generated torsion module is a finite direct sum of cyclic modules. Moreover, by the Chinese Remainder Theorem, each cyclic module is a finite direct sum of modules of the form $R/\mathfrak{p}^l$ with $\mathfrak{p}$ a maximal ideal of R and $l \geqslant 1$.

We have a decomposition of categories $\mathcal{H} = \mathcal{H}_+ \vee \mathcal{H}_0$, where $\mathcal{H}_+$ denotes the full subcategory of $\mathcal{H}$ consisting of all projective modules and $\mathcal{H}_0$ consists of all torsion modules.

Note that $\mathcal{H}_0$ is an exact abelian subcategory and therefore hereditary. Moreover, there exists an Auslander–Reiten translation τ in $\mathcal{H}_0$, which is by definition an autoequivalence of $\mathcal{H}_0$ such that there exists a natural isomorphism $D\,\mathrm{Ext}^1_{\mathcal{H}}(X,Y) \cong \mathrm{Hom}_{\mathcal{H}}(Y,\tau X)$. This identity is usually called Serre duality and implies the existence of Auslander–Reiten sequences in $\mathcal{H}_0$.

Let $\max(R)$ denote the set of all maximal ideals of R. Then all the indecomposables in $\mathcal{H}_0$ are given by $\{R/\mathfrak{p}^l \mid \mathfrak{p} \in \max(R),\ l \geqslant 1\}$. Denote by $\mathcal{H}_\mathfrak{p}$ the subcategory of $\mathcal{H}_0$ consisting of all $\mathfrak{p}$-torsion modules, i.e. the additive category generated by $\{R/\mathfrak{p}^l \mid l \geqslant 1\}$ that is an exact abelian subcategory of $\mathcal{H}_0$. Note that $\mathcal{H}_\mathfrak{p}$ is τ-invariant and $\mathcal{H}_0 = \coprod_{\mathfrak{p}\in\max(R)} \mathcal{H}_\mathfrak{p}$.

One can show that $\mathcal{H}_\mathfrak{p}$ is equivalent to the subcategory of $\mathrm{mod}(R_\mathfrak{p})$ that consists of torsion $R_\mathfrak{p}$-modules, where $R_\mathfrak{p}$ is the localization of R at $\mathfrak{p}$; and $\mathcal{H}_\mathfrak{p}$ is obviously a full subcategory of $\mathrm{mod}(\hat{R}_\mathfrak{p})$, where $\hat{R}_\mathfrak{p}$ is the completion of R with respect to $\mathfrak{p}$, i.e. the inverse limit $\varprojlim R/\mathfrak{p}^l$. Note that we have an isomorphism $R/\mathfrak{p} \cong R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ of residue fields, and we denote it by $k_\mathfrak{p}$.

Fix an element $x \in \mathfrak{p}\setminus\mathfrak{p}^2$. Since $R_\mathfrak{p}$ is a discrete valuation ring, the multiplication with x gives us an isomorphism $R/\mathfrak{p}^l \cong \mathfrak{p}/\mathfrak{p}^{l+1}$ of R-modules for any $l \geqslant 1$. Thus, we have in $\mathcal{H}_\mathfrak{p}$ Auslander–Reiten sequences

$$\sigma^1_\mathfrak{p}\colon 0 \to R/\mathfrak{p} \to R/\mathfrak{p}^2 \to R/\mathfrak{p} \to 0$$

and

$$\sigma^l_\mathfrak{p}\colon 0 \to R/\mathfrak{p}^l \to R/\mathfrak{p}^{l-1} \oplus R/\mathfrak{p}^{l+1} \to R/\mathfrak{p}^l \to 0$$

for all $l \geqslant 2$, where the morphism from $R/\mathfrak{p}^{l+1}$ to $R/\mathfrak{p}^l$ is the natural quotient map, and we use the isomorphism $R/\mathfrak{p}^l \cong \mathfrak{p}/\mathfrak{p}^{l+1}$ induced by the multiplication with x. This says that the Auslander–Reiten quiver of $\mathcal{H}_\mathfrak{p}$ is a tube of τ-period 1. For the Auslander–Reiten sequences for Dedekind domains, see also [**2**, Example 3.1]. Note that we have a natural equivalence $\tau \cong \mathrm{Id}_{\mathcal{H}_\mathfrak{p}}$.

It is easy to show that $\Omega(R/\mathfrak{p}^s) \cong R_\mathfrak{p}$ for any $s \geqslant 1$, where $\Omega\colon \mathrm{mod}(R_\mathfrak{p}) \to \mathrm{mod}(R_\mathfrak{p})$ is the syzygy functor. We have a presentation of $R/\mathfrak{p}^s$

$$0 \longrightarrow R_\mathfrak{p} \xrightarrow{l_{x^s}} R_\mathfrak{p} \longrightarrow R/\mathfrak{p}^s \longrightarrow 0,$$

where l_{x^s} denotes the multiplication by x^s. Note that we have an isomorphism $R/\mathfrak{p}^s \cong R_\mathfrak{p}/\mathfrak{p}^s R_\mathfrak{p}$. Now the above exact sequence induces an epimorphism

$$\mathrm{Hom}_R(R_\mathfrak{p}, R/\mathfrak{p}^i) \to \mathrm{Ext}^1_R(R/\mathfrak{p}^s, R/\mathfrak{p}^i),$$

and when $i \leqslant s$ this induces an isomorphism. Now it is easy to show that the Auslander–Reiten sequence $\sigma^s_\mathfrak{p}$ corresponds to the composition $R_\mathfrak{p} \to R/\mathfrak{p} \cong \mathfrak{p}^{s-1}/\mathfrak{p}^s \hookrightarrow R/\mathfrak{p}^s$. Moreover, all the Auslander–Reiten sequences with starting term $R/\mathfrak{p}^s$ are given by the morphisms of the form $R_\mathfrak{p} \to \mathrm{Soc}(R/\mathfrak{p}^s) \hookrightarrow R/\mathfrak{p}^s$, which equals $\lambda\sigma^s_\mathfrak{p}$ for some $0 \neq \lambda \in k_\mathfrak{p}$.

On the other hand, we identify $\mathrm{Hom}_{\boldsymbol{D}^b(\mathcal{H})}(M, \Sigma N) = \mathrm{Ext}^1_{\mathcal{H}}(M,N)$ for any abelian category $\mathcal{H}$ and any objects $M, N \in \mathcal{H}$. Now we can write down the graded centre of $\boldsymbol{D}^b(\mathcal{H}_\mathfrak{p})$ explicitly by using the notion of a trivial extension.

Let A be an arbitrary ring and let M be an A–A-bimodule. The trivial extension ring of A by the bimodule M, denoted by $T(A,M)$, is defined to be the ring whose additive ring is $A \oplus M$ with multiplication given by

$$(a,m)\cdot(a',m') = (aa', am' + ma')$$

for all $a, a' \in A$ and $m, m' \in M$. Note that $T(A,M)$ can be identified with the subring of the upper triangular ring

$$\begin{pmatrix} A & M \\ 0 & A \end{pmatrix}$$

that consists of all the matrices with equal diagonal entries.

The trivial extension ring is a positively graded ring which is concentrated in degree 0 and 1 with $T(A,M)^0 = A$ and $T(A,M)^1 = M$. Conversely, let $A = A^0 \oplus A^1$ be an arbitrary positively graded ring which is concentrated in degree 0 and 1. Then $A \cong T(A^0, A^1)$ as graded rings, where the A^0-bimodule structure on A^1 is induced by the multiplication of A. If A is commutative and M an A-module, one can also define the trivial extension $T(A,M)$, where M is viewed as an A–A-bimodule. Note that in this case $T(A,M)$ is always graded commutative.

Proposition 3.5. *Let R be a Dedekind domain, $\mathfrak{p}$ a maximal ideal of R and $k_{\mathfrak{p}}$ the residue field. Then, as a graded ring,*

$$Z^*(\boldsymbol{D}^b(\mathcal{H}_{\mathfrak{p}})) \cong T\biggl(\hat{R}_{\mathfrak{p}}, \prod_{l\in\mathbb{Z},\, l\geqslant 1} k_{\mathfrak{p}}\biggr),$$

where $k_{\mathfrak{p}}$ is viewed as a simple $\hat{R}_{\mathfrak{p}}$-module.

