

PRE-RESOLUTIONS OF NONCOMMUTATIVE ISOLATED SINGULARITIES

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ABSTRACT. We introduce the notion of right pre-resolutions (quasi-resolutions) for noncommutative isolated singularities, which is a weaker version of quasi-resolutions introduced by Qin-Wang-Zhang in [QWZ]. We prove that right quasi-resolutions for noetherian bounded below and locally finite graded algebra with right injective dimension 2 are always Morita equivalent. When we restrict to noncommutative quadric hypersurfaces, we prove that a noncommutative quadric hypersurface, which is a noncommutative isolated singularity, always admits a right pre-resolution. Besides, we provide a method to verify whether a noncommutative quadric hypersurface is an isolated singularity. An example of noncommutative quadric hypersurfaces with detailed computations of indecomposable maximal Cohen-Macaulay modules and right pre-resolutions is included as well.

0. INTRODUCTION

Let R be a commutative normal Gorenstein domain. Van den Bergh introduced the notion of noncommutative crepant resolutions of R in [VdB]. Roughly speaking, a noncommutative crepant resolution of R is an R -algebra of the form $\Lambda = \text{End}_R(M)$, where M is a reflexive R -module. Iyama-Reiten extended the notion of noncommutative crepant resolutions to module-finite commutative algebras over a noetherian commutative Cohen-Macaulay ring (cf. [IR]). Let R be a commutative Cohen-Macaulay equi-codimensional normal Gorenstein domain with a canonical module. It has been proven that noncommutative crepant resolutions of R are always derived equivalent provided $\dim R \leq 3$ (cf. [IR, Corollary 8.8] and [IW, Theorem 1.5]).

In order to study noncommutative singularities, Qin-Wang-Zhang extended the notion of noncommutative resolutions to noncommutative algebras which are possibly not module-finite over their centers (cf. [QWZ]). Let A be a (both left and right) noetherian algebra over a field k , and let ∂ be a symmetric dimension function of the category of right A -modules. Two right A -modules M and N are said to be s -isomorphic (cf. [QWZ]) if there is a right A -module P and two homomorphisms $f: P \rightarrow M$ and $g: P \rightarrow N$ such that the ∂ -dimensions of the kernels and the cokernels of f and g are no larger than s . The following definition was given in [QWZ].

Definition 0.1. Let A be a (both left and right) noetherian algebra with ∂ -dimension d . If there is a noetherian Auslander regular ∂ -Cohen-Macaulay algebra B with ∂ -dimension d and two finitely generated bimodules ${}_B M_A$ and ${}_A N_B$ such that $M \otimes_A N$ is $(d-2)$ -isomorphic to B and $N \otimes_B M$ is $(d-2)$ -isomorphic to A as bimodules, then B is called a *noncommutative quasi-resolution* of A .

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Qin-Wang-Zhang proved that noncommutative quasi-resolutions of a noetherian algebra A with ∂ -dimension 2 are Morita equivalent. If ∂ -dimension of A is 3, then noncommutative quasi-resolutions of A are derived equivalent (with further assumptions on A , cf. [QWZ, Theorem 0.6]). Thus, they generalized the corresponding results in [IR] and [IW].

If further, A is Auslander Gorenstein and ∂ -Cohen-Macaulay, then the algebra B in Definition 0.1 is isomorphic to the endomorphism algebra $\text{End}_A(U)$ for some bimodule ${}_B U_A$ which is reflexive on both sides (cf. [QWZ, Corollary 3.13]).

Unlike the commutative case, given a noncommutative noetherian algebra A and a finitely generated right A -module U , it is usually a tough task to check whether $\text{End}_A(U)$ is a noetherian algebra. In this sense, to find a noncommutative quasi-resolution of a noetherian algebra is not an easy job in general.

In this paper, we only consider the noncommutative resolutions of noncommutative graded isolated singularities (cf. Section 2), which allows us to drop some restrictions on the algebras as given in Definition 0.1.

Now let A be a bounded below graded algebra, that is, $A = \bigoplus_{i \in \mathbb{Z}} A_i$ with $A_i = 0$ for $i \ll 0$. Assume A is right noetherian and locally finite. Let $\text{gr } A$ be the category of finitely generated right graded A -modules, and $\text{tor } A$ the subcategory of $\text{gr } A$ consisting of finite dimensional modules. Let $\text{qgr } A = \text{gr } A / \text{tor } A$. We introduce the following weaker version of noncommutative resolutions of right noetherian algebras, which is much closer to Van den Bergh's original definition of noncommutative crepant resolution (cf. [VdB]).

Definition 0.2. (More precisely, see Definition 2.1) Let A be a right noetherian graded algebra which is bounded below and locally finite with injective dimension $\text{injdim } A_A = d < \infty$. If there is a small maximal Cohen-Macaulay (cf. Section 2) module M_A with $B = \text{End}_A(M)$ such that

- (i) right global dimension $\text{r.gldim}(B) = d$,
- (ii) the functor $\underline{\text{Hom}}_A(M, -): \text{gr } A \rightarrow \text{gr } B$ induces an equivalence $\text{qgr } A \cong \text{qgr } B$,

then we call B a *right pre-resolution* of A .

If further, B is *right generalized Artin-Schelter regular* (cf. Definition 1.3), then we call B a *right quasi-resolution* of A .

Then we have the following result (cf. Theorem 2.3) parallel to the ones in [IR, IW, QWZ]. We remark that our proof is quite different from theirs. In fact, the algebras considered in [IR, IW, QWZ] are assumed to be noetherian on both sides, while we only assume the right noetherianness here.

Theorem 0.3. *Let A be a right noetherian graded algebra which is bounded below and locally finite with injective dimension $\text{injdim } A_A = 2$. If A has a right quasi-resolution, then*

- (i) A is *CM-finite*, that is, there are only finitely many nonisomorphic indecomposable maximal Cohen-Macaulay (MCM) right A -modules (up to degree shifts);
- (ii) any two right quasi-resolutions of A are graded Morita equivalent.

If A is an AS-Gorenstein algebra (cf. Definition 1.1) which is a noncommutative isolated singularity, then the CM-finiteness will induce the existence of right pre-resolutions, as shown in the following results (cf. Theorems 3.2 and 3.6).

Theorem 0.4. *Let A be an AS-Gorenstein algebra which is a noncommutative isolated singularity.*

- (i) *Let M_A be an MCM module. Then $\underline{\text{End}}_A(M)$ is a right noetherian graded algebra.*
- (ii) *Assume that A is CM-finite and $\text{injdim } A \geq 2$. Let $\{P_0 = A, P_1, \dots, P_n\}$ be the set of all the nonisomorphic indecomposable MCM modules (up to degree shifts). Let $M = \bigoplus_{i=1}^n P_n \oplus A$. Then $B := \underline{\text{End}}_A(M)$ is a right pre-resolution of A .*

Let S be a quantum polynomial algebra (cf. Section 4), that is, S is a Koszul AS-regular algebra with Hilbert series $H_S(t) = \frac{1}{(1-t)^n}$ for some $n \geq 1$. Pick a central regular element $\varpi \in A$ of degree 2. The quotient algebra $A = S/S\varpi$ is called a noncommutative quadric hypersurface. Note that a noncommutative quadric hypersurface is CM-finite if and only if it is a noncommutative isolated singularity (cf. [MU, Theorem 4.13]). Hence a noncommutative quadric hypersurface A which is also a noncommutative isolated singularity always admits a right pre-resolution, and to find such a pre-resolution it suffices to compute all the indecomposable MCM-modules of A .

Assume A is a noncommutative quadric hypersurface with injective dimension d . Let $\Omega^d(\mathbb{k}_A)$ be the d -th syzygy of the trivial module \mathbb{k}_A . Set

$$\mathbb{M} := \Omega^d(\mathbb{k}_A)(d).$$

Then \mathbb{M} is a Koszul MCM module. Associated to A , Smith-Van den Bergh constructed a finite dimensional algebra $C(A)$, and proved that the stable category of MCM modules over A is equivalent to the derived category of $C(A)$. In this paper, we prove the following results, which provides a relatively easy way to compute the algebra $C(A)$ and to find all the indecomposable MCM modules of A . Consequently, we show a method to construct a right pre-resolution of A in case A is a CM-finite.

Theorem 0.5. *(cf. Theorems 4.6 and 4.11) Let S be a quantum polynomial algebra, and let $\varpi \in A$ be a central regular element of degree 2. Set $A := S/S\varpi$. We have the follow consequences.*

- (i) $\text{End}_{\text{gr } A}(\mathbb{M}) \cong C(A)$;
- (ii) A is a noncommutative isolated singularity if and only if $\text{End}_{\text{gr } A}(\mathbb{M})$ is semisimple.
- (iii) Assume that A is a noncommutative isolated singularity. Then $B = \underline{\text{End}}_A(\mathbb{M} \oplus A)$ is a right pre-resolution of A .

We remark that the second statement of the above theorem follows from [HY, Theorem 6.3] (see also [MU, Theorem 4.13]).

In the view of Theorem 0.5, some properties of indecomposable MCM modules of A are obtained in Section 4. We provide a concrete example of quadric hypersurfaces with detailed computations of indecomposable MCM modules and right pre-resolutions in the last section.

1. PRELIMINARIES

Throughout, \mathbb{k} will be a field of characteristic zero and all algebras considered are over \mathbb{k} .

Let A be a \mathbb{Z} -graded \mathbb{k} -algebra. We denote by $\text{Gr } A$ the category of right graded A -modules, and by $\text{Hom}_{\text{Gr } A}(M, N)$ the set of homogeneous right A -module homomorphisms which preserve the degrees of elements for $M, N \in \text{Gr } A$. For $k \in \mathbb{Z}$, $M(k)$ is the right graded A -module such that $M(k)_n = M_{n+k}$. We write $\underline{\text{Hom}}_A(M, N) = \bigoplus_{k \in \mathbb{Z}} \text{Hom}_{\text{Gr } A}(M, N(k))$, and $\underline{\text{Ext}}_A^i(M, N)$ for the derived functor of $\underline{\text{Hom}}_A(M, N)$. In particular, we write $\underline{\text{End}}_A(M)$ for $\underline{\text{Hom}}_A(M, M)$.

