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ON CENTERS AND DIRECT SUM DECOMPOSITIONS OF HIGHER DEGREE FORMS

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Abstract. Higher degree forms are homogeneous polynomials of degree $d > 2$, or equivalently symmetric d-linear spaces. In this paper we investigate direct sum decompositions of higher degree forms, namely expressing as sums of forms in disjoint sets of variables, by their centers. Let $\mathrm{V}_{n,d}$ denote the affine space of n-variate degree d forms. We show that within $\mathrm{V}_{n,d}$ the central forms, a priori absolutely indecomposable, make a dense open subset. The algebraic structure of the centers of higher degree forms is studied. We show that the center algebra of almost every form is semisimple, and that the center algebra of a form is nonsemisimple if and only if the form is a limit of direct sums form. We also emphasize that the problem of direct sum decompositions of higher degree forms can be boiled down to some standard tasks of linear algebra, in particular the simultaneous upper-triangularization of a set of commuting matrices. Simple criteria and algorithms for direct sum decompositions of higher degree forms are also provided.

1. Introduction

One of the central problems of classical invariant theory is the equivalence of higher degree forms under linear changes of variables. Direct sum decomposition is a natural step of dimension reduction as it provides the separation of variables. Direct sum decomposition of higher degree forms also plays important roles in many problems of such subjects as commutative algebra, geometric invariant theory, multilinear algebra and computational complexity.

In his pioneering work [3], Harrison initiated to generalize Witt's algebraic theory of quadratic forms to the higher degree situation. One of his important results is that any nondegenerate higher degree form admits a unique decomposition into a direct sum of indecomposable forms. For each nondegenerate higher degree form f a commutative algebra $Z(f)$, the so-called center, was introduced to deal with its direct sum decomposition.

Centers of higher degree forms, possibly in different terminologies, with connection to their direct sum decompositions were rediscovered independently several times by other authors, for example [12, 14, 8]. Direct sum decompositions of homogeneous polynomials are also considered in depth in the so-called Sebastiani-Thom type problems of algebraic geometry, where a decomposable polynomial is called of Sebastiani-Thom type. In [18], some sufficient conditions are provided for the direct sum decomposability of homogeneous polynomials via their Jacobian ideals.

On the other hand, recently direct sum decompositions of higher degree forms are also approached through apolarity, see for example [8, 1, 17]. In [2], the direct sum decomposition of a smooth form is interpreted in terms of the product factorization of its associated form and an algorithm for computing direct sum decompositions is provided. In these aforementioned works, criteria and algorithms of direct sum decompositions of higher degree forms involve sophisticated tools of Gröbner bases and associated forms.

The main aim of the present paper is to stress that the theory of Harrison's centers is highly effective for direct sum decompositions of higher degree forms both theoretically and computationally. There are many research works (see e.g. [7, 10, 11, 13, 15]) along the line of [3, 4], however general algebraic structures of centers and algorithms for direct sum decompositions via centers were not systematically pursued before. We show that almost all forms have trivial center (i.e. isomorphic to the ground field), a priori absolutely indecomposable. We also prove that the center algebra of a higher degree form is semisimple if and only if

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the form is not a limit of direct sums form [1]. As limit of direct sums forms are nullforms [2, 9], almost all higher degree forms have semisimple center. For forms with semisimple center, we give an elementary criterion for the direct sum decomposability which is equivalent to computing the rank of a finite set of vectors. Moreover, we show through a simple algorithm that direct sum decompositions for arbitrary higher degree forms can be boiled down to some standard tasks of linear algebra, specifically the computations of eigenvalues and eigenvectors for which there are efficient algorithms and well-established softwares. Some complicated examples treated in [1, 2, 17] are easily rehandled by this approach.

The rest of the paper is organized as follows. Section 2 is devoted to some preliminaries on centers and direct sum decompositions of higher degree forms. Main results are presented in Section 3. We conclude the paper with some examples in Section 4. Throughout, let $d \geq 3$ be an integer and let $\Bbbk$ be a field with $\operatorname{char} \Bbbk = 0$, or $\operatorname{char} \Bbbk > d$.

2. Preliminaries

A form of degree d in n variables is an element of the polynomial ring $\Bbbk[x_1, \cdots, x_n]$ which is a sum of monomials of degree d. A form f is called a direct sum if, after an invertible linear change of variables, it can be written as a sum of $t \geq 2$ nonzero forms in disjoint sets of variables as follows

$$f = f_1(x_1, \cdots, x_{a_1}) + \cdots + f_t(x_{a_{t-1}+1}, \cdots, x_n). \tag{2.1}$$

If this is not the case, then f is said to be indecomposable. On the other extreme, if the f_i's are forms in only one variable, then f is said to be diagonalizable.