Proof. Note that elements in $\hat{R}_{\mathfrak{p}}$ are by definition sequences $q = (q_i)_{i\in\mathbb{Z}, i\geqslant 0}$ with $q_i \in R/\mathfrak{p}^i$ and satisfying $\pi_{i,j}(q_i) = q_j$ for all $i > j$, where $\pi_{i,j}\colon R/\mathfrak{p}^i \to R/\mathfrak{p}^j$ is the natural quotient map. Now the collection of morphisms $\{l_{q_i}\colon R/\mathfrak{p}^i \to R/\mathfrak{p}^i,\ i \in \mathbb{N}\}$ determines uniquely an element in $Z^0(\boldsymbol{D}^b(\mathcal{H}_{\mathfrak{p}}))$, where l_{q_i} is given by multiplication with q_i, and it is easy to show this correspondence gives a bijection between $\hat{R}_{\mathfrak{p}}$ and $Z^0(\boldsymbol{D}^b(\mathcal{H}_{\mathfrak{p}}))$, which means that $Z^0(\boldsymbol{D}^b(\mathcal{H}_{\mathfrak{p}})) \cong \hat{R}_{\mathfrak{p}}$.

Now we consider the degree-1 component of the graded centre. For any $l \in \mathbb{Z}$, $l \geqslant 1$, we define $\eta^l_{\mathfrak{p}} \in Z^1(\boldsymbol{D}^b(\mathcal{H}_{\mathfrak{p}}))$ by setting $(\eta^l_{\mathfrak{p}})_{R/\mathfrak{p}^s} = 0$ for all $s \neq l$, $(\eta^l_{\mathfrak{p}})_{R/\mathfrak{p}^l} = \sigma^l_{\mathfrak{p}}$ and $(\eta^l_{\mathfrak{p}})_{\Sigma^i(R/\mathfrak{p}^s)} = (-1)^i \Sigma^i(\eta^l_{\mathfrak{p}})_{R/\mathfrak{p}^s}$ for all i, s. To show that $\eta^l_{\mathfrak{p}} \in Z^1(\boldsymbol{D}^b(\mathcal{H}_{\mathfrak{p}}))$, it suffices to show that $\eta^l_{\mathfrak{p}}$ is a natural transformation. For this, one needs to check that, for all i, j, m, n and $f\colon \Sigma^i R/\mathfrak{p}^m \to \Sigma^j R/\mathfrak{p}^n$, the equality

$$\Sigma f \circ (\eta^l_{\mathfrak{p}})_{\Sigma^i R/\mathfrak{p}^m} = (\eta^l_{\mathfrak{p}})_{\Sigma^j R/\mathfrak{p}^n} \circ f$$

holds. This is clear since both sides of the above equality vanish, unless f is an isomorphism, where we use the fact that any $\sigma^l_{\mathfrak{p}}$ is given by an almost split sequence.

The argument above shows that if, for each l, we fix an Auslander–Reiten sequence, say $\lambda_l \eta^l_{\mathfrak{p}}$ for some $\lambda_l \in k_{\mathfrak{p}}$, with starting term $R/\mathfrak{p}^l$, then we obtain an element $\Sigma_l \lambda_l \eta^l_{\mathfrak{p}}$ in

$Z^1(\boldsymbol{D}^b(\mathcal{H}_\mathfrak{p}))$. The infinite product makes sense, since when it is applied to any object in $\boldsymbol{D}^b(\mathcal{H}_\mathfrak{p})$ it becomes a finite sum.

Conversely, let $\eta \in Z^1(\boldsymbol{D}^b(\mathcal{H}_\mathfrak{p}))$. We claim that $\eta_{R/\mathfrak{p}^l}$ either corresponds to an Auslander–Reiten sequence or is zero for all l. This can be done by induction on l. Clearly, it holds for $l = 1$. Assume that it holds for all $l \leqslant n-1$; we will show that it is also true for $l = n$.

From the exact sequence $0 \to \mathfrak{p}^{n-1}/\mathfrak{p}^n \to R/\mathfrak{p}^n \xrightarrow{\pi} R/\mathfrak{p}^{n-1} \to 0$ we obtain an exact sequence

$$\operatorname{Ext}^1_R(R/\mathfrak{p}^n, R/\mathfrak{p}) \to \operatorname{Ext}^1_R(R/\mathfrak{p}^n, R/\mathfrak{p}^n) \xrightarrow{f} \operatorname{Ext}^1_R(R/\mathfrak{p}^n, R/\mathfrak{p}^{n-1}) \to 0.$$

It is easy to show that $\operatorname{Ker}(f)$ is one dimensional and spanned by the Auslander–Reiten sequences with starting term $R/\mathfrak{p}^n$. Since η is a natural transformation, the map $\pi\colon R/\mathfrak{p}^n \to R/\mathfrak{p}^{n-1}$ yields $\Sigma\pi \circ \eta_{R/\mathfrak{p}^n} = \eta_{R/\mathfrak{p}^{n-1}} \circ \pi = 0$, where the last equality holds since $\eta_{R/\mathfrak{p}^{n-1}}$ is given by some Auslander–Reiten sequence or zero. Thus, $\eta_{R/\mathfrak{p}^n}$ corresponds to an Auslander–Reiten sequence by the above argument. The proposition now follows. □

Next we combine Lemma 3.1, Proposition 3.4 and the condition that $\operatorname{mod}(R) = \mathcal{H}_+ \vee \mathcal{H}_0$ with $\mathcal{H}_0 = \coprod_{\mathfrak{p}\in\max(R)} \mathcal{H}_\mathfrak{p}$. This gives the following proposition. Note that $\mathcal{H}_+$ consists of free modules and hence $Z^1(\Sigma^*\mathcal{H}_+) = 0$.

Proposition 3.6. *Let R be a Dedekind domain and let $\max(R)$ be the set of all maximal ideals. Then, as a graded ring*

$$Z^*(\boldsymbol{D}^b(\operatorname{mod}(R))) \cong T\Bigg(R, \prod_{\mathfrak{p}\in\max(R)} \prod_{l\in\mathbb{Z}, l\geqslant 1} k_\mathfrak{p}\Bigg),$$

where each $k_\mathfrak{p} \cong R/\mathfrak{p}$ is viewed as a simple R-module.

4. Tame hereditary algebras and weighted projective lines

This section deals with the derived category for some further classes of hereditary categories. We consider either the category of modules $\operatorname{mod}(A)$ of a tame hereditary algebra A or the category $\operatorname{Coh}(\mathbb{X})$ for a weighted projective line $\mathbb{X}$ of non-negative Euler characteristic. Unfortunately, our methods do not work for the wild cases. What we want to emphasize is that tubes are of special importance in our calculations.

Throughout this section, k denotes an algebraically closed field and all categories considered are assumed to be k-linear; therefore, the graded centres are k-algebras. Note that most results hold for an arbitrary base field k; however, the proofs would require modifications.

We begin by studying tubes. The tubes occurring in this section are different from those for Dedekind domains and we will use a different method to deal with them. Note that one can use completed path algebras to unify the proofs.

Let $\mathcal{C}$ be a uniserial hom-finite hereditary length k-category. Recall that a length category is an abelian category such that any object has a composition series of finite length. Note that a length category is always a Krull–Remak–Schmidt category, i.e. any object can be written as a finite direct sum of indecomposables and the endomorphism ring of any indecomposable object is local. Following [**1**], a locally finite abelian category is called uniserial if any indecomposable object of finite length has a unique composition series.

It follows from [**16**, Theorem 2.13], that any hom-finite length category is equivalent to the category of finite-length comodules of some basic coalgebra. And since k is assumed to be algebraic closed any basic coalgebra is pointed, that is, it can be realized as a subcoalgebra of certain path coalgebra of some quiver.