Let $\text{PHom}_{\text{Gr } A}(M, N)$ be the subset of $\text{Hom}_{\text{Gr } A}(M, N)$ consisting of homomorphisms which factor through some graded projective modules, and let $\underline{\text{PHom}}_A(M, N) = \bigoplus_{k \in \mathbb{Z}} \text{PHom}_{\text{Gr } A}(M, N(k))$. The set of graded stable homomorphism is denoted by $\underline{\text{SHom}}_A(M, N) = \underline{\text{Hom}}_A(M, N) / \underline{\text{PHom}}_A(M, N)$.

Let A be a right noetherian graded algebra, and let $\text{gr } A$ be the full subcategory of $\text{Gr } A$ consisting of finitely generated modules. A right graded module $M \in \text{Gr } A$ is called a *torsion* module if for every $m \in M$, the right submodule mA is finite dimensional. Let $\text{Tor } A$ be the full subcategory of $\text{Gr } A$ consisting of all the torsion modules, and let $\text{tor } A$ be the full subcategory of $\text{Tor } A$ consisting of finite dimensional ones. Since A is right noetherian, $\text{Tor } A$ (resp. $\text{tor } A$) is a Serre subcategory of $\text{Gr } A$ (resp. $\text{gr } A$). The quotient categories $\text{QGr } A = \text{Gr } A / \text{Tor } A$ and $\text{qgr } A = \text{gr } A / \text{tor } A$ are both abelian and $\text{qgr } A$ is an abelian subcategory of $\text{QGr } A$. Let $\pi: \text{Gr } A \rightarrow \text{QGr } A$ be the projection functor. Then π has a right adjoint functor $\omega: \text{QGr } A \rightarrow \text{Gr } A$ such that $\pi\omega \cong \text{id}$.

For $M \in \text{Gr } A$, we write $\mathcal{M} = \pi(M)$. The Hom-sets in $\text{QGr } A$ is defined by

$$\text{Hom}_{\text{QGr } A}(\mathcal{M}, \mathcal{N}) = \varinjlim \text{Hom}_{\text{Gr } A}(M', N/K),$$

where the limit runs over all the pairs (M', K) with $K \subseteq N$ and $M' \subseteq M$ such that both K and M/M' are torsion modules. We refer to [AZ] for more information about the quotient categories.

A graded algebra A is locally finite if $\dim A_n < \infty$ for all $n \in \mathbb{Z}$, and A is *bounded below* if $A_n = 0$ for all $n < 0$. If $A_0 = \mathbb{k}$ and $A_n = 0$ for all $n < 0$, then A is said to be *connected* graded.

We recall the following classical definition (cf. [AS]).

Definition 1.1. Let A be a (both left and right) noetherian connected graded algebra. A is called an *Artin-Schelter Gorenstein* algebra of dimension d if

- (i) $\text{injdim } {}_A A = \text{injdim } A_A = d < \infty$;
- (ii) $\underline{\text{Ext}}_A^i(\mathbb{k}, A) = 0$ for all $i \neq d$, and $\underline{\text{Ext}}_A^d(\mathbb{k}, A) = \mathbb{k}(l)$ for some l ;
- (iii) the left version of (ii) is satisfied.

If further, $\text{gldim } A = d$, then A is called an *Artin-Schelter regular* algebra. The integer l is usually called the *Gorenstein parameter* of A .

We need a more general version of Artin-Schelter Gorenstein algebras in this paper. Without otherwise statement, we always assume that A is a right noetherian graded algebra which is locally finite and bounded below. Let $J(A)$ be the graded Jacobson radical of A . We recall the following well-known facts without giving a proof.

Lemma 1.2. *Retain the notations as above. Then the following statements hold true.*

- (i) $A/J(A)$ is finite dimensional.
- (ii) Let $J(A_0)$ be the Jacobson radical of A_0 . Then $J(A_0) = J(A) \cap A_0$.
- (iii) There is an integer n_0 such that $J(A) \supseteq A_{\geq n_0}$, and $\bigcap_{n \geq 0} J(A)^n = 0$.

We generalize Artin-Schelter Gorenstein algebras to right noetherian bounded below algebras.

Definition 1.3. Let A be as above and $J = J(A)$ be the graded Jacobson radical of A . We call A a *right generalized Artin-Schelter Gorenstein algebra* of dimension d if

- (i) $\text{injdim } A_A = d < \infty$,
- (ii) $\underline{\text{Ext}}_A^i(A/J, A) = 0$ for all $i \neq 0$,
- (iii) as a left A -module $\underline{\text{Ext}}_A^d(A/J, A)$ is annihilated by J , and $\underline{\text{Ext}}_A^d(A/J, A)$ is invertible as a graded A/J - A/J -bimodule.

If further, the right global dimension $\text{r.gldim}(A) = d$, then we call A a *right generalized Artin-Schelter regular algebra*.

Below, the phase ‘‘Artin-Schelter’’ is simply denoted by ‘‘AS’’, and ‘‘generalized Artin-Schelter’’ by ‘‘GAS’’.

Remark 1.4. The notion of GAS-Gorenstein algebras is a slight generalization of the ones in [RR, Definition 1.4] and [MM, Definition 3.1], where the algebras considered are \mathbb{N} -graded.

Let $\Gamma: \text{gr } A \rightarrow \text{gr } A$ be the torsion functor, that is, $\Gamma(M) = \{m \in M \mid \dim_{\mathbb{k}}(mA) < \infty\}$. The i -th right derived functor of Γ is denoted by $R^i\Gamma$. By Lemma 1.2, we have $\Gamma \cong \lim_{n \rightarrow \infty} \underline{\text{Hom}}_A(A/J^n, -)$. For $M \in \text{gr } A$, the *depth* of M is defined to be

$$\text{depth}(M) = \min\{i \mid R^i\Gamma(M) \neq 0\}.$$

Then $\text{depth}(M)$ is either a non-negative integer or ∞ . The following lemma is classical for connected graded algebras, and the proof for connected case applies also to our case.

Lemma 1.5. $\text{depth}(M_A) = \min\{i \mid \underline{\text{Ext}}_A^i(A/J, M) \neq 0\}$.

Recall that $\pi: \text{Gr } A \rightarrow \text{QGr } A$ has a right adjoint functor $\omega: \text{QGr } A \rightarrow \text{Gr } A$. The following lemma is clear.

Lemma 1.6. *Let M_A be a finitely generated module. If $\text{depth}(M_A) \geq 2$, then $\omega\pi(M) \cong M$.*

A useful homological identity in the theory of AS-Gorenstein algebras is Auslander-Buchsbaum formula (cf. [Jo, Theorem 3.2] for a noncommutative version), which provides an effective way to calculate the depth of a module over a local ring. For our purpose, it will be handfull to have a more general version of the Auslander-Buchsbaum formula for right GAS-Gorenstein algebras. We mention that the proof is a modification of that of [Jo, Theorem 3.2].

Theorem 1.7 (Auslander-Buchsbaum formula). *Let A be a right GAS-Gorenstein algebra, and $M_A \in \text{gr } A$. Suppose that the projective dimension $\text{projdim}(M_A) < \infty$. Then*

$$\text{projdim}(M_A) + \text{depth}(M_A) = \text{depth}(A_A).$$

Proof. If $\text{injdim } A_A = 0$, then it is clear. Assume $\text{injdim}(A_A) = d \geq 1$ and $\text{projdim}(M_A) = p$. Take a graded projective resolution of the right module A/J :

$$\dots \rightarrow P^{-2} \rightarrow P^{-1} \rightarrow P^0 \rightarrow A/J \rightarrow 0,$$

where each P^i is finitely generated for all i . Applying $\underline{\text{Hom}}_A(-, A)$ to the resolution, we obtain the following sequence:

$$(1.7.1) \quad 0 \longrightarrow \underline{\text{Hom}}_A(P^0, A) \longrightarrow \underline{\text{Hom}}_A(P^{-1}, A) \longrightarrow \underline{\text{Hom}}_A(P^{-2}, A) \longrightarrow \cdots .$$

Take a minimal graded projective resolution

$$(1.7.2) \quad 0 \rightarrow Q^{-p} \rightarrow \cdots \rightarrow Q^{-1} \rightarrow Q^0 \rightarrow M \rightarrow 0$$

of M . By taking the tensor product of (1.7.2) and (1.7.1) we obtain a double complex

$$\begin{array}{ccccccc} & \vdots & & \vdots & & \vdots & \\ & \downarrow & & \downarrow & & \downarrow & \\ Q^{-2} \otimes_A \underline{\text{Hom}}_A(P^0, A) & \longrightarrow & Q^{-2} \otimes_A \underline{\text{Hom}}_A(P^{-1}, A) & \longrightarrow & Q^{-2} \otimes_A \underline{\text{Hom}}_A(P^{-2}, A) & \longrightarrow & \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ Q^{-1} \otimes_A \underline{\text{Hom}}_A(P^0, A) & \longrightarrow & Q^{-1} \otimes_A \underline{\text{Hom}}_A(P^{-1}, A) & \longrightarrow & Q^{-1} \otimes_A \underline{\text{Hom}}_A(P^{-2}, A) & \longrightarrow & \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ Q^0 \otimes_A \underline{\text{Hom}}_A(P^0, A) & \longrightarrow & Q^0 \otimes_A \underline{\text{Hom}}_A(P^{-1}, A) & \longrightarrow & Q^0 \otimes_A \underline{\text{Hom}}_A(P^{-2}, A) & \longrightarrow & \cdots , \end{array}$$

which implies a convergent spectral sequence

$$(1.7.3) \quad E_2^{rs} = \text{Tor}_r^A(M, \underline{\text{Ext}}_A^s(A/J, A)) \implies \underline{\text{Ext}}_A^{s-r}(A/J, M).$$

By assumption, $\underline{\text{Ext}}_A^s(A/J, A) = 0$ for $s \neq d$ and $\underline{\text{Ext}}_A^d(A/J, A) \cong V$ for some graded invertible A/J - A/J -bimodule V . Thus the spectral sequence (1.7.3) collapses at the second level. Since the resolution (1.7.2) is minimal, we have $\text{Tor}_s^A(M, V) \cong Q^{-s} \otimes_A V$, and hence $\text{Tor}_s^A(M, V) \neq 0$ for each $0 \leq s \leq p$ for V is invertible. Therefore, $\underline{\text{Ext}}_A^{d-p}(A/J, M) \neq 0$ and $\underline{\text{Ext}}_A^i(A/J, M) = 0$ for all $i < d - p$. By Lemma 1.5, $\text{depth}(M) = d - p$. The Auslander-Buchsbaum formula follows. \square

2. NONCOMMUTATIVE RESOLUTIONS

Noncommutative crepant resolutions for commutative Gorenstein algebras were introduced by Van den Bergh (cf. [VdB]). Later, Qin-Wang-Zhang introduced noncommutative quasi-resolutions for noncommutative Auslander Gorenstein algebra in [QWZ]. In this section, we will modify the definition of noncommutative resolutions of [QWZ] and give a weaker version of noncommutative resolutions for noncommutative isolated singularities.