For convenience, a general form of degree d in n variables is written in the symmetric way:

$$f(x_1, \cdots, x_n) = \sum_{1 \leq i_1, \cdots, i_d \leq n} a_{i_1 \cdots i_d} x_{i_1} \cdots x_{i_d} \tag{2.2}$$

where the $a_{i_1 \cdots i_d}$'s are symmetric in the sense that they remain unchanged under any permutation of their subscripts. The resulting symmetric d-tensor $A = (a_{i_1 \cdots i_d})_{1 \leq i_1, \cdots, i_d \leq n}$ is called the symmetric tensor of f.

Corresponding to forms of degree d there are associated symmetric d-linear spaces. Let V be a vector space over $\Bbbk$ of dimension n with a basis $e_1, \cdots, e_n$. Define $\Theta \colon V \times \cdots \times V \longrightarrow \Bbbk$ by $\Theta(e_{i_1}, \cdots, e_{i_d}) = a_{i_1 \cdots i_d}$. The pair (V, Θ) is called the associated symmetric d-linear space of f under the basis $e_1, \cdots, e_n$. One can recover the form f from (V, Θ) as

$$f(x_1, \cdots, x_n) = \Theta \left(\sum_{1 \leq i \leq n} x_i e_i, \ \ldots, \ \sum_{1 \leq i \leq n} x_i e_i \right).$$

Nonzero subspaces $V_1, \cdots, V_t$ of (V, Θ) are said to be orthogonal, if $\Theta(v_1, \cdots, v_d) = 0$ unless all the v_i's are in the same V_s for some $1 \leq s \leq t$. If $V = V_1 \oplus \cdots \oplus V_t$ for $t \geq 2$ nonzero orthogonal subspaces, then call (V, Θ) decomposable. Otherwise, call (V, Θ) indecomposable. Clearly, the orthogonal decompositions of (V, Θ) are in bijection with the direct sum decompositions of its associated form f.

A form f is said to be nondegenerate, if no variable can be removed by an invertible linear change of variables. This is equivalent to saying, in terms of symmetric d-linear spaces, that $\Theta(u, v_2, \cdots, v_d) = 0$ for all $v_2, \cdots, v_d \in V$ implies $u = 0$. For the associated symmetric d-tensor A, let A_{i_1} denote the $(d-1)$-tensor $A = (a_{i_1 \cdots i_d})_{1 \leq i_2, \cdots, i_d \leq n}$. Then f is nondegenerate if and only if the A_{i_1}'s are linearly independent in the space of $(d-1)$-tensors. Moreover, the form f involves essentially $\operatorname{Rank}\{A_1, \cdots, A_n\}$ variables. See [6] for more details.

According to [3, 4], the center of a higher degree form can be defined in the following three equivalent ways. Suppose f is a form of degree d in n variables. By H we denote its Hessian matrix $(\frac{\partial^2 f}{\partial x_i \partial x_j})_{1 \leq i, \, j \leq n}$. Then the center of f is defined as

$$Z(f) := \{X \in \Bbbk^{n \times n} \mid (HX)^T = HX\}. \tag{2.3}$$

Let $A = (a_{i_1 \cdots i_d})_{1 \leq i_1, \cdots, i_d \leq n}$ be the associated symmetric d-tensor of f and $A^{(i_3 \cdots i_d)}$ the $n \times n$ matrix $(a_{i_1 i_2 i_3 \cdots i_d})_{1 \leq i_1, i_2 \leq n}$. Then the center of A is defined by

$$Z(A) := \{X \in \Bbbk^{n \times n} \mid X^T A^{(i_3 \cdots i_d)} = A^{(i_3 \cdots i_d)} X, \ \forall 1 \leq i_3, \cdots, i_d \leq n\}. \tag{2.4}$$

In terms of the associated symmetric d-linear space, the center is defined as

$$Z(V,\Theta) := \{\phi \in \mathrm{End}(V) \mid \Theta(\phi(v_1), v_2, \cdots, v_d) = \Theta(v_1, \phi(v_2), \cdots, v_d),\ \forall v_1, v_2, \cdots, v_d \in V\}. \tag{2.5}$$

The following are some useful facts about centers and direct sum decompositions of higher degree forms obtained in [3].

Proposition 2.1. *Suppose f is a nondegenerate form of degree d in n variables. Then*

(1) *The center $Z(f)$ is a commutative subalgebra of the full matrix algebra $\Bbbk^{n\times n}$.*
(2) *There is a one-to-one correspondence between direct sum decompositions of f and complete sets of orthogonal idempotents of $Z(f)$.*
(3) *The decomposition of f into a direct sum of indecomposable forms is unique up to equivalence and permutation of indecomposable summands.*
(4) *If $K/\Bbbk$ is a field extension, by f_K it is meant treating $f \in K[x_1, \cdots, x_n]$, then $Z(f_K) \cong Z(f) \otimes_{\Bbbk} K$.*

Proof. For later applications, we include a proof of item (2). Proofs of other items can be found in [3]. It is enough to prove the correspondence between direct sum decompositions into two terms and complete sets of pairwise orthogonal idempotents. This can be easily extended to the general situation.