A quiver $Q = (Q_0, Q_1, s, t)$ is by definition an oriented graph, where Q_0 is the set of vertices, Q_1 the set of edges that are usually called arrows, s and t are two maps from Q_1 to Q_0 such that, for each arrow α, $s(\alpha)$ and $t(\alpha)$ denote respectively the starting vertex and the terminating vertex of α. A path in Q is a sequence of arrows $\alpha_1\alpha_2\cdots\alpha_n$ with $t(\alpha_i) = s(\alpha_{i+1})$ for $1 \leqslant i \leqslant n-1$; $s(\alpha_1)$ and $t(\alpha_n)$ are called the starting vertex and the terminating vertex, respectively, and n is the length. Each vertex v can be viewed as a path of length 0 which starts and terminates at v.

It is well known that there are a path algebra and a path coalgebra structure on the vector space kQ with their basis consisting of all paths in Q, and the multiplication and comultiplication are given by composing and splitting the paths. Denote by kQ^a and (kQ^c, Δ, ϵ) the path algebra and path coalgebra of Q, respectively.

We denote the category of k-representations by $\mathrm{Rep}(Q)$ and the subcategory of locally nilpotent representations by $\mathrm{NRep}(Q)$. As usual, we denote the subcategories consisting of finite-length objects by $\mathrm{rep}(Q)$ and $\mathrm{nrep}(Q)$. It is well known that $\mathrm{Rep}(Q)$ is equivalent to the module category of the path algebra kQ^a and $\mathrm{NRep}(Q)$ is equivalent to the comodule category of the path coalgebra kQ^c.

Let $n, m \in \mathbb{Z} \cup \{-\infty, +\infty\}$ with $n \leqslant m$. We use $A_{[n,m]}$ to denote the following quiver. The vertices are indexed by $\{i \in \mathbb{Z} \mid n \leqslant i \leqslant m\}$ and for each $n \leqslant i \leqslant m-1$ there is exactly one arrow which starts at the vertex i and terminates at $i+1$. Now denote the quivers $A_{[-\infty,0]}$, $A_{[-\infty,+\infty]}$, $A_{[0,+\infty]}$ and $A_{[0,n]}$ for any $n \geqslant 1$ by A^{∞}, A^{∞}_{∞}, A_{∞} and A_n, respectively. Also, we denote by Z_n the basic cycle of length n for any $n \geqslant 1$, i.e. the quiver obtained from A_n by gluing the vertices 0 and n.

The following classification is a special case of [**6**, Theorem 2.10 (i)].

Lemma 4.1. *Let $\mathcal{C}$ be a uniserial hereditary length k-category. Then $\mathcal{C}$ is equivalent to* $\mathrm{nrep}(Q)$ *for some quiver Q, where Q is a disjoint union of quivers of type A^{∞}, A^{∞}_{∞}, A_{∞}, A_n or Z_n.*

The idea of the proof is easy. As shown in [**16**], $\mathcal{C}$ is equivalent to the category of finite-length comodules of some pointed coalgebra C, and that $\mathcal{C}$ is hereditary means that C must be a path coalgebra and hence $\mathcal{C}$ is given by some $\mathrm{nrep}(Q)$. The fact that $\mathcal{C}$ is uniserial implies that, for each vertex $v \in Q_0$, there is at most one arrow starting at v and at most one arrow terminating at v, and hence the lemma follows.

Now suppose that Q is one of A^{∞}, A^{∞}_{∞}, A_{∞} and A_n. Then the category $\mathrm{nrep}(Q)$ is directed. More explicitly, any indecomposable object $M \in \mathrm{nrep}(Q)$ is a stone, i.e. $\mathrm{End}_{\mathcal{C}}(M) \cong k$ and $\mathrm{Ext}^1_{\mathcal{C}}(M, M) = 0$. This has the following consequence.

Proposition 4.2. *$Z^*(\boldsymbol{D}^b(\mathrm{nrep}(Q))) \cong k$ for $Q = A^\infty$, A^∞_∞, A_∞ and A_n.*

The only case left is $Q = Z_n$, where $n \geqslant 1$ is a positive integer. It is well known that $\mathrm{nrep}(Z_n)$ is a tube of τ-period n. We need to fix some notation. Denote by S_i the simple representation with respect to the vertex i, and $M_i^{[l]}$ the indecomposable representation with socle S_i and of length l, for any i and $l \geqslant 1$. In the category $\mathrm{nrep}(Q)$, there is neither a non-zero projective object nor a non-zero injective object, while in the category $\mathrm{NRep}(Q)$ there are enough injective objects, and we denote by $M_i^{[\infty]}$ or simply M_i the indecomposable injective module with socle S_i. Note that $\{M_i^{[l]} \mid 0 \leqslant i \leqslant n-1,\ 1 \leqslant l \leqslant \infty\}$ gives a complete set of isoclasses of indecomposables in $\mathrm{NRep}(Q)$.

There is a monomorphism $i_s^l \colon M_s^{[l]} \to M_s^{[l+1]}$ and an epimorphism $\pi_s^{[l]} \colon M_s^{[l]} \to M_{s-1}^{[l-1]}$ for any s and l, and any morphism between the indecomposables is a linear combination of compositions of such morphisms. For convenience, we again write π_s^∞ as π_s, and we set $M_s^{[l]} = 0$ and $i_s^l = \pi_s^l = 0$ for $l \leqslant 0$. More generally, we can define monomorphisms

$$i_s^{l,t} = i_s^{l+t-1} \circ \cdots \circ i_s^l \colon M_s^{[l]} \to M_s^{[l+t]}, \quad \forall 0 \leqslant s \leqslant n-1,\ l \geqslant 1,\ t \geqslant 1,$$

and epimorphisms

$$\pi_s^{l,t} = \pi_{s-t+1}^{l-t+1} \circ \cdots \circ \pi_s^l \colon M_s^{[l]} \to M_{s-t}^{[l-t]}, \quad \forall 0 \leqslant s \leqslant n-1,\ l \geqslant 1,\ 1 \leqslant t \leqslant l-1.$$

We also set $i_s^{l,\infty}$ to be the inclusion $M_s^{[l]} \to M_s$ and $i_s^{l,0} = \pi_s^{l,0} = \mathrm{id}_{M_s^{[l]}}$. Note that we have the equality of morphisms

$$\pi_s^{l+1} \circ i_s^l = i_{s-1}^{l-1} \circ \pi_s^l \colon M_s^{[l]} \to M_{s-1}^{[l]}$$

for all s and l. The syzygy functor Ω^{-1} is given by $\Omega^{-1}(M_s^{[l]}) = M_{s-l}$, $\Omega^{-1}(i_s^l) = \pi_{s-l}$ and $\Omega^{-1}(\pi_s^l) = \mathrm{id}_{M_{s-l}}$. In the case $n = 1$, the subscript s is omitted for simplicity.

Lemma 4.3. *Let $Q = Z_n$. Then $Z^0(\boldsymbol{D}^b(\mathrm{nrep}(Q))) \cong k[\![\xi]\!]$, where ξ is the natural transformation from the identity functor to itself, which is given by*

$$\xi_{M_s^{[l]}} = i_{s-n}^{l-n,n} \circ \pi_s^{l,n} \colon M_s^{[l]} \to M_s^{[l]}.$$

It is easy to check that ξ is a natural transformation. Moreover, the infinite sum $\sum_{m \geqslant 0} \lambda_m \xi^m$ gives a natural transformation, where $\lambda_m \in k$ for all m. Observe that this does make sense, because the sum is indeed a finite sum when applied to any object in $\mathrm{nrep}(Q)$. To show that this gives all the natural transformations, one just uses the fact that $\{\xi^m_{M_s^{[l]}},\ m \geqslant 0\}$ spans $\mathrm{End}_{\mathcal{C}}(M_s^{[l]})$ for any $M_s^{[l]} \in \mathrm{nrep}(Q)$.