Let A be a right noetherian graded algebra which is bounded below and locally finite. Recall that A is called a *noncommutative isolated singularity* (cf. [U]), if $\text{qgr } A$ has finite global dimension, i.e., there is an integer $n_0 \geq 0$ such that $\text{Ext}_{\text{qgr } A}^i(\mathcal{M}, \mathcal{N}) = 0$ for all $i > n_0$ and $M, N \in \text{gr } A$.

Recall that a finitely generated right graded A -module M is said to be *small* if the following conditions are satisfied:

- (i) $\underline{\text{End}}_A(M)$ is a right noetherian graded algebra, and
- (ii) $\underline{\text{Hom}}_A(M, N)$ is a finitely generated right graded $\underline{\text{End}}_A(M)$ -module for any $N \in \text{gr}(A)$.

Let M_A be small and let $B = \underline{\text{End}}_A(M)$. Since M is finitely generated and A is bounded below and locally finite, B is also bounded below and locally finite, and we have an additive functor

$$F = \underline{\text{Hom}}_A(M, -): \text{gr } A \longrightarrow \text{gr } B.$$

If K is a finite dimensional right graded A -module, then $\underline{\text{Hom}}_A(M, K)$ is a finite dimensional right graded B -module. For $X, Y \in \text{gr } A$, let $f: X \rightarrow Y$ be a homomorphism such that both $\ker f$ and $\text{coker } f$ are finite dimensional. Let $f_* = \underline{\text{Hom}}_A(M, f)$. It is not hard to see both $\ker f_*$ and $\text{coker } f_*$ are finite dimensional. Therefore, the functor F induces a functor $\mathcal{F}: \text{qgr } A \longrightarrow \text{qgr } B$ which fits into the following commutative diagram

$$(2.0.1) \quad \begin{array}{ccc} \text{gr } A & \xrightarrow{F} & \text{gr } B \\ \pi \downarrow & & \downarrow \pi \\ \text{qgr } A & \xrightarrow{\mathcal{F}} & \text{qgr } B. \end{array}$$

Suppose that A has finite injective dimension $\text{injdim } A_A = d$. Recall that a finitely generated right graded A -module M is called a *maximal Cohen-Macaulay module* (MCM module, for simplicity) if $R^i\Gamma(M) = 0$ for all $i \neq d$.

Definition 2.1. Let A be a right noetherian graded algebra which is bounded below and locally finite with injective dimension $\text{injdim } A_A = d < \infty$. If there is a small MCM module M_A such that

- (i) right global dimension $\text{r.gldim}(B) = d$,
- (ii) the functor \mathcal{F} , as in the diagram (2.0.1), is an equivalence,

then we call B a *right pre-resolution* of A .

If further, B is right GAS-regular, then we call B a *right quasi-resolution* of A .

Remark 2.2. (1) A right noetherian graded algebra which admits a right pre-resolution is automatically a noncommutative isolated singularity. This follows from the well-known fact that the global dimension of $\text{qgr } B$ is no greater than the one of $\text{gr } B$, see for instance [AZ, Section 7].

(2) The above definition is a modification of [QWZ, Definition 0.5] for noncommutative isolated singularities, where the algebra B is assumed to be a (both left and right) noetherian Auslander regular \mathbb{N} -graded algebra. We will show some examples of right pre-resolutions and right quasi-resolutions for right GAS-Gorenstein algebras in the subsequent sections. We consider non-positively graded algebras because a noncommutative resolution of a noetherian algebra may not positively graded in general.

A right noetherian graded locally finite algebra with finite right injective dimension is said to be *CM-finite* if it has, up to degree shifts, finitely many nonisomorphic indecomposable MCM modules.

Our main result of this section is as follows. It may be viewed as a noncommutative version of [IW, Theorem 1.5] in dimension 2 case.

Theorem 2.3. *Let A be a right noetherian graded algebra which is bounded below and locally finite with injective dimension $\text{injdim } A_A = 2$. Assume that A has a right quasi-resolution. Then*

- (i) A is CM-finite;

(ii) any two right quasi-resolutions of A are graded Morita equivalent.

Proof. (i) By assumption, there exists some small MCM module M_A such that $B = \underline{\text{End}}_A(M)$ is a right quasi-resolution of A . Consider the functor $F = \underline{\text{Hom}}_A(M, -): \text{gr } A \rightarrow \text{gr } B$ and the induced functor $\mathcal{F}: \text{qgr } A \rightarrow \text{qgr } B$. Then we have commutative diagram (2.0.1). Take an MCM A -module N , and let $X = \underline{\text{Hom}}_A(M, N)$. Then $X \in \text{gr } B$. Since \mathcal{F} is an equivalence, we have

$$\text{Hom}_{\text{qgr } A}(\pi(M), \pi(N)(k)) \cong \text{Hom}_{\text{qgr } B}(\mathcal{F}(\pi(M)), \mathcal{F}(\pi(N)(k))), \text{ for all } k \in \mathbb{Z}.$$

By definition

$$\begin{aligned} \text{Hom}_{\text{qgr } B}(\mathcal{F}(\pi(M)), \mathcal{F}(\pi(N)(k))) &= \text{Hom}_{\text{qgr } B}(\pi(\underline{\text{Hom}}_A(M, M)), \pi(\underline{\text{Hom}}_A(M, N))(k)) \\ &= \text{Hom}_{\text{qgr } B}(\pi(B), \pi(X)(k)). \end{aligned}$$

Let J be the graded Jacobson radical of B . Since B/J is finite dimensional and X is finitely generated, we have

$$\text{Hom}_{\text{qgr } B}(\pi(B), \pi(X)(k)) = \lim_{n \rightarrow \infty} \text{Hom}_{\text{gr } B}(J^n, X(k)).$$

From the exact sequence $0 \rightarrow J^n \rightarrow B \rightarrow B/J^n \rightarrow 0$ we obtain the following exact sequence

$$(2.3.1) \quad \begin{aligned} 0 \longrightarrow \lim_{n \rightarrow \infty} \text{Hom}_{\text{gr } B}(B/J^n, X(k)) \longrightarrow \text{Hom}_{\text{gr } B}(B, X(k)) \longrightarrow \lim_{n \rightarrow \infty} \text{Hom}_{\text{gr } B}(J^n, X(k)) \\ \longrightarrow \lim_{n \rightarrow \infty} \text{Ext}_{\text{gr } B}^1(B/J^n, X(k)) \longrightarrow 0. \end{aligned}$$

By the commutative diagram (2.0.1), we have the following commutative diagram

$$(2.3.2) \quad \begin{array}{ccc} \text{Hom}_{\text{gr } B}(B, X(k)) & \longrightarrow & \lim_{n \rightarrow \infty} \text{Hom}_{\text{gr } B}(J^n, X(k)) \\ \uparrow = & & \uparrow = \\ \text{Hom}_{\text{gr } B}(B, X(k)) & \longrightarrow & \text{Hom}_{\text{qgr } B}(\pi(B), \pi(X)(k)) \\ \uparrow F & & \uparrow \mathcal{F} \\ \text{Hom}_{\text{gr } A}(M, N(k)) & \longrightarrow & \text{Hom}_{\text{qgr } A}(\pi(M), \pi(N)(k)). \end{array}$$

Note that $X_k = \underline{\text{Hom}}_A(M, N)(k) = \text{Hom}_{\text{gr } A}(M, N(k))$. It follows that the maps in the left column of the diagram (2.3.2) are bijective. Since \mathcal{F} is an equivalence, the maps in the right column are bijective. Since N is MCM, then $\text{depth}(N) = 2$. It follows from Lemma 1.6 that $\text{Hom}_{\text{qgr } A}(\pi(M), \pi(N)(k)) \cong \text{Hom}_{\text{gr } A}(M, N(k))$. Therefore the bottom map in the diagram (2.3.2) is also bijective. Hence the top map in the diagram (2.3.2) is an isomorphism.

By the exact sequence (2.3.1), we have, for all $k \in \mathbb{Z}$, $\lim_{n \rightarrow \infty} \text{Hom}_{\text{gr } B}(B/J^n, X(k)) = 0$ and $\lim_{n \rightarrow \infty} \text{Ext}_{\text{gr } B}^1(B/J^n, X(k)) = 0$. Hence $\Gamma(X) = R^1\Gamma(X) = 0$. It follows that $\text{depth}(X) \geq 2$. Since $\text{r.gldim}(B) = \text{injdim } A_A = 2$, X is a projective B -module by Auslander-Buchsbaum formula (cf. Lemma 1.2). Hence $X \in \text{add}(B)$, where $\text{add}(B)$ is the category of direct summands of direct sums of finite copies of degree shifts of B . Therefore $\pi(X) \in \text{add}(\pi(B))$ and hence $\pi(N) \in \text{add}(\pi(M))$ for \mathcal{F} is an equivalence. Since both M and N are MCM, it follows that $\omega\pi(M) \cong M$ and $\omega\pi(N) \cong N$ (cf. Lemma 1.6). Hence $N \in \text{add}(M)$, which implies that every indecomposable MCM module is a direct summand of M (up to a degree shift). Hence A is CM-finite.

(ii) Suppose that N' is another small MCM module such that $B' = \underline{\text{End}}_A(M)$ is a right quasi-resolution of A . By the proof of (i), $N' \in \text{add}(M)$ and $M \in \text{add}(N')$. Therefore $\text{add}(M) = \text{add}(N')$. Hence B and B' are graded Morita equivalent. \square

Remark 2.4. In [QWZ], the authors extended the theory of noncommutative crepant resolutions for commutative algebras to noncommutative settings. They proved that noncommutative quasi-resolutions for a noetherian \mathbb{N} -graded algebra with Gelfand-Kirillov dimension 2 are always Morita equivalent (cf. [QWZ, Theorem 0.6(1)]), which extensively generalizes a similar result in commutative case (cf. [IW, Theorem 1.5]). We remark that a noncommutative quasi-resolution B in [QWZ] is assumed to be (left and right) noetherian \mathbb{N} -graded Auslander regular and Cohen-Macaulay. In contrast, we assume that the resolution B is right GAS-regular. Moreover, the method we used are different from that of [QWZ].