Clearly, we have an exact sequence

$$0 \to M_s^{[l]} \xrightarrow{i_s^{l,\infty}} M_s \xrightarrow{\pi_s^{\infty,l}} M_{s-l} \longrightarrow 0$$

for any $M_s^{[l]}$. This induces an epimorphism

$$\mathrm{Hom}_{\mathcal{C}}(M_r^{[m]}, M_{s-l}) \twoheadrightarrow \mathrm{Ext}^1_{\mathcal{C}}(M_r^{[m]}, M_s^{[l]}),$$

which is an isomorphism when $m \leqslant l$. In particular, we can identify $\mathrm{Hom}_{\mathcal{C}}(M_s^{[l]}, M_{s-l})$ with $\mathrm{Ext}^1_{\mathcal{C}}(M_s^{[l]}, M_s^{[l]})$.

The following lemma is needed to describe the degree 1 component of the graded centre of $\boldsymbol{D}^b(\mathrm{nrep}(Q))$.

Lemma 4.4. *Let $Q = Z_n$ and $n \geqslant 1$. If $n \geqslant 2$, then $Z^1(\boldsymbol{D}^b(\mathrm{nrep}(Q))) = 0$; if $n = 1$, then as a k-vector space,*

$$Z^1(\boldsymbol{D}^b(\mathrm{nrep}(Q))) \cong \prod_{l \in \mathbb{Z},\, l \geqslant 1} k \cdot \eta^l,$$

where η^l is given by $(\eta^l)_{M^{[l]}} = i^{1,\infty} \circ \pi^{l,l-1}$ and $(\eta^l)_{M^{[a]}} = 0$ for $a \neq l$.

Proof. First we consider the case $n \geqslant 2$. Fix $\eta \in Z^1(\boldsymbol{D}^b(\mathrm{nrep}(Q)))$. We show that $\eta_{M_s^{[l]}} = 0$ for all s and l by using induction on l. Clearly, $\eta_{M_s^{[1]}} = 0$ for all s, since there is no self-extension for the simple objects if $n \geqslant 2$.

Now assume that the assertion holds for $l - 1$. Applying the naturality of η to the injection $i_s^{l-1} \colon M_s^{[l-1]} \to M_s^{[l]}$, we get the equality

$$\eta_{M_s^{[l]}} \circ i_s^{l-1} = \Sigma i_s^{l-1} \circ \eta_{M_s^{[l-1]}} = 0.$$

We claim that this equality holds only if $\eta_{M_s^{[l]}} = 0$. Otherwise, if

$$\eta_{M_s^{[l]}} \colon M_s^{[l]} \to M_{s-l}$$

is non-zero, then the dimension of $\mathrm{Im}(\eta_{M_s^{[l]}})$ is at least n since $M_{s-l}^{[n]}$ is the minimal submodule of M_{s-l} with the same top as $M_s^{[l]}$; thus, the dimension of $\mathrm{Im}(\eta_{M_s^{[l]}} \circ i_s^{l-1})$ is at least $n - 1$ and hence non-zero (here we see the difference between the $n = 1$ and $n \geqslant 2$ cases), and now we use the isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(M_s^{[l-1]}, M_{s-l}) \cong \mathrm{Ext}^1_{\mathcal{C}}(M_s^{[l-1]}, M_s^{[l]})$$

to obtain that the left-hand side of the above equality is non-zero; this introduces a contradiction.

Now we assume that $n = 1$. Note that any Auslander–Reiten sequence with starting term $M_s^{[l]}$ is given by a non-zero multiple of $\eta^l_{M^{[l]}}$. Now we can use the same argument as in the proof of Proposition 3.5. □

Combining Lemmas 4.3 and 4.4, we get the following.

Proposition 4.5. *Let $Q = Z_n$ and $n \geqslant 1$. If $n \geqslant 2$, then $Z^*(\boldsymbol{D}^b(\mathrm{nrep}(Q))) \cong k[\![\xi]\!]$ is a graded k-algebra concentrated in degree 0; if $n = 1$, we have an isomorphism*

$$Z^*(\boldsymbol{D}^b(\mathrm{nrep}(Q))) \cong T\left(k[\![\xi]\!], \prod_{l \in \mathbb{Z},\, l \geqslant 1} k\right)$$

of graded algebras, where k is viewed as the unique simple $k[\![\xi]\!]$-module on which ξ acts trivially. Moreover, we have an isomorphism of graded algebras

$$T\left(k[\![\xi]\!], \prod_{l \in \mathbb{Z},\, l \geqslant 1} k\right) \cong k[\![\xi]\!][\eta]/(\eta^2),$$

where ξ is of degree 0 and η is of degree 1.

Remark 4.6. In case that $n = 1$, we know that $\mathrm{nrep}(Z_1)$ is equivalent to the category of finite-dimensional nilpotent $k[x]$-modules, which is just the category of finitely generated (x)-torsion modules over the Dedekind domain $k[x]$. Thus, Lemma 3.5 applies and we get the same result. More generally, one can consider the completed path algebra $k\hat{Z}_n$ of the quiver Z_n. Then $\mathrm{nrep}(Z_n)$ is equivalent to the category of finite-dimensional nilpotent modules over $k\hat{Z}_n$, and the centre of $k\hat{Z}_n$ is isomorphic to $k[\![x]\!]$.

Remark 4.7. Combining Lemma 4.1 with Propositions 4.2 and 4.5, we have now a description of $Z^*(\boldsymbol{D}^b(\mathcal{C}))$ for any uniserial hereditary length k-category $\mathcal{C}$.

Next we consider the category of finite-dimensional modules over finite-dimensional hereditary k-algebras. Since k is assumed to be algebraically closed, we need only consider the path algebras. Now let Q be a finite, connected quiver without oriented cycles and let $A = kQ$ be the path algebra. Note that in this case the centre of the algebra is the field k. First we consider the finite-type case.

Proposition 4.8. *Let Q be a quiver such that the path algebra kQ is of finite-representation type. Then $Z^*(\boldsymbol{D}^b\mathrm{mod}(kQ)) \cong k$.*

Proof. The proof is almost the same as that of Proposition 4.2. If A is of finite-representation type, then any indecomposable A-module M is a stone. In particular, $\mathrm{Hom}_{\boldsymbol{D}^b(A)}(M, \Sigma^1 M) = \mathrm{Ext}^1_A(M, M) = 0$. □

Next we consider the tame case. Let τ be the Auslander–Reiten translation in $\mathrm{mod}(A)$. The Auslander–Reiten quiver of $\mathrm{mod}(A)$ consists of the pre-projective part, the pre-injective part and the regular part. Recall that a A-module M is pre-projective if and only if $\tau^n M = 0$ for sufficiently large n; and M is pre-injective if and only if $\tau^{-n} M = 0$ for sufficiently large n. Modules without pre-projective and pre-injective summands are called regular modules. Denote by $\mathcal{P}$, $\mathcal{R}$ and $\mathcal{I}$ the full subcategories of pre-projective modules, regular modules and pre-injective modules, respectively. We have the decomposition $\mathrm{mod}(A) = \mathcal{P} \vee \mathcal{R} \vee \mathcal{I}$. Let $\eta \in Z^1(\boldsymbol{D}^b(\mathrm{mod}(A)))$. Since pre-projective and pre-injective modules have no self-extensions, we get the following easy lemma by applying Proposition 3.4.

Lemma 4.9. *Let Q be a quiver such that the path algebra kQ is of tame representation type, and let $\mathcal{R}$ denote the full subcategory of $\mathrm{mod}(kQ)$ consisting of regular modules. Then $Z^1\boldsymbol{D}^b(\mathrm{mod}(kQ)) \cong Z^1(\boldsymbol{D}^b(\mathcal{R}))$.*

Recall that for a tame quiver the regular part of the Auslander–Reiten quiver is a disjoint union of tubes, and there are neither morphisms nor extensions between different tubes, i.e. $\mathcal{R} = \coprod_{\mathfrak{t}\in\mathfrak{T}} \mathcal{R}_{\mathfrak{t}}$, where $\mathfrak{T}$ is an index set for all the tubes, $\mathcal{R}_{\mathfrak{t}} \cong \mathrm{nrep}(Z_{p(\mathfrak{t})})$ and $p(\mathfrak{t})$ denotes the τ-period of $\mathcal{R}_{\mathfrak{t}}$. Each tube is an abelian subcategory and we have $\boldsymbol{D}^b(\mathcal{R}) \cong \coprod_{\mathfrak{t}\in\mathfrak{T}} \boldsymbol{D}^b(\mathcal{R}_{\mathfrak{t}})$. Applying Propositions 4.5 and 3.4, we get the following.