3. ENDOMORPHISM RINGS OF CM-FINITE AS-GORENSTEIN ALGEBRAS

In this section, A is an AS-Gorenstein algebra. We will show that the endomorphism ring of an MCM module over a noncommutative isolated singularity is always right noetherian, which suggests the existence of resolutions for noncommutative isolated singularities.

Lemma 3.1. *Let A be an AS-Gorenstein algebra which is a noncommutative isolated singularity. Let M_A be an MCM module, and let N_A be a finitely generated graded A -module. Then $\underline{\text{SHom}}_A(M, N)$ is finite dimensional.*

Proof. Assume that the injective dimension $\text{injdim}(A_A) = d$. Since M is an MCM module, there is an exact sequence

$$0 \rightarrow M \xrightarrow{\tau} P^{-d} \xrightarrow{\partial^{-d}} P^{-d+1} \rightarrow \dots \rightarrow P^0 \xrightarrow{\partial^0} P^1 \rightarrow \dots,$$

where P^i is a finitely generated graded projective A -module for all $i \geq -d$. Let $X = \text{im } \partial^0$. Then $\underline{\text{Ext}}_A^{d+1}(X, N) = \underline{\text{Hom}}_A(M, N) / \text{im}(\tau^*)$, where $\tau^*: \underline{\text{Hom}}_A(P^{-d}, N) \rightarrow \underline{\text{Hom}}_A(M, N)$ is the induced map. Moreover $\text{im}(\tau^*) = \underline{\text{PHom}}_A(M, N)$ since M is an MCM module. Hence $\underline{\text{Ext}}_A^{d+1}(X, N) = \underline{\text{Hom}}_A(M, N) / \text{im}(\tau^*) = \underline{\text{SHom}}_A(M, N)$. By assumption A is a noncommutative isolated singularity, it follows that $\underline{\text{Ext}}_A^{d+1}(X, N)$ is finite dimensional (cf. [SvdB, Proposition 4.3]). \square

We fix a notation which will be used in the proof of the next theorem. For an integer i and a graded A -module X , we define a graded homomorphism $s^i: X(i) \rightarrow X$ by setting $s^i(x) = x$ for all $x \in X$. Then s^i is a graded homomorphism of degree i .

Theorem 3.2. *Let A be an AS-Gorenstein algebra which is a noncommutative isolated singularity. Let M_A be an MCM module. Then $\underline{\text{End}}_A(M)$ is a right noetherian graded algebra.*

Proof. Set $B = \underline{\text{End}}_A(M)$. Let I be a graded right ideal of B and let S be a finite set consisting of homogeneous elements of I . We write $M^S = \sum_{b \in S} b(M)$. Then M^S is a submodule of M . Consider the set

$$\mathcal{X} = \{M^S \mid S \text{ finite set of homogeneous elements of } I\}.$$

Since A is noetherian and M is finitely generated, the set \mathcal{X} has a maximal object. Let M^{S_0} be a maximal object in \mathcal{X} . For every element $b' \in I$, consider the set $S_1 = S_0 \cup \{b'\}$. Since $M^{S_1} \supseteq M^{S_0}$,

and by assumption M^{S_0} is maximal, it follows $M^{S_1} = M^{S_0}$. Hence $b'(M) \subseteq \sum_{b \in S_0} b(M)$ and $M^{S_0} = IM$. Set $N = IM = M^{S_0}$.

Assume $S_0 = \{b_1, \dots, b_n\}$, and the degrees of b_1, \dots, b_n are k_1, \dots, k_n respectively. Let

$$\phi: M(-k_1) \oplus M(-k_2) \oplus \cdots \oplus M(-k_n) \longrightarrow N$$

be the homomorphism defined by the homomorphisms $\{b_1 s^{-k_1}, \dots, b_n s^{-k_n}\}$. Then ϕ is a homomorphism of degree 0.

We next prove that the right ideal I is finitely generated. There two different situations.

Case 1. For $b' \in I$, assume the degree of the homomorphism b' is k , and assume that the composition $M(-k) \xrightarrow{s^{-k}} M \xrightarrow{b'} N$ factors through a graded projective module, that is, there is a graded projective module P such that $b' s^{-k}$ is the composition $M(-k) \xrightarrow{r} P \xrightarrow{t} N$, where both r and t are homomorphisms of degree 0. Since ϕ is an epimorphism, there is a homomorphism $f: P \rightarrow N$ such that $t = \phi f$. Then $b' s^{-k} = tr = \phi fr$. Let $h = fr$. Then h is a morphism from $M(-k)$ to $M(-k_1) \oplus M(-k_2) \oplus \cdots \oplus M(-k_n)$. Let $h': M \rightarrow M(-k_1) \oplus M(-k_2) \oplus \cdots \oplus M(-k_n)$ be the homomorphism such that $h = h' s^{-k}$. Then $b' = \phi h'$. Let $p_i: M(-k_1) \oplus M(-k_2) \oplus \cdots \oplus M(-k_n) \rightarrow M(-k_i)$ be the projection map. Then $b' = \sum_{i=1}^n (b_i s^{-k_i})(p_i h')$. For each i , let $h'_i = s^{-k_i} p_i h'$. Then h'_i is an endomorphism of M . Hence, in this case, $b' = \sum_{i=1}^n b_i h'_i \in \sum_{i=1}^n b_i B$.

Case 2. Assume b' does not factor through any projective modules. Since A is a noncommutative isolated singularity, $\underline{\text{SHom}}_A(M, N)$ is finite dimensional by Lemma 3.1. Note that $\text{im}(b) = bM \subseteq N$ for any $b \in I$, thus we may view b as an element in $\text{Hom}_A(M, N)$ and identify I with a subspace of $\text{Hom}_A(M, N)$. Then

$$\bar{I} = I / (I \cap \underline{\text{PHom}}_A(M, N)) \cong (I + \underline{\text{PHom}}_A(M, N)) / \underline{\text{PHom}}_A(M, N)$$

is finite dimensional, for the latter is a subspace of $\underline{\text{SHom}}_A(M, N)$. Choose homogeneous elements $f_1, \dots, f_m \in I$ such that their images $\bar{f}_1, \dots, \bar{f}_m$ in \bar{I} form a basis of \bar{I} . Let \bar{b}' be the image of b' in the quotient space \bar{I} . Since $\bar{f}_1, \dots, \bar{f}_m$ is a basis, we may write $\bar{b}' = l_1 \bar{f}_1 + \cdots + l_m \bar{f}_m$ for some $l_1, \dots, l_m \in \mathbb{k}$. Then $b' - (l_1 f_1 + \cdots + l_m f_m) \in I \cap \underline{\text{PHom}}_A(M, N)$. By Case 1, $b' - (l_1 f_1 + \cdots + l_m f_m) = \sum_{i=1}^n b_i g_i$ for some $g_1, \dots, g_n \in B$. It follows that $b' = l_1 f_1 + \cdots + l_m f_m + \sum_{i=1}^n b_i g_i$.

Summarizing, the right ideal I is generated by $f_1, \dots, f_m, b_1, \dots, b_n$. □

The proof of Theorem 3.2 also implies the following result.

Corollary 3.3. *Retain the same notations as in Theorem 3.2. Let N be a finitely generated right graded A -module. Then $\underline{\text{Hom}}_A(M, N)$ is a finitely generated right graded $\underline{\text{End}}_A(M)$ -module.*

We conclude the following result which suggests noncommutative resolutions for noncommutative singularities.

Proposition 3.4. *Let A be an AS-Gorenstein algebra and $M_A \in \text{gr } A$ be an MCM module. Assume that A is a noncommutative isolated singularity. Then $B := \underline{\text{End}}_A(M \oplus A)$ is a right noetherian graded algebra, and moreover, there is an equivalence of abelian categories $\text{qgr } B \cong \text{qgr } A$.*

Proof. The graded algebra B may be written as a matrix algebra

$$B = \begin{pmatrix} \underline{\text{End}}_A(M) & M \\ M^\vee & A \end{pmatrix},$$

where $M^\vee = \underline{\text{Hom}}_A(M, A)$. Define a map

$$\varphi: M \otimes_A M^\vee \longrightarrow \underline{\text{End}}_A(M), m_1 \otimes_A f \mapsto [m_2 \mapsto m_1 f(m_2)].$$

The multiplication of the above matrix algebra reads as

$$\begin{pmatrix} g_1 & m_1 \\ f_1 & a_1 \end{pmatrix} \begin{pmatrix} g_2 & m_2 \\ f_2 & a_2 \end{pmatrix} = \begin{pmatrix} g_1 g_2 + \varphi(m_1 \otimes_A f_2) & g_1(m_2) + m_1 a_2 \\ f_1 g_2 + a_1 f_1 & f_1(m_2) + a_1 a_2 \end{pmatrix}.$$

Let $e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then $A \cong eBe$, and $BeB = \begin{pmatrix} \varphi(M \otimes_A M^\vee) & M \\ M^\vee & A \end{pmatrix}$. It is easy to see $\varphi(M \otimes_A M^\vee) = \underline{\text{PHom}}_A(M, M)$. Then

$$B/BeB \cong \underline{\text{SHom}}_A(M, M).$$

By Lemma 3.1, B/BeB is finite dimensional. By the graded version of the proof of [BHZ, Lemma 2.3], we obtain the desired equivalence $\text{qgr } B \cong \text{qgr } A$. \square

Lemma 3.5. *Let A be an AS-Gorenstein algebra which is a noncommutative isolated singularity. Let M_A be an MCM module. For $X \in \text{gr } A$, define*

$$\varphi_X: X \otimes_A \underline{\text{Hom}}_A(M, A) \longrightarrow \underline{\text{Hom}}_A(M, X), x \otimes f \mapsto [m \mapsto x f(m)].$$

Then both $\ker(\varphi)$ and $\text{coker}(\varphi)$ are finite dimensional.

Proof. Let $0 \rightarrow K \rightarrow P \rightarrow X \rightarrow 0$ be an exact sequence with P a finitely generated graded projective A -module. Then we have the following commutative diagram

$$\begin{array}{ccccccc} K \otimes_A \underline{\text{Hom}}_A(M, A) & \longrightarrow & P \otimes_A \underline{\text{Hom}}_A(M, A) & \longrightarrow & X \otimes_A \underline{\text{Hom}}_A(M, A) & \longrightarrow & 0 \\ & & \varphi_K \downarrow & & \varphi_P \downarrow & & \varphi_X \downarrow \\ 0 & \longrightarrow & \underline{\text{Hom}}_A(M, K) & \longrightarrow & \underline{\text{Hom}}_A(M, P) & \longrightarrow & \underline{\text{Hom}}_A(M, X) \end{array}$$

with exact rows. Note that φ_P is an isomorphism. By Snake Lemma, $\ker(\varphi_X) \cong \text{coker}(\varphi_K)$.

Since $\text{im}(\varphi_X) = \underline{\text{PHom}}_A(M, X)$, it follows from Lemma 3.1 that $\text{coker}(\varphi_X)$ is finite dimensional. Similarly $\text{coker}(\varphi_K)$ and hence $\ker(\varphi_X)$ is also finite dimensional. \square

Theorem 3.6. *Let A be an AS-Gorenstein algebra with $\text{injdim } A \geq 2$ which is a noncommutative isolated singularity. Assume that A is CM-finite. Let $\{P_0 = A, P_1, \dots, P_n\}$ be the set of all the nonisomorphic indecomposable MCM modules (up to degree shifts). Let $M = \bigoplus_{i=1}^n P_n \oplus A$. Then $B := \underline{\text{End}}_A(M)$ is a right pre-resolution of A .*

Proof. By Theorem 3.2, B is right noetherian. Since M is an MCM A -module and A is a noncommutative isolated singularity, it follows from Corollary 3.3 that $\underline{\text{Hom}}_A(M, K)$ is finitely generated for every finitely generated right graded A -module K . Therefore M is a small A -module.

Set $F = \underline{\text{Hom}}_A(M, -): \text{gr } A \rightarrow \text{gr } B$ and $\mathcal{F}: \text{qgr } A \rightarrow \text{qgr } B$ to be the induced functor. We next show that \mathcal{F} is an equivalence. By the proof of Proposition 3.4 (see also [BHZ, Lemma 2.3]), the

functor $G = - \otimes_A M^\vee: \text{gr } A \rightarrow \text{gr } B$ induces an equivalence of abelian categories $\mathcal{G}: \text{qgr } A \rightarrow \text{qgr } B$. Now for any $X \in \text{gr } A$,

$$\mathcal{G}(\pi(X)) = \pi(X \otimes_A M^\vee) = \pi(X \otimes_A \underline{\text{Hom}}_A(M, A)).$$

Let $\varphi_X: X \otimes_A \underline{\text{Hom}}_A(M, A) \rightarrow \underline{\text{Hom}}_A(M, X)$ be the map as in Lemma 3.5. Then both $\ker(\varphi_X)$ and $\text{coker}(\varphi_X)$ are finite dimensional, and φ_X induces a natural isomorphism

$$\pi(X \otimes_A \underline{\text{Hom}}_A(M, A)) \cong \pi(\underline{\text{Hom}}_A(M, X)).$$

It follows that \mathcal{F} is natural isomorphic to \mathcal{G} , and hence an equivalence.

By [CKWZ, Theorem 5.4] (where A is assumed to be Cohen-Macaulay, but the proof applies in our case. See also [Le] for commutative case), $\text{r.gldim}(B) = d$. Therefore all conditions in Definition 2.1 are satisfied. By definition B is a right pre-resolution of A . \square

4. NONCOMMUTATIVE QUADRIC HYPERSURFACES

In this section, we focus on noncommutative resolutions of noncommutative quadric hypersurfaces. Let us recall some terminologies.

Let A be a locally finite connected graded algebra. A graded A -module M_A is called a *Koszul module* (cf. [P]) if M_A has a linear projective resolution:

$$\cdots \rightarrow P^{-n} \rightarrow \cdots \rightarrow P^{-1} \rightarrow P^0 \rightarrow M \rightarrow 0,$$

where P^{-n} is a graded projective module generated in degree n for all $n \geq 0$. A left Koszul A -module is defined similarly. If the trivial module \mathbb{k}_A is a Koszul module, then A is called a *Koszul Algebra*. It is known that a Koszul algebra A must be quadratic, that is, A may be written as $A = T(V)/(R)$, where V is a finite dimensional vector space, and the generating relations R is contained in $V \otimes V$. The *quadratic dual* of A is defined to be the graded algebra $A^\perp = T(V^*)/(R^\perp)$, where R^\perp is the orthogonal dual of R in the space $V^* \otimes V^*$. Note that A^\perp is also a Koszul algebra.

For a locally finite graded algebra A , the Hilbert series of A is defined to be the formal power series:

$$H_A(t) = \sum_{i \in \mathbb{Z}} (\dim A_i) t^i.$$

A noetherian connected graded algebra S is called a *quantum polynomial algebra* if the following conditions are satisfied:

- (i) S is a Koszul AS-regular algebra;
- (ii) the Hilbert series of S is $H_S(t) = \frac{1}{(1-t)^d}$ for some $d \geq 1$.

Let S be a quantum polynomial algebra. Suppose $w \in S_2$ is a central regular element in S of degree two. The quotient algebra $A = S/Sw$ is usually called a *noncommutative quadric hypersurface*.

The following properties of noncommutative quadric hypersurfaces are well known (see [SvdB, Lemma 5.1(1)], [HY, Lemma 1.2] for instance). Note that for a quantum polynomial algebra the Gorenstein parameter coincides with the global dimension.

Lemma 4.1. *Assume S is a quantum polynomial algebra with global dimension $d + 1$ ($d \geq 0$). Let $w \in S_2$ be a central regular element of S , and let $A = S/Sw$.*

- (i) A is a Koszul algebra.
- (ii) A is AS-Gorenstein of injective dimension d with Gorenstein parameter $d - 1$.
- (iii) There is a central regular element $\varpi \in A_2^!$ such that $S^! \cong A^!/A^!\varpi$.

Setup 4.2. In the rest of this section, S is a quantum polynomial algebra of global dimension $d + 1$ with $d \geq 0$, and $w \in S_2$ is a central regular element. Set $A = S/Sw$.

We recall some results obtained in [SvdB]. Let $D^b(\text{gr } A)$ be the bounded derived category of $\text{gr } A$. There is a Koszul duality (cf. [SvdB, Subsection 2.4], or [BGS, Section 3] for general situation):

$$(4.2.1) \quad K: D^b(\text{gr } A) \longrightarrow D^b(\text{gr } A^!).$$

Notice that

$$(4.2.2) \quad K(\mathbb{k}) = A^!, \quad K(A) = \mathbb{k} \quad \text{and} \quad K(M(1)) = K(M)[-1](1) \quad \text{for all } M \in D^b(\text{gr } A).$$

The above duality induces the following duality:

$$(4.2.3) \quad \overline{K}: D^b(\text{gr } A)/\text{per } A \longrightarrow D^b(\text{gr } A^!)/D_{\text{tor}}^b(\text{gr } A^!),$$

where $\text{per } A$ is the full subcategory of $D^b(\text{gr } A)$ consisting of perfect complexes, and $D_{\text{tor}}^b(\text{gr } A^!)$ is the full subcategory of $D^b(\text{gr } A^!)$ consisting of complexes with finite dimensional total cohomology. Notice that $D^b(\text{gr } A^!)/D_{\text{tor}}^b(\text{gr } A^!) \cong D^b(\text{qgr } A^!)$. Let $\text{mcm } A$ be the full subcategory of $\text{gr } A$ consisting of all the MCM modules, and let $\underline{\text{mcm}} A$ be the stable category. Then there is a natural equivalence of triangulated categories [B, Theorem 4.4.1(2)]:

$$(4.2.4) \quad G: \underline{\text{mcm}} A \longrightarrow D^b(\text{gr } A)/\text{per } A.$$

Combining the functors in (4.2.3) and (4.2.4), we obtain the following Buchweitz's duality (cf. [SvdB, Theorem 3.2]):

$$(4.2.5) \quad B: \underline{\text{mcm}} A \longrightarrow D^b(\text{qgr } A^!).$$

By Lemma 4.1(iii), there is a central regular element $\varpi \in A_2^!$. Let $A^![\varpi^{-1}]$ be the localization of $A^!$ by the multiplication system defined by ϖ . Then $A^![\varpi^{-1}]$ is a \mathbb{Z} -graded algebra. Define a finite dimensional algebra (cf. [SvdB, Subsection 5.1]):

$$(4.2.6) \quad C(A) = A^![\varpi^{-1}]_0,$$

which is the degree zero part of $A^![\varpi^{-1}]$. Since S is a quantum polynomial algebra, it follows that (cf. [SvdB]):

$$(4.2.7) \quad \dim C(A) = \sum_{i \geq 0} \dim S_{2i}^! = \frac{1}{2} \dim S^!.$$

Notice that $A^!/A^!\varpi$ is finite dimensional. It follows that $M \in \text{tor } A^!$ if and only if any $m \in M$ is annihilated by some power of ϖ . Therefore, there is an equivalence:

$$L: D^b(\text{qgr } A^!) \longrightarrow D^b(\text{mod } C(A)),$$

where $\text{mod } C(A)$ is the category of finite dimension modules. Notice that

$$(4.2.8) \quad L(\pi(A^!)) = C(A).$$

Combining L with the Buchweitz's duality, we obtain the following equivalence (cf. [SvdB, Proposition 5.2]):

$$(4.2.9) \quad F: \underline{\text{mcm}}A \xrightarrow{B} D^b(\text{qgr } A^!) \xrightarrow{L} D^b(\text{mod } C(A)).$$

We may put the above mentioned triangle equivalences in the following commutative diagram

$$\begin{array}{ccccc} & & \xrightarrow{F} & & \\ & & \text{---} & & \\ \underline{\text{mcm}}A & \xrightarrow{B} & D^b(\text{qgr } A^!) & \xrightarrow{L} & D^b(\text{mod } C(A)) . \\ \downarrow G & & \downarrow \cong & & \\ D^b(\text{gr } A)/\text{per } A & \xrightarrow{\overline{K}} & D^b(\text{gr } A^!)/D_{\text{tor}}^b(\text{gr } A^!) & & \end{array}$$

We remark that the finite dimensional algebra $C(A)$ may be obtained from a Clifford deformation of the Frobenius algebra S^1 (cf. [HY]). In this section, we will give a new method to obtain the finite dimensional algebra $C(A)$.