Proposition 4.10. *Let Q be a tame quiver, and let $\mathfrak{T}_1$ be the index set of all homogeneous tubes. Then*

$$Z^*(\boldsymbol{D}^b(\mathrm{mod}(kQ))) \cong T\left(k, \prod_{\mathfrak{t}\in\mathfrak{T}_1} \prod_{m\geqslant 0} k\right).$$

Next we consider the weighted projective lines over the field k. Recall that a weighted projective line $\mathbb{X}$ is defined through the attached category $\mathrm{Coh}(\mathbb{X})$ of coherent sheaves, which is a small k-category satisfying certain axioms. This concept was introduced by Geigle and Lenzing in [**7**] to study the interaction between pre-projective modules and regular modules for tame hereditary algebras. For a definition we refer the reader to [**12**, § 10], where one can also find most references about this subject.

Weighted projective lines play an important role in the classification of hereditary categories. By a theorem of Happel [**8**], any connected, Ext-finite, hereditary abelian k-category which has a tilting complex is derived to be equivalent to either the category $\mathrm{mod}(A)$ for some finite-dimensional hereditary algebra A or the category $\mathrm{Coh}(\mathbb{X})$ for some weighted projective line $\mathbb{X}$.

First we recall some basic facts. Let $\mathbb{X}$ be a weighted projective line and let $\mathcal{H} = \mathrm{Coh}(\mathbb{X})$ be the category of coherent sheaves. The category $\mathcal{H}$ has Serre duality, i.e. there exist an equivalence $\tau\colon \mathcal{H} \to \mathcal{H}$ and a natural isomorphism $D\,\mathrm{Ext}^1_{\mathcal{H}}(X,Y) \cong \mathrm{Hom}_{\mathcal{H}}(Y, \tau X)$. We denote by $\mathcal{H}_0$ the full subcategory consisting of all objects of finite length. Then $\mathcal{H}_0$ is a hereditary abelian subcategory and $\mathcal{H}_0 = \coprod_{x\in C} \mathcal{U}_x$ for some index set C, where $\mathcal{U}_x$ is a tube with finite τ-period $p(x)$. Members of C are called the points of $\mathcal{H}$. Note that there are only finitely many points with $p(x) > 1$.

We denote by $\mathcal{H}_+$ the subcategory consisting of all objects without a simple subobject. Objects of $\mathcal{H}_+$ are called vector bundles. Any indecomposable object of $\mathcal{H}$ is either of finite length or a vector bundle. There is a linear form $\mathrm{rk}\colon \mathrm{K}_0(\mathcal{H}) \to \mathbb{Z}$, called rank, which is τ-invariant, vanishes on objects of $\mathcal{H}_0$ and takes positive values on objects of $\mathcal{H}_+$. Objects of $\mathcal{H}_+$ of rank 1 are called line bundles, and by definition $\mathcal{H}$ contains a line bundle. For any vector bundle E, we have a filtration $E_0 \subseteq E_1 \subset \cdots \subseteq E_r = E$ with the line bundle factors E_i/E_{i-1}, where $r = \mathrm{rk}(E)$.

For any line bundle L and any point $x \in C$, $\sum_{S\in\mathcal{U}_x} \dim_k \mathrm{Hom}_{\mathcal{H}}(L,S) = 1$, where S runs through all simple objects in $\mathcal{U}_x$. Clearly, we have $\mathcal{H} = \mathcal{H}_+ \vee \mathcal{H}_0$, and therefore any non-zero morphism between line bundles is a monomorphism.

Now we consider the graded centre of $\boldsymbol{D}^b(\mathcal{H})$. Note that one can define the Euler characteristic $\chi_{\mathcal{H}}$ for $\mathcal{H}$. If $\chi_{\mathcal{H}} > 0$, then $\mathcal{H}$ is derived as equivalent to the category $\mathrm{mod}(A)$ for some finite-dimensional tame hereditary algebra A, and in this case the graded centre has been computed. Firstly, we have the following easy lemma.

Lemma 4.11. *Let $\mathbb{X}$ be a weighted projective line. Then $Z^0(\boldsymbol{D}^b(\mathrm{Coh}(\mathbb{X}))) = k$.*

Proof. We denote $\mathrm{Coh}(\mathbb{X})$ by $\mathcal{H}$ as before. Since $\mathcal{H}$ contains a line bundle, we choose one and denote it by L. Let $\eta\colon \mathrm{Id}_{\mathcal{H}} \to \mathrm{Id}_{\mathcal{H}}$ be a natural transformation. To prove the lemma, it suffices to show that if $\eta_L = 0$, then $\eta = 0$.

Now assume that $\eta_L = 0$. Let $x \in C$ be an arbitrary point, and let $S \in \mathcal{U}_x$ be the simple object with $\mathrm{Hom}_{\mathcal{H}}(L,S) \neq 0$. Note that such S exists and is unique. By Proposition 4.3, $\eta_{\mathcal{U}_x} = 0$ if and only if $\eta_{S^{[mr+1]}} = 0$ for all $m \geqslant 0$, where $r = p(x)$ is the τ-period and $S^{[mr+1]}$ is the object in $\mathcal{U}_x$ with socle S and of length $mr+1$. By using induction we have $\dim_k \mathrm{Hom}_{\mathcal{H}}(L, S^{[mr+1]}) = m+1$ for any $m \geqslant 0$. We claim that there exists an epimorphism from L to $S^{[mr+1]}$. Otherwise, all morphisms will factor through

$S^{[(m-1)r+1]}$, and hence

$$\dim_k \operatorname{Hom}_{\mathcal{H}}(L, S^{[mr+1]}) = \dim_k \operatorname{Hom}_{\mathcal{H}}(L, S^{[(m-1)r+1]}) = m,$$

which gives a contradiction.

Let $f\colon L \to S^{[mr+1]}$ be an epimorphism. Since η is a natural transformation, we have $\eta_{S^{[mr+1]}} \circ f = f \circ \eta_L = 0$, and hence $\eta_{S^{[mr+1]}} = 0$. Now we have shown that $\eta_{\mathcal{H}_0} = 0$. Conversely, using a similar argument, one can show that if $\eta_{\mathcal{H}_0} = 0$, then $\eta_N = 0$ for any line bundle N. Since any vector bundle E has a filtration with line bundle factors, we get, using the Five Lemma, that $\eta_E = 0$. This completes the proof. □

Combining the above results with Proposition 3.4, Lemma 4.4 and Proposition 4.5, we obtain the following embedding of algebras.

Lemma 4.12. *Let $\mathbb{X}$ be a weighted projective line, let $\mathcal{H} = \operatorname{Coh}(\mathbb{X})$ and let C_1 be the set of points of τ-period 1. Then the algebra*

$$Z = T\biggl(k, \prod_{x \in C_1} \prod_{m \geqslant 0} k\biggr)$$

is isomorphic to a subalgebra of $Z^*(\boldsymbol{D}^b(\mathcal{H}))$.

In the tubular case, i.e. $\chi_{\mathcal{H}} = 0$, we have $\mathcal{H} = \bigvee_{q \in \mathbb{Q} \cup \{\infty\}} \mathcal{H}^{\langle q \rangle}$, where for each q we have $\mathcal{H}^{\langle q \rangle} \cong \mathcal{H}_0$. In fact, one can define the slope for objects of $\mathcal{H}$ and, roughly speaking, for any $q \in \mathbb{Q}$, $\mathcal{H}^{\langle q \rangle}$ is just given by objects of slope q, and $\mathcal{H}^{\langle \infty \rangle} = \mathcal{H}_0$. With this decomposition of categories, we have the following proposition.

Proposition 4.13. *Let $\mathbb{X}$ be a weighted projective line of Euler characteristic 0, $\mathcal{H} = \operatorname{Coh}(\mathbb{X})$ and C_1 the set of points of τ-period 1. Then*

$$Z^*(\boldsymbol{D}^b(\mathcal{H})) \cong T\biggl(k, \prod_{q \in \mathbb{Q} \cup \{\infty\}} \prod_{x \in C_1} \prod_{m \geqslant 0} k\biggr).$$

5. The graded centre of $\boldsymbol{D}^b(\mathbf{mod}(k[x]/(x^2)))$

In this section, we study the ring of dual numbers, which is by definition the k-algebra $A = k[x]/(x^2)$, where k is an arbitrary base field. Set $\mathcal{C} = \operatorname{mod}(A)$ and $\mathcal{P}$ the full subcategory of $\mathcal{C}$ consisting of projective modules. One has a complete description of the indecomposable objects of $\boldsymbol{D}^b(\mathcal{C}) = \boldsymbol{K}^{+,b}(\mathcal{P})$, and therefore one can write down the elements in $Z^*(\boldsymbol{D}^b(\mathcal{C}))$ explicitly. By Theorem 2.5, we need only to consider the category $\boldsymbol{K}^b(\mathcal{P})$.

The indecomposable objects in $\boldsymbol{K}^b(\mathcal{P})$ are well understood (see, for example, [**11**]). They are given by $\{A_m^n \mid -\infty < m \leqslant n < \infty\}$, where A_m^n is the complex

$$\cdots \longrightarrow 0 \longrightarrow \underbrace{A}_{n} \xrightarrow{x} A \xrightarrow{x} \cdots \xrightarrow{x} \underbrace{A}_{m} \longrightarrow 0 \longrightarrow \cdots,$$

that is, $(A_m^n)_i = A$ for $m \leqslant i \leqslant n$ and 0 otherwise, and $d_i^{A_m^n} = x$ for all $m < i \leqslant n-1$, where we use x to denote the multiplication map l_x. If we allow n to take the value ∞, then we get all indecomposable objects in $\boldsymbol{K}^{+,b}(\mathcal{P})$, in fact $A_m^\infty \cong \Sigma^m S$ in the derived category, where S is the simple A-module, regarded as a stalk complex concentrated in degree 0. Note that $\Sigma A_m^n \cong A_{m+1}^{n+1}$. The following lemma is basic for our computations.

Lemma 5.1. *Let $-\infty < m \leqslant n < \infty$ and let $-\infty < m' \leqslant n' < \infty$. If $(m,n) \neq (m',n')$, then $\mathrm{Hom}_{\boldsymbol{K}^b(\mathcal{P})}(A_m^n, A_{m'}^{n'})$ is at most one dimensional. The morphisms between indecomposable objects in $\boldsymbol{K}^b(\mathcal{P})$ are linear combinations of compositions of the following four classes of morphisms:*

(a) $\pi_m^{n,n'} \colon A_m^n \to A_m^{n'}$ *for $m \leqslant n' \leqslant n$, $(\pi_m^{n,n'})_m = x$ and $(\pi_m^{n,n'})_i = 0$ for all $i \neq m$;*

(b) $\pi_{m,m'}^n \colon A_m^n \to A_{m'}^n$ *for $m \leqslant m' \leqslant n$, $(\pi_{m,m'}^n)_i = 1$ for all $m' \leqslant i \leqslant n$;*

(c) $i_m^{n,n'} \colon A_m^n \to A_m^{n'}$ *for $m \leqslant n \leqslant n'$, $(i_m^{n,n'})_i = 1$ for all $m \leqslant i \leqslant n$;*

(d) $i_{m,m'}^n \colon A_m^n \to A_{m'}^n$ *for $m' \leqslant m \leqslant n$, $(i_{m,m'}^n)_m = x$, and $(i_{m,m'}^n)_i = 0$ for all $i \neq m$.*

The morphisms in the lemma look as follows.

(a)
$$\begin{tikzcd}
0 \arrow[r] & \overbrace{A}^{n} \arrow[r] & \cdots \arrow[r] & \overbrace{A}^{n'} \arrow[r] \arrow[d, "0"] & \cdots \arrow[r] & \overbrace{A}^{m} \arrow[r] \arrow[d, "x"] & 0 \\
 & & 0 \arrow[r] & A \arrow[r] & \cdots \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

(b)
$$\begin{tikzcd}
0 \arrow[r] & \overbrace{A}^{n} \arrow[r] \arrow[d, "1"] & \cdots \arrow[r] & \overbrace{A}^{m'} \arrow[r] \arrow[d, "1"] & \cdots \arrow[r] & \overbrace{A}^{m} \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r] & \cdots \arrow[r] & A \arrow[r] & 0 & &
\end{tikzcd}$$

(c)
$$\begin{tikzcd}
 & & 0 \arrow[r] & A \arrow[r] \arrow[d, "1"] & \cdots \arrow[r] & A \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & \underbrace{A}_{n'} \arrow[r] & \cdots \arrow[r] & \underbrace{A}_{n} \arrow[r] & \cdots \arrow[r] & \underbrace{A}_{m} \arrow[r] & 0
\end{tikzcd}$$

(d)
$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "0"] & \cdots \arrow[r] & A \arrow[r] \arrow[d, "x"] & 0 & & \\
0 \arrow[r] & \underbrace{A}_{n} \arrow[r] & \cdots \arrow[r] & \underbrace{A}_{m} \arrow[r] & \cdots \arrow[r] & \underbrace{A}_{m'} \arrow[r] & 0
\end{tikzcd}$$

The proof is straightforward and left to the reader. For any $m \leqslant n < \infty$, the space $\mathrm{Hom}_{\boldsymbol{K}^b(\mathcal{P})}(A_m^n, A_m^n)$ is two dimensional, and we denote the morphism $i_{m,m}^n = \pi_m^{n,n}$ by x_m^n. Now let $\eta \colon \mathrm{Id}_{\boldsymbol{K}^b(\mathcal{P})} \to \mathrm{Id}_{\boldsymbol{K}^b(\mathcal{P})}$ be a natural transformation. Clearly, η is uniquely given by some datum $\{\mu_m^n, \lambda_m^n \in k,\ -\infty < m \leqslant n < \infty\}$ with $\eta_{A_m^n} = \mu_m^n \cdot 1 + \lambda_m^n x_m^n$.

Proposition 5.2. *Let* $\eta\colon \mathrm{Id}_{\boldsymbol{K}^b(\mathcal{P})} \to \mathrm{Id}_{\boldsymbol{K}^b(\mathcal{P})}$ *be a natural transformation and let* $\{\mu_m^n, \lambda_m^n\}$ *be the corresponding datum.*

(i) *We have* $\mu_m^n = \mu_{m'}^{n'}$ *for any* m, m', n *and* n'. *Conversely, any datum of the form* $\{\mu, \lambda_m^n \in k,\ \infty < m \leqslant n < \infty\}$ *arises as the datum of some natural transformation* $\eta\colon \mathrm{Id}_{\boldsymbol{K}^b(\mathcal{P})} \to \mathrm{Id}_{\boldsymbol{K}^b(\mathcal{P})}$ *by setting* $\eta_{A_m^n} = \mu + \lambda_m^n x_m^n$ *for any* m *and* n.

(ii) *If* $\eta \in Z^0(\boldsymbol{K}^b(\mathcal{P}))$, *then* $\lambda_m^n = \lambda_{m+r}^{n+r}$ *for any* m, n *and* r, *and any elements in* $Z^0(\boldsymbol{K}^b(\mathcal{P}))$ *are obtained in this way.*

(iii) *As an algebra,* $Z^0(\boldsymbol{K}^b(\mathcal{P})) \cong T(k, \prod_{r\geqslant 0} k)$, *where* $T(k, \prod_{r\geqslant 0} k)$ *is viewed as a graded algebra concentrated in degree* 0.