Taking a minimal graded projective resolution of \mathbb{k}_A as follows:

$$(4.2.10) \quad \cdots \longrightarrow P^{-d} \xrightarrow{\partial^{-d}} P^{-d+1} \xrightarrow{\partial^{-d+1}} \cdots \longrightarrow P^0 \longrightarrow \mathbb{k}_A \longrightarrow 0.$$

Let $\Omega^d(\mathbb{k}_A) = \ker \partial^{-d+1}$ be the d th syzygy of the trivial module. Since A is a Koszul algebra, $\Omega^d(\mathbb{k}_A)$ is generated in degree d . We fix a notion as follows:

$$\mathbb{M} := \Omega^d(\mathbb{k}_A)(d).$$

The following properties of \mathbb{M} is clear.

Lemma 4.3. *Retain the notation as above.*

- (i) \mathbb{M} is a Koszul A -module.
- (ii) \mathbb{M} is an MCM module.

We investigate more properties of \mathbb{M} . For a right graded A -module X , let $X^\vee = \underline{\text{Hom}}_A(X, A)$ denote the dual module. Clearly X^\vee is a left graded module in an obvious way.

Proposition 4.4. *Retain the notations as above. Then $\mathbb{M}^\vee(1)$ is a left Koszul A -module.*

Proof. Applying the functor $\underline{\text{Hom}}_A(-, A)$ to the exact sequence (4.2.10), we obtain the sequence

$$(4.4.1) \quad 0 \rightarrow \underline{\text{Hom}}_A(P^0, A) \rightarrow \underline{\text{Hom}}_A(P^{-1}, A) \rightarrow \cdots \rightarrow \underline{\text{Hom}}_A(P^{-d+1}, A) \xrightarrow{\iota^\vee} \underline{\text{Hom}}_A(\Omega(\mathbb{k}_A), A) \rightarrow 0.$$

Since A is AS-Gorenstein of injective dimension d with Gorenstein parameter $d - 1$, the sequence (4.4.1) is exact except at the last position, where the cohomology group is ${}_A\mathbb{k}(d - 1)$. Note that $\mathbb{M}^\vee(d) = \underline{\text{Hom}}_A(\Omega(\mathbb{k}_A), A)$. The sequence (4.4.1) implies an exact sequence of left A -modules

$$(4.4.2) \quad 0 \rightarrow K \rightarrow \mathbb{M}^\vee(d) \rightarrow {}_A\mathbb{k}(d - 1) \rightarrow 0,$$

where $K = \text{im } \iota^\vee$. Note that K has a projective resolution

$$0 \rightarrow \underline{\text{Hom}}_A(P^0, A) \rightarrow \underline{\text{Hom}}_A(P^{-1}, A) \rightarrow \cdots \rightarrow \underline{\text{Hom}}_A(P^{-d+1}, A) \rightarrow K \rightarrow 0.$$

Hence $K(-d + 1)$ is a left Koszul A -module, and the Koszulity of $\mathbb{M}^\vee(1)$ follows from (4.4.2). \square

Proposition 4.5. *Retain the notations as above. We have the following properties.*

- (i) $\underline{\text{Ext}}_A^i(\mathbb{k}_A, \mathbb{M}) = 0$ for $i < d$.
- (ii) *The graded vector space $\underline{\text{Ext}}_A^d(\mathbb{k}_A, \mathbb{M})$ is concentrated in degree $-d$.*

Proof. (i) follows from the fact that \mathbb{M} is an MCM module.

(ii) Since A is AS-Gorenstein, $R\underline{\text{Hom}}_A(-, A): D^b(\text{gr } A) \rightarrow D^b(\text{gr } A^\circ)$ is a duality (cf. [Y]), where A° is the opposite algebra of A . Since \mathbb{M} is an MCM module, it follows that $R\underline{\text{Hom}}_A(\mathbb{M}, A) \cong \underline{\text{Hom}}_A(\mathbb{M}, A) = \mathbb{M}^\vee$ in $D^b(\text{gr } A^\circ)$. Note that $R\underline{\text{Hom}}_A(\mathbb{k}_A, A) \cong {}_A\mathbb{k}[-d](d-1)$. Therefore we have

$$\begin{aligned} \underline{\text{Ext}}_A^d(\mathbb{k}_A, \mathbb{M})_i &= \text{Hom}_{D^b(\text{gr } A)}(\mathbb{k}_A, \mathbb{M}[d](i)) \\ &\cong \text{Hom}_{D^b(\text{gr } A^\circ)}(R\underline{\text{Hom}}_A(\mathbb{M}, A)[-d], R\underline{\text{Hom}}_A(\mathbb{k}_A, A)(i)) \\ &\cong \text{Hom}_{D^b(\text{gr } A^\circ)}(\mathbb{M}^\vee, {}_A\mathbb{k}(d+i-1)) \\ &\cong \text{Hom}_{D^b(\text{gr } A^\circ)}(\mathbb{M}^\vee(1), {}_A\mathbb{k}(d+i)) \\ &\cong \text{Hom}_{\text{gr } A^\circ}(\mathbb{M}^\vee(1), {}_A\mathbb{k}(d+i)). \end{aligned}$$

By Proposition 4.4, $\mathbb{M}^\vee(1)$ is generated in degree one. Therefore, $\text{Hom}_{\text{gr } A^\circ}(\mathbb{M}^\vee(1), {}_A\mathbb{k}(d+i)) = 0$ for $i \neq -d$. Hence $\underline{\text{Ext}}_A^d(\mathbb{k}_A, \mathbb{M})$ is concentrated in degree $-d$. \square

Theorem 4.6. *Retain the notations as above. We have*

- (i) $\text{End}_{\text{gr } A}(\mathbb{M}) \cong C(A)$;
- (ii) *A is a noncommutative isolated singularity if and only if $\text{End}_{\text{gr } A}(\mathbb{M})$ is semisimple.*

Proof. (i) From the resolution (4.2.10), we have the following exact triangle in $D^b(\text{gr } A)$:

$$\Omega^d(\mathbb{k}_A)[d-1] \rightarrow P^\cdot \rightarrow \mathbb{k}_A \rightarrow \Omega^d(\mathbb{k}_A)[d],$$

where P^\cdot is the complex $0 \rightarrow P^{-d+1} \xrightarrow{\partial^{-d+1}} \dots \xrightarrow{\partial^1} P^0 \rightarrow 0$. Therefore

$$(4.6.1) \quad \Omega^d(\mathbb{k}_A)[d] \cong \mathbb{k}_A$$

in the quotient category $D^b(\text{gr } A)/\text{per } A$. Let F be the equivalence as in (4.2.9). Considering the equivalence functors (4.2.1)–(4.2.5), we have $F(\Omega^d(\mathbb{k}_A)[d]) = LB(\Omega^d(\mathbb{k}_A)[d]) = L\overline{K}G(\Omega^d(\mathbb{k}_A)[d])$. By (4.6.1), $G(\Omega^d(\mathbb{k}_A)[d]) \cong \mathbb{k}_A$. By (4.2.1) and (4.2.2), we have $F(\Omega^d(\mathbb{k}_A)[d]) \cong L\overline{K}(\mathbb{k}_A) \cong L(\pi(A^1))$. By (4.2.8), $L(\pi(A^1)) \cong C(A)$. Finally, we obtain that

$$(4.6.2) \quad F(\Omega^d(\mathbb{k}_A)[d]) \cong C(A).$$

Since F is an equivalence, we have

$$\text{End}_{\underline{\text{mcm}}(A)}(\Omega^d(\mathbb{k}_A)[d]) \cong \text{End}_{D^b(\text{mod } C(A))}(C(A)) \cong C(A),$$

whereas in the triangulated category $\underline{\text{mcm}}A$, we have

$$\text{End}_{\underline{\text{mcm}}(A)}(\mathbb{M}) = \text{End}_{\underline{\text{mcm}}(A)}(\Omega^d(\mathbb{k}_A)(d)) \cong \text{End}_{\underline{\text{mcm}}(A)}(\Omega^d(\mathbb{k}_A)) \cong \text{End}_{\underline{\text{mcm}}(A)}(\Omega^d(\mathbb{k}_A)[d]).$$

Therefore,

$$\text{End}_{\underline{\text{mcm}}(A)}(\mathbb{M}) \cong C(A).$$

By Proposition 4.4, $\mathbb{M}^\vee(1)$ is a Koszul module. In particular, \mathbb{M}^\vee is generated in degree 1 and $\text{Hom}_{\text{gr } A}(\mathbb{M}, A) = 0$, thus $\text{End}_{\underline{\text{mcm}}A}(\mathbb{M}) = \text{End}_{\text{gr } A}(M)$ and the desired isomorphism (i) follows.

The statement (ii) follows from [HY, Theorem 6.3] (see also, [MU, Theorem 4.13]). \square

Since A is a Koszul algebra, we may compute \mathbb{M} and $\text{End}_{\text{gr } A}(\mathbb{M})$ by Koszul resolution of A . Now assume $A = T(V)/(R)$ for some finite dimension vector space V with generating relations $R \subseteq V \otimes V$. Let $C_0 = \mathbb{k}$, $C_1 = V$, $C_2 = R$ and $C_n = \bigcap_{i+j=n-2} V^{\otimes i} \otimes R \otimes V^{\otimes j}$ for $n > 2$. Then the minimal projective resolution of \mathbb{k}_A reads as follows (cf. [BGS]):

$$(4.6.3) \quad \cdots \longrightarrow C_n \otimes A \xrightarrow{\partial^{-n}} C_{n-1} \otimes A \xrightarrow{\partial^{-(n-1)}} \cdots \xrightarrow{\partial^{-1}} C_0 \otimes A \longrightarrow \mathbb{k}_A \longrightarrow 0,$$

where the differential is defined as follows: if $\sum_{i=1}^m x_{1i} \otimes \cdots \otimes x_{ni} \in C_n$, for every $a \in A$,

$$\partial^{-n} \left(\left(\sum_{i=1}^m x_{1i} \otimes \cdots \otimes x_{ni} \right) \otimes a \right) = \sum_{i=1}^m (x_{1i} \otimes \cdots \otimes x_{n-1,i}) \otimes x_{ni} a.$$

By the above resolution, $\Omega^d(\mathbb{k}_A) \cong \text{im } \partial^{-d}$. From the exact sequence

$$0 \rightarrow \text{im } \partial^{-d-1} \rightarrow C_d \otimes A \rightarrow \Omega^d(\mathbb{k}_A) \rightarrow 0,$$

we obtain

$$(4.6.4) \quad \text{End}_{\text{gr } A}(\Omega^d(\mathbb{k}_A)) = \{f \in \text{End}_{\text{gr } A}(C_d \otimes A) \mid f(\text{im } \partial^{-d-1}) \subseteq \text{im } \partial^{-d-1}\}.$$

Notice that the restriction of ∂^{-d-1} on $(C_{d+1} \otimes A)_{d+1} \cong C_{d+1}$ is injective, and every element $f \in \text{End}_{\text{gr } A}(C_d \otimes A)$ is defined by its restriction on C_d . It follows an isomorphism

$$(4.6.5) \quad \{f \in \text{End}_{\text{gr } A}(C_d \otimes A) \mid f(\text{im } \partial^{-d-1}) \subseteq \text{im } \partial^{-d-1}\} \cong \{f \in \text{End}_{\mathbb{k}}(C_d) \mid (f \otimes 1)(C_{d+1}) \subseteq C_{d+1}\}.$$

Combining (4.6.4) and (4.6.5), we have the following isomorphism.

Proposition 4.7. *Write $A = T(V)/(R)$. We have*

$$\text{End}_{\text{gr } A}(\mathbb{M}) \cong \{f \in \text{End}_{\mathbb{k}}(C_d) \mid (f \otimes 1)(C_{d+1}) \subseteq C_{d+1}\},$$

where $C_n = \bigcap_{i+j=n-2} V^{\otimes i} \otimes R \otimes V^{\otimes j}$ for $n = d, d+1$, and $f \otimes 1$ is viewed as a linear map in $\text{End}_{\mathbb{k}}(C_d \otimes V)$.

Remark 4.8. The proposition above provides a relatively easy way to compute the endomorphism ring of \mathbb{M} , especially when d is small. We will compute a detailed example of noncommutative quadric hypersurface of dimension 2 in the next section. According to Theorem 4.6 and the proposition above, to find whether A is a noncommutative isolated singularity is a linear algebra problem.

Let us check the MCM modules when A is a noncommutative isolated singularity.

Lemma 4.9. *Assume that A is a noncommutative isolated singularity. Each nonprojective indecomposable MCM A -module is isomorphic to a direct summand of \mathbb{M} (up to a degree shift).*

Proof. By the isomorphism (4.6.2) in the proof of Theorem 4.6, $F(\Omega^d(\mathbb{k}_A)) \cong C(A)[-d]$. Since $C(A)$ is semisimple and F is an equivalence, all the indecomposable objects in $\underline{\text{mcm}}A$ are direct summands of $\Omega^d(\mathbb{k}_A)$ (up to degree shifts). Notice that the class of nonprojective indecomposable MCM module over A is in one-to-one correspondence to the class of indecomposable objects in $\underline{\text{mcm}}A$ (cf. [SvdB, Lemma 3.4]). Since \mathbb{M} is a shift of $\Omega^d(\mathbb{k}_A)$, the result follows. \square

We have the following properties of indecomposable MCM modules.

Proposition 4.10. *Retain the notation as above and keep Setup 4.2. Assume that A is a noncommutative isolated singularity.*

- (i) *For $n \geq d$, $\dim \Omega^n(\mathbb{k}_A)_n = \frac{1}{2} \dim S^!$, where $\Omega^n(\mathbb{k}_A)_n$ is the degree n part of the graded module $\Omega^n(\mathbb{k}_A)$.*
- (ii) $\dim \mathbb{M}_0 = \dim \text{End}_{\text{gr } A}(\mathbb{M})$.
- (iii) *Suppose that $\text{End}_{\text{gr } A}(\mathbb{M})$ is a direct product of \mathbb{k} . If N is an indecomposable MCM module, then $N(i) \cong A/xA$ for some $i \in \mathbb{Z}$ and some element $x \in A_1$. Moreover, $\Omega(\mathbb{M}) \cong \mathbb{M}(-1)$.*

Proof. (i) By Lemma 4.1(ii), we have the following exact sequence

$$0 \longrightarrow A^!(-2) \xrightarrow{\cdot \varpi} A^! \longrightarrow S^! \longrightarrow 0.$$

Then we have $\dim A_i^! = \dim A_{i-2}^! + \dim S_i^!$ for all $i \geq 2$, and $\dim A_i^! = \dim S_i^!$ for $i = 0, 1$. Then by an iterative computation, we have

$$\dim A_n^! = \begin{cases} \dim S_0^! + \dim S_2^! + \cdots + S_n^!, & \text{when } n \text{ is even;} \\ \dim S_1^! + \dim S_3^! + \cdots + S_n^!, & \text{when } n \text{ is odd.} \end{cases}$$

Since S is a quantum polynomial algebra, $H_{S^!}(t) = (1+t)^{d+1}$. Therefore $\dim A_n^! = \frac{1}{2} \dim S^!$ for $n \geq d$. Since A is a Koszul algebra and $\dim A_n^! = \dim \text{Hom}_{\text{gr } A}(\Omega^n(\mathbb{k}_A)(n), \mathbb{k}_A) = \dim \Omega^n(\mathbb{k}_A)_n$.

(ii) By Theorem 4.6, $\text{End}_{\text{gr } A}(\mathbb{M}) \cong C(A)$. Hence $\dim \text{End}_{\text{gr } A}(\mathbb{M}) = \dim C(A) = \frac{1}{2} \dim S^!$ (see Equation (4.2.7)). Since $\mathbb{M} = \Omega^d(\mathbb{k}_A)(d)$, the identity follow from (i).

(iii) By Lemma 4.9, it suffices to show that the result holds for each indecomposable direct summand N of \mathbb{M} .

Assume $\mathbb{M} = \mathbb{M}^1 \oplus \cdots \oplus \mathbb{M}^s$, where each \mathbb{M}^i is indecomposable. By assumption $\text{End}_{\text{gr } A}(\mathbb{M})$ is a direct product of \mathbb{k} , which forces that $\text{End}_{\text{gr } A}(\mathbb{M}_i) = \mathbb{k}$ for all i and $\text{Hom}_{\text{gr } A}(\mathbb{M}_i, \mathbb{M}_j) = 0$ for $i \neq j$. Hence $s = \dim \text{End}_{\text{gr } A}(\mathbb{M})$, and \mathbb{M} has no projective direct summands, otherwise if some \mathbb{M}_i is projective, then $\text{Hom}_{\text{gr } A}(\mathbb{M}_i, \mathbb{M}_j) \neq 0$ for any j . Since \mathbb{M} is a Koszul module, each \mathbb{M}^i is a Koszul module. Hence \mathbb{M}^i is generated in degree 0 for every i . By (i) and (ii), $s = \dim \mathbb{M}_0 = \frac{1}{2} \dim S^!$. Therefore, each \mathbb{M}^i is a cyclic module. Hence we have the following exact sequence

$$0 \rightarrow \Omega(\mathbb{M}^i) \rightarrow A \rightarrow \mathbb{M}^i \rightarrow 0, \text{ for all } 1 \leq i \leq s.$$

Then $\Omega^{d+1}(\mathbb{k}_A)(d) \cong \Omega(\mathbb{M}) \cong \bigoplus_{i=1}^s \Omega(\mathbb{M}^i)$. By (i), $\dim \Omega^{d+1}(\mathbb{k}_A)_{d+1} = s$. Hence $\dim \Omega(\mathbb{M}^i)_1 = 1$ for all $1 \leq i \leq s$ since each \mathbb{M}^i is not projective. Note that $\Omega(\mathbb{M}^i)$ is generated in degree 1. It follows that there is an element $x_i \in A_1$ such that $\Omega(\mathbb{M}^i) \cong x_i A$ for all $1 \leq i \leq s$. Hence $\mathbb{M}^i \cong A/x_i A$.

Since $\dim \text{End}_{\text{gr } A}(\mathbb{M}) = \dim \mathbb{M}_0$, $\text{Hom}_{\text{gr } A}(\mathbb{M}^i, \mathbb{M}^j) = 0$ if $i \neq j$. Hence $\mathbb{M}^i \not\cong \mathbb{M}^j$ if $i \neq j$, which implies $\Omega(\mathbb{M}^i) \not\cong \Omega(\mathbb{M}^j)$ for $i \neq j$ since $\Omega = [-1]$ is the suspension functor in the triangulated category $\underline{\text{mcm}}A$. Since each $\Omega(\mathbb{M}^i)$ is indecomposable and generated in degree 1, it follows that the set $\{\Omega(\mathbb{M}^1), \dots, \Omega(\mathbb{M}^s)\} = \{\mathbb{M}^1(-1), \dots, \mathbb{M}^s(-1)\}$ by Lemma 4.9. Therefore $\Omega(\mathbb{M}) \cong \mathbb{M}(-1)$. \square

Lemma 4.9 also provides a way to construct noncommutative resolution of noncommutative isolated singularities.

Theorem 4.11. *Keep the notions in Setup 4.2. Assume that A is a noncommutative isolated singularity. Then $B = \underline{\text{End}}_A(\mathbb{M} \oplus A)$ is a right pre-resolution of A . Moreover, B is concentrated in nonnegative degrees.*

Proof. That B is a right pre-resolution of A follows from Theorem 3.6 and Lemma 4.9. Since \mathbb{M} is a Koszul module, B is concentrated in nonnegative degrees. \square

Remark 4.12. The algebra B is isomorphic to the matrix algebra $\begin{pmatrix} \underline{\text{End}}_A(\mathbb{M}) & \mathbb{M} \\ \mathbb{M}^\vee & A \end{pmatrix}$. By Proposition 4.4, \mathbb{M}^\vee is concentrated in degrees not less than 1. By Proposition 4.5(ii), every element $f \in \underline{\text{End}}_A(\mathbb{M})_{\geq 1}$ factors through a projective module. By [McR, Proposition 7.5.1],

$$B_0 = \begin{pmatrix} \text{End}_{\text{gr}A}(\mathbb{M}) & \mathbb{M}_0 \\ 0 & \mathbb{k} \end{pmatrix}$$

is of global dimension 1, since $\text{End}_{\text{gr}A}(\mathbb{M})$ is semisimple by assumption. In the next section, we give a concrete example with detailed computations of elements of B .

5. AN EXAMPLE

In this section, we give a detailed computation of indecomposable MCM module of an explicit noncommutative quadric hypersurfaces. Let $\mathbb{k} = \mathbb{C}$, let $S = \mathbb{k}\langle x, y, z \rangle / (R)$, where $R = \text{span}\{xz + zx, yz + zy, x^2 + y^2\}$. Then S is a quantum polynomial algebra of global dimension 3, which is an AS-regular algebra of type S_2 as listed in [AS, Table 3.11, P.183].