Proof. (i) Apply the naturality of η to get $i_m^{n,n'} \circ \eta_m^n = \eta_m^{n'} \circ i_m^{n,n'}$ and $\pi_{m,m'}^n \circ \eta_m^n = \eta_m^{n'} \circ \pi_{m,m'}^n$. From this follows that $\mu_m^n = \mu_{m'}^{n'}$ for any m, m', n and n'.

Conversely, for any datum of the form $\{\mu, \lambda_m^n \in k,\ \infty < m \leqslant n < \infty\}$, we claim that the η constructed above is indeed a natural transformation. In fact, one can easily show that the equalities $f \circ \eta_m^n = \eta_{m'}^{n'} \circ f$ hold for those morphisms $f\colon A_m^n \to A_{m'}^{n'}$ listed in Lemma 5.1. Now, using the fact that $\boldsymbol{K}^b(\mathcal{P})$ is a Krull–Remak–Schmidt category and any morphism is some linear combination of compositions of morphisms listed in Lemma 5.1, we get that $\eta_Y \circ f = f \circ \eta_X$ holds for any morphism $f\colon X \to Y$ in the category $\boldsymbol{K}^b(\mathcal{P})$. Thus, η is a natural transformation.

(ii) Use the fact that by definition $\eta \in Z^0(\boldsymbol{K}^b(\mathcal{P}))$ if and only if $\Sigma\eta = \eta\Sigma$, and this is equivalent to the requirement that $\lambda_m^n = \lambda_{m+1}^{n+1}$ for any m and n.

(iii) is an easy consequence of (ii). In fact we can explicitly write down the elements in $Z^0(\boldsymbol{K}^b(\mathcal{P}))$. For any $r \geqslant 0$, let $\eta_r \in Z^0(\boldsymbol{K}^b(\mathcal{P}))$ denote the natural transformation obtained by setting $(\eta_r)_{A_m^n} = x_m^n$ for $m - n = r$ and 0 otherwise. Thus, as vector spaces,

$$Z^0(\boldsymbol{K}^b(\mathcal{P})) = k \cdot 1 \oplus \prod_{r\geqslant 0} k \cdot \eta_r.$$

By direct computation, the multiplication satisfies $\eta_r\eta_{r'} = 0$ for any r and r' and the isomorphism in (iii) follows. □

Now we consider the natural transformations from the identity functor to Σ^t for any positive integer $t > 0$. Note that $\mathrm{Hom}_{\boldsymbol{K}^b(\mathcal{P})}(A_m^n, \Sigma^t A_m^n) = 0$ for any m, n with $n < m+t$, and in the case $n \geqslant m + t$, the morphism space is one dimensional with basis element $f_{t;m}^n = i_{m+t}^{n,n+t} \circ \pi_{m,m+t}^n$. Let $\zeta\colon \mathrm{Id}_{\boldsymbol{D}^b(\mathcal{P})} \to \Sigma^t$ be a natural transformation; it is uniquely determined by the datum $\{\lambda_{t;m}^n,\ n \geqslant m + t\}$, where $\zeta_{A_m^n} = \lambda_{t;m}^n f_{t;m}^n$. Applying the naturality of ζ to the morphisms $i_m^{n,n'}$ and $\pi_{m,m'}^n$, one gets $f_{t;m}^n = f_{t;m'}^{n'}$ for any m, m', n and n'. Thus, we get the following lemma.

Lemma 5.3. *Let* $t > 0$. *All natural transformations from* $\mathrm{Id}_{\boldsymbol{K}^b(\mathcal{P})}$ *to* Σ^t *form a one-dimensional* k*-space with a basis element* ζ_t, *where* ζ_t *is given by* $(\eta_t)_{A_m^n} = f_{t;m}^n$ *for all* $n \geqslant m + t$ *and* 0 *otherwise. Moreover, the multiplication satisfies* $\zeta_t\zeta_{t'} = \zeta_{t+t'}$ *for any* $t, t' > 0$ *and* $\zeta_t\eta_r = \eta_r\zeta_t = 0$ *for any* $t > 0$ *and* $r \geqslant 0$, *where the* η_r *are given as in the proof of Proposition 5.2.*

Note that $\Sigma\zeta_t = (-1)^t\zeta_t\Sigma$ if and only if either $\mathrm{char}(k) = 2$ or $\mathrm{char}(k) \neq 2$ and t is even. Combined with the last lemma and Proposition 5.2, we get the graded centre of $\boldsymbol{D}^b(\mathrm{mod}(A))$.

Proposition 5.4. *Let k be an arbitrary base field. Then, as a graded algebra,*

$$Z^*(\boldsymbol{D}^b(\mathrm{mod}(k[x]/(x^2)))) \cong T\biggl(k[\zeta], \prod_{r\geqslant 0} k\biggr),$$

where k is identified with $k[\zeta]/(\zeta)$ as a $k[\zeta]$-module,

$$Z^0(\boldsymbol{D}^b(\mathrm{mod}(k[x]/(x^2)))) \cong T\biggl(k, \prod_{r\geqslant 0} k\biggr),$$

and ζ is of degree 2 if $\mathrm{char}(k) \neq 2$, and of degree 1 if $\mathrm{char}(k) = 2$.

6. The graded centre of the stable category $\underline{\mathrm{mod}}(\boldsymbol{k}[\boldsymbol{x}]/(\boldsymbol{x}^{\boldsymbol{n}}))$

Another important class of triangulated categories is the stable categories of self-injective algebras. We calculate the graded centres in some special cases, namely for the algebras of the form $k[x]/(x^n)$ with $n \geqslant 2$. These calculations are based on the fact that the indecomposable objects and their morphisms are well understood. Note that algebras of the form $k[x]/(x^n)$ are Brauer tree algebras, and we refer the reader to [**9**] for the calculation of the graded centres of their stable module categories.

Let $A = k[x]/(x^n)$ with k an arbitrary base field. It is well known that A is uniserial and that all the indecomposable objects in $\mathrm{mod}(A)$ are of the form $A_l = A/x^lA = x^{n-l}A$ with $1 \leqslant l \leqslant n$. There are epimorphisms $\pi^l_r = l_{x^{l-r}}\colon A_l \twoheadrightarrow A_r$ for $l \geqslant r$ and monomorphisms $i^r_l\colon A_l \hookrightarrow A_r$ for $l \leqslant r$. For any l and r, $\mathrm{Hom}_A(A_l, A_r)$ has a basis $\{f^{l,r}_s = i^r_s \circ \pi^l_s \mid 1 \leqslant s \leqslant \min(l, r)\}$. Moreover, the syzygy functor Ω is given by $\Omega(A_l) = A_{n-l}$ and

$$\Omega(f^{l,r}_s) = \pi^{n-s}_{n-r} \circ i^{n-s}_{n-l} = f^{n-l,n-r}_{n-r-l+s} \quad \text{for all } 1 \leqslant l \leqslant n-1,\ r, s \leqslant l.$$

Now let $\mathcal{C} = \underline{\mathrm{mod}}(A)$ be the stable category. One knows that $\mathcal{C}$ is a triangulated category with suspension functor $\Sigma = \Omega^{-1} = \Omega$. In particular, we have $\Omega^2 = \mathrm{Id}_{\mathcal{C}}$ in $\mathcal{C}$. The indecomposable objects in $\mathcal{C}$ are given by $A_l = A/x^lA = x^{n-l}A$ with $1 \leqslant l < n$, and $\bar{f}^{l,r}_s = 0$ if and only if $l + r - n \geqslant s$. Consequently, $\mathrm{Hom}_{\mathcal{C}}(A_l, A_{n-l}) = \mathrm{Hom}_A(A_l, A_{n-l})$.

For any self-injective ring R, let $Z(R)$ denote the graded centre of R. There is a canonical morphism from $Z(R)$ to $Z^0(\underline{\mathrm{mod}}(R))$. As we will show below, this map is not injective in general. The more interesting question is whether or not it is surjective. In the case $A = k[x]/(x^n)$, the answer is affirmative. In fact, for an arbitrary uniserial self-injective algebra, all natural transformations from the identity functor to itself for the stable category come from the centre of the algebra.