The following facts were proved in [HY, Section 9], see also [Hu] for a complete classification of noncommutative conics.

Lemma 5.1. *Let $\varpi = x^2 + z^2 \in S_2$. Then*

- (i) ϖ is a central regular element of S ,
- (ii) $A = S/S\varpi$ is a noncommutative isolated singularity,
- (iii) $C(A) \cong \mathbb{k}^4$, where $C(A)$ is the algebra defined in (4.2.6).

Consider the Koszul resolution

$$\cdots \longrightarrow (R \otimes V \cap V \otimes R) \otimes A \xrightarrow{\partial^{-3}} R \otimes A \xrightarrow{\partial^{-2}} V \otimes A \xrightarrow{\partial^{-1}} A \longrightarrow \mathbb{k}_A \longrightarrow 0$$

of \mathbb{k}_A , where $V = \text{span}\{x, y, z\}$. Clearly R has a basis

$$x \otimes z + z \otimes x, y \otimes z + z \otimes y, x \otimes x + y \otimes y, x \otimes x + z \otimes z,$$

and by direct calculation, $R \otimes V \cap V \otimes R$ has a basis

$$\begin{aligned} & (x \otimes z + z \otimes x) \otimes x + (y \otimes z + z \otimes y) \otimes y + (x \otimes x + y \otimes y) \otimes z, \\ & 2(x \otimes z + z \otimes x) \otimes x + (y \otimes z + z \otimes y) \otimes y + (x \otimes x + y \otimes y) \otimes z + (x \otimes x + z \otimes z) \otimes z, \\ & (y \otimes z + z \otimes y) \otimes z - (x \otimes x + y \otimes y) \otimes y + (x \otimes x + z \otimes z) \otimes y, \\ & (x \otimes z + z \otimes x) \otimes z + (y \otimes z + z \otimes y) \otimes z - (x \otimes x + y \otimes y) \otimes y + (x \otimes x + z \otimes z) \otimes (x + y). \end{aligned}$$

Set $\mathbb{M} = \Omega^2(\mathbb{k}_A)(2) = \text{im } \partial^{-2}(2)$. Then \mathbb{M} is a Koszul module which is generated by $\frac{1}{2} \dim S^1 (= 4)$ elements, see Proposition 4.10(i). The generating relations of \mathbb{M} is equal to $\text{im } \partial^{-3}(2)$ which is also generated by 4 elements. Thus we may write \mathbb{M} as a quotient module of a free module:

$$0 \longrightarrow K \longrightarrow m_1 A \oplus m_2 A \oplus m_3 A \oplus m_4 A \longrightarrow \mathbb{M} \longrightarrow 0,$$

where $\{m_1, m_2, m_3, m_4\}$ is a free basis, and K is the submodule generated by

$$\begin{aligned} r_1 &= m_1x + m_2y + m_3z, \\ r_2 &= 2m_1x + m_2y + m_3z + m_4z, \\ r_3 &= m_2z - m_3y + m_4y, \\ r_4 &= m_1z + m_2z - m_3y + m_4(x + y). \end{aligned}$$

To find indecomposable MCM A -modules, we only need to find a set of primitive idempotents of $\text{End}_{\text{gr } A}(\mathbb{M})$ by Lemma 4.9. Let $F = m_1A \oplus m_2A \oplus m_3A \oplus m_4A$. Note that degree one part of K is $K_1 = \text{span}\{r_1, r_2, r_3, r_4\}$. We have

$$\text{End}_{\text{gr } A}(\mathbb{M}) = \{\theta \in \text{End}_{\text{gr } A}(F) \mid \theta(r_i) \in K_1, i = 1, 2, 3, 4\}.$$

By some computations on linear equations, we have

$$\text{End}_{\text{gr } A}(\mathbb{M}) = \left\{ \begin{pmatrix} b+d & 0 & a & a \\ 0 & b & c & 0 \\ 0 & -c & b & 0 \\ a & c & d & b+d \end{pmatrix} : a, b, c, d \in \mathbb{k} \right\}.$$

We have the following complete set of primitive idempotents in $\text{End}_{\text{gr } A}(\mathbb{M})$:

$$\begin{aligned} e_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2}i & 0 \\ 0 & -\frac{1}{2}i & \frac{1}{2} & 0 \\ 0 & \frac{1}{2}i & -\frac{1}{2} & 0 \end{pmatrix}, & e_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2}i & 0 \\ 0 & \frac{1}{2}i & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2}i & -\frac{1}{2} & 0 \end{pmatrix}, \\ e_3 &= \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, & e_4 &= \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}. \end{aligned}$$

Therefore, we have the following nonprojective nonisomorphic indecomposable MCM modules:

$$\begin{aligned} \mathbb{M}^1 &= \mathbb{M}e_1 = (\overline{m}_2 - i\overline{m}_3 + i\overline{m}_4)A, \\ \mathbb{M}^2 &= \mathbb{M}e_2 = (\overline{m}_2 + i\overline{m}_3 - i\overline{m}_4)A, \\ \mathbb{M}^3 &= \mathbb{M}e_3 = (\overline{m}_1 + \overline{m}_4)A, \\ \mathbb{M}^4 &= \mathbb{M}e_4 = (\overline{m}_1 - \overline{m}_4)A, \end{aligned}$$

where $\overline{m}_1, \overline{m}_2, \overline{m}_3, \overline{m}_4$ are the image of m_1, m_2, m_3, m_4 in \mathbb{M} , and $i = \sqrt{-1}$ is a square root of -1 .

Since $\mathbb{M}^1, \dots, \mathbb{M}^4$ are Koszul modules, a straightforward check shows that

$$\mathbb{M}^1 \cong A/(y + iz)A, \quad \mathbb{M}^2 \cong A/(y - iz)A, \quad \mathbb{M}^3 \cong A/(x + z)A, \quad \mathbb{M}^4 \cong A/(x - z)A.$$

Note that $y + iz, y - iz, x + z, x - z$ are nilpotent elements in A . Also, we have

$$\begin{aligned} A/(y + iz)A &\cong (y + iz)A(1), & A/(y - iz)A &\cong (y - iz)A(1), \\ A/(x + z)A &\cong (x + z)A(1), & A/(x - z)A &\cong (x - z)A(1). \end{aligned}$$

Summarizing, we have the following conclusion.

Proposition 5.2. *The quadric hypersurface A has nonprojective nonisomorphic indecomposable MCM modules (up to degree shifts):*

$$\begin{aligned}\mathbb{M}^1 &\cong A/(y+iz)A \cong (y+iz)A(1), \\ \mathbb{M}^2 &\cong A/(y-iz)A \cong (y-iz)A(1), \\ \mathbb{M}^3 &\cong A/(x+z)A \cong (x+z)A(1), \\ \mathbb{M}^4 &\cong A/(x-z)A \cong (x-z)A(1).\end{aligned}$$

By Theorem 4.11, A has a right pre-resolution $\underline{\text{End}}_A(\mathbb{M} \oplus A)$. Write $u_1 = y + iz, u_2 = y - iz, u_3 = x + z, u_4 = x - z$. Since A/u_iA is an MCM module for each $1 \leq i \leq 4$, the map $\tau^\vee: \underline{\text{Hom}}_A(A, A) \rightarrow \underline{\text{Hom}}_A(u_iA, A)$ induced from the inclusion map $\tau: u_iA \rightarrow A$ is surjective. Therefore, for each homogeneous element $f \in \underline{\text{Hom}}_A(u_iA, A)$, there is a homogeneous element $a \in A$ such that $f(u_i) = au_i$. On the other hand, for each homogeneous element $a \in A$, there is a graded right A -module morphism $f: u_iA \rightarrow A$ defined by $f(u_i) = au_i$. Hence, we obtain $\underline{\text{Hom}}_A(u_iA, A) \cong Au_i(1)$.

Since $\underline{\text{mcm}}A$ is semisimple and $C(A) \cong \text{End}_A(\mathbb{M}) \cong \mathbb{k}^4$, by the equivalence functor (4.2.9), we have $\underline{\text{Ext}}_A^1(u_iA, u_jA) = 0$ for all $i \neq j$ and $\underline{\text{Ext}}_A^1(u_iA, u_iA) \cong \mathbb{k}$. Therefore, the exact sequence $0 \rightarrow u_iA \rightarrow A \rightarrow A/u_iA \rightarrow 0$ induces a surjective map $\underline{\text{Hom}}_A(A, u_jA) \rightarrow \underline{\text{Hom}}_A(u_iA, u_jA)$ for $i \neq j$. Then we obtain $\underline{\text{Hom}}_A(u_iA, u_jA) = u_jAu_i(1)$ for $i \neq j$. Similarly, we have $\underline{\text{End}}_A(u_iA) = \mathbb{k} \oplus u_iAu_i(1)$ for $i = 1, 2, 3, 4$. Then $\underline{\text{End}}_A(\mathbb{M} \oplus A)$ is isomorphic to the following algebra

$$\begin{pmatrix} u_1Au_1(1) & u_1Au_2(1) & u_1Au_3(1) & u_1Au_4(1) & u_1A(1) \\ u_2Au_1(1) & u_2Au_2(1) & u_2Au_3(1) & u_2Au_4(1) & u_2A(1) \\ u_3Au_1(1) & u_3Au_2(1) & u_3Au_3(1) & u_3Au_4(1) & u_3A(1) \\ u_4Au_1(1) & u_4Au_2(1) & u_4Au_3(1) & u_4Au_4(1) & u_4A(1) \\ Au_1 & Au_2 & Au_3 & Au_4 & A_{\geq 1} \end{pmatrix} \oplus \begin{pmatrix} \mathbb{k} & 0 & 0 & 0 & 0 \\ 0 & \mathbb{k} & 0 & 0 & 0 \\ 0 & 0 & \mathbb{k} & 0 & 0 \\ 0 & 0 & 0 & \mathbb{k} & 0 \\ 0 & 0 & 0 & 0 & \mathbb{k} \end{pmatrix},$$

where the multiplication is defined as below. For consistency of notations we set $u_5 = 1$. We simply write elements in the left matrix as $(u_ia_{ij}u_j)$. Then

$$(u_ia_{ij}u_j)(u_ib_{ij}u_j) = \left(\sum_{k=1}^5 u_ia_{ik}u_kb_{kj}u_j \right).$$

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