Proposition 6.1. *Let $A = k[x]/(x^n)$ with $n \geqslant 2$ and $\mathcal{C} = \underline{\mathrm{mod}}(A)$. Then $Z^0(\mathcal{C}) \cong k[x]/(x^{[n/2]})$, where $[n/2]$ denotes the maximal integer which is no larger than $\frac{1}{2}n$.*

Proof. Note that $A_{[n/2]}$ is of special importance since $\mathrm{End}_{\mathcal{C}}(A_{[n/2]})$ is of maximal dimension among the indecomposable objects. Let η be a natural transformation from $\mathrm{Id}_{\mathcal{C}}$ to $\mathrm{Id}_{\mathcal{C}}$. We will show that η is uniquely determined by $\eta_{A_{[n/2]}}$.

We fix some $a \in A$ such that $\eta_{A_{[n/2]}} = \bar{l}_a$, where l_a is given by the multiplication with a as before. Since η is a natural transformation, we have $\pi^l_{[n/2]} \circ \eta_{A_l} = \bar{l}_a \circ \pi^l_{[n/2]}$ for $l > [\frac{1}{2}n]$ and $i^{[n/2]}_l \circ \eta_{A_l} = \bar{l}_a \circ i^{[n/2]}_l$ for any $l < [\frac{1}{2}n]$. Now it is easy to show that $\eta_{A_l} = \bar{l}_a$ for any l, since the solutions of the equations above are unique. Therefore, we have an epimorphism from A to $Z^0(\mathcal{C})$, and easy computations show that $l_{x^{[n/2]}} = 0$ in $\mathcal{C}$. □

Next we will compute the natural transformations from the identity functor to Ω. The following lemma is easy.

Lemma 6.2. *Let $\zeta\colon \mathrm{Id}_{\mathcal{C}} \to \Omega$ be a natural transformation. Then, for any $1 \leqslant l < n$, we have $\zeta_{A_l} = \lambda_l \cdot \bar{f}^{l,n-l}_1$ for some $\lambda_l \in k$. And conversely, any family $\{\lambda_l,\ 1 \leqslant l < n\}$ induces a natural transformation ζ by setting $\zeta_{A_l} = \lambda_l \cdot \bar{f}^{l,n-l}_1$ for any l. Moreover, $\zeta \in Z^1(\mathcal{C})$ if and only if $\lambda_l = -\lambda_{n-l}$ for any l.*

Proof. We use induction on l. Clearly, we have $\zeta_{A_1} = \lambda_1 \cdot \bar{f}^{1,n-1}_1$. Now assume that $\zeta_{A_s} = \lambda_s \cdot \bar{f}^{s,n-s}_1$ for some $\lambda_s \in k$, and consider the inclusion i^{s+1}_s. One obtains

$$\zeta_{A_{s+1}} \circ \bar{i}^{s+1}_s = \bar{\pi}^{n-s}_{n-s-1} \circ \zeta_{A_s} = 0,$$

and hence $\zeta_{A_{s+1}} = \lambda_{s+1} \cdot \bar{f}^{s+1,n-s-1}_1$. The remaining part is straightforward. □

Now let ζ_s denote the natural transformation given by $(\zeta_s)_{A_l} = \delta^l_s \bar{f}^{s,n-s}_1$ for any $1 \leqslant l < n$. We also denote by t the identity map from $\mathrm{Id}_{\mathcal{C}}$ to $\Omega^2 = \mathrm{Id}_{\mathcal{C}}$ and view it as an element in $Z^2(\mathcal{C})$.

Let $\tilde{Z}^*(\mathcal{C})$ be the $\mathbb{Z}$-graded space with $\tilde{Z}^n(\mathcal{C})$ consisting of all natural transformations from $\mathrm{Id}_{\mathcal{C}}$ to Ω^n. Note that $\tilde{Z}^*(\mathcal{C})$ forms a graded algebra and $Z^*(\mathcal{C})$ is a subalgebra of $\tilde{Z}^*(\mathcal{C})$.

Observe that the case $n = 2$ is slightly different. In fact, in this case, not only Ω^2 but also the shift functor Ω itself is equivalent to the identity functor. We deal with this case separately. With the above notation, we get the following results.

Proposition 6.3. *Let $\mathcal{C} = \underline{\mathrm{mod}}(k[x]/(x^2))$. Then $\tilde{Z}^*(\mathcal{C}) = k[\zeta_1, \zeta_1^{-1}]$ with ζ_1 of degree 1. We have $Z^*(\mathcal{C}) = \tilde{Z}^*(\mathcal{C})$ if $\mathrm{char}(k) = 2$, and $Z^*(\mathcal{C}) = k[\zeta_1^2]$ if $\mathrm{char}(k) \neq 2$.*

Note that ζ_1^2 equals t as defined above, and clearly ζ_1^{-1} is of degree -1. The proof follows directly from Proposition 6.1 and Lemma 6.2.

Proposition 6.4. *Let $\mathcal{C} = \underline{\mathrm{mod}}(k[x]/(x^n))$ and $n \geqslant 3$. Then we have*

$$\tilde{Z}^*(\mathcal{C}) = k[x, \zeta_1, \ldots, \zeta_{n-1}, t, t^{-1}]/\langle x^{[n/2]}, x\zeta_s, \zeta_s x, \zeta_s\zeta_{s'}\rangle,$$

where x, each ζ_s and t are of degree 0, 1 and 2, respectively. Moreover, $Z^*(\mathcal{C})$ is the subalgebra generated by x, t, t^{-1} and $\zeta_s - \zeta_{n-s}$ with $1 \leqslant s \leqslant [\frac{1}{2}n]$ if either n is odd or $\mathrm{char}(k) \neq 2$; if $\mathrm{char}(k) = 2$ and n is even, then $Z^*(\mathcal{C})$ is the subalgebra generated by x, t, t^{-1}, $\zeta_{[n/2]}$ and $\zeta_s - \zeta_{n-s}$ with $1 \leqslant s \leqslant [\frac{1}{2}n]$.

Corollary 6.5. *Let* $\mathcal{C} = \underline{\mathrm{mod}}(k[x]/(x^n))$ *and* $n \geqslant 3$. *Then, as a graded algebra,*

$$Z^*(\mathcal{C}) = k[x, \zeta_1, \dots, \zeta_l, t, t^{-1}]/\langle x^{[n/2]}, x\zeta_s, \zeta_s x, \zeta_s\zeta_{s'}\rangle$$

with x, *each* η_s *and* t *of degree* 0, 1 *and* 2, *respectively, where* $l = [\frac{1}{2}(n-1)]$ *if either* n *is odd or* $\mathrm{char}(k) \neq 2$, *and* $l = [\frac{1}{2}n]$ *if* $\mathrm{char}(k) = 2$ *and* n *is even.*

Remark 6.6. For a self-injective algebra A, one has

$$\boldsymbol{D}^b(\mathrm{mod}(A))/\boldsymbol{K}^b(\mathrm{proj}\, A) \cong \underline{\mathrm{mod}}(A).$$

We have already seen that $Z^*(\boldsymbol{D}^b(\mathrm{mod}(A))) \cong Z^*(\boldsymbol{K}^b(\mathrm{proj}\, A))$, but what can we say about the ring homomorphism $\pi_*\colon Z^*(\boldsymbol{D}^b(\mathrm{mod}(A))) \to Z^*(\underline{\mathrm{mod}}(A))$?

For the algebra $A = k[x]/(x^2)$ we can describe π_* explicitly, since both graded centres are known. Recall that

$$Z^*(\boldsymbol{D}^b(\mathrm{mod}(k[x]/(x^2)))) = \left(k \oplus \prod_{r\geqslant 0} k\cdot\eta_r\right)[\zeta]/\langle \eta_r\eta_r', \eta_r\zeta\rangle$$

and $Z^*(\underline{\mathrm{mod}}(A)) = k[t, t^{-1}]$. We know that in this case π_* is neither injective nor surjective. Explicitly, $\mathrm{Im}(\pi_*) = k[t]$ and $\mathrm{Ker}(\pi_*) = \prod_{r\geqslant 0} k\cdot\eta_r$.

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