Suppose $f(x_1, \cdots, x_n) = f_1(x_1, \cdots, x_a) + f_2(x_{a+1}, \cdots, x_n)$ is a direct sum decomposition. Let (V, Θ) be the associated symmetric d-linear space with basis $e_1, \cdots, e_n$. Under these assumptions, we have

$$\begin{aligned} f_1(x_1, \cdots, x_a) &= \Theta\left(\sum_{i=1}^{a} x_i e_i, \cdots, \sum_{i=1}^{a} x_i e_i\right), \\ f_2(x_{a+1}, \cdots, x_n) &= \Theta\left(\sum_{i=a+1}^{n} x_i e_i, \cdots, \sum_{i=a+1}^{n} x_i e_i\right). \end{aligned}$$

Note in particular that

$$\Theta(e_{i_1}, e_{i_2}, \cdots, e_{i_d}) = 0 \ \text{ unless } \ 1 \le i_1, i_2, \cdots, i_d \le a \ \text{ or } \ a+1 \le i_1, i_2, \cdots, i_d \le n. \tag{2.6}$$

Let V_1 (resp. V_2) be the subspace of V spanned by $e_1, \cdots, e_a$ (resp. $e_{a+1}, \cdots, e_n$) and let $\epsilon_i : V \twoheadrightarrow V_i$ be the natural projections. Then for each $v \in V$, we have $v = \epsilon_1(v) + \epsilon_2(v)$. Clearly $1 = \epsilon_1 + \epsilon_2$. It remains to prove that $\epsilon_i \in Z(V, \Theta)$. Indeed, with (2.6) and the d-linearity of Θ we have

$$\begin{aligned} \Theta(\epsilon_1(v_1), v_2, \cdots, v_d) &= \Theta(\epsilon_1(v_1), \epsilon_1(v_2) + \epsilon_2(v_2), \cdots, \epsilon_1(v_d) + \epsilon_2(v_d)) \\ &= \Theta(\epsilon_1(v_1), \epsilon_1(v_2), \cdots, \epsilon_1(v_d)) \\ &= \Theta(\epsilon_1(v_1) + \epsilon_2(v_1), \epsilon_1(v_2), \cdots, \epsilon_1(v_d) + \epsilon_2(v_d)) \\ &= \Theta(v_1, \epsilon_1(v_2), \cdots, v_d). \end{aligned}$$

Similarly, we have $\epsilon_2 \in Z(V, \Theta)$.

Conversely, suppose $\{\epsilon_1, \epsilon_2\}$ is a pair of orthogonal idempotents of the center $Z(V, \Theta)$ and $1 = \epsilon_1 + \epsilon_2$. Let $V_i = \epsilon_i(V)$. Then it is clear that $V = V_1 \oplus V_2$. Assume that $e_1, \cdots, e_a$ (resp. $e_{a+1}, \cdots, e_n$) are a basis of V_1 (resp. V_2). As $\epsilon_1, \epsilon_2 \in Z(V, \Theta)$, it follows by the definition of centers that (2.6) holds. Now under the basis $e_1, \cdots, e_n$ we have

$$\begin{aligned} f(x_1, \cdots, x_a, x_{a+1}, \cdots, x_n) &= \Theta\left(\sum_{i=1}^{n} x_i e_i, \cdots, \sum_{i=1}^{n} x_i e_i\right) \\ &= \Theta\left(\sum_{i=1}^{a} x_i e_i, \cdots, \sum_{i=1}^{a} x_i e_i\right) + \Theta\left(\sum_{i=a+1}^{n} x_i e_i, \cdots, \sum_{i=a+1}^{n} x_i e_i\right). \end{aligned}$$

This gives rise to a direct sum decomposition of f. $\square$

3. Main Results

We are mainly concerned about the algebraic structure of the center algebra of an arbitrary higher degree forms and the application to the problem of direct sum decompositions. First of all, there is no loss of generality in assuming that the forms are nondegenerate. In order to take advantage of tools of algebraic geometry and for simplicity, we assume further that the ground field $\mathbb{k}$ is algebraically closed. For forms over an arbitrary ground field $\mathbb{k}$, one may study their direct sum decompositions on the algebraic closure $\overline{\mathbb{k}}$ first. Then by Harrison's uniqueness result of decompositions, detect the exact situation on the original ground field directly. Examples will be provided in the next section to elucidate the procedure. From now on, let $\mathrm{V}_{n,d} \subset \mathbb{k}[x_1, \cdots, x_n]$ denote the linear space of forms of degree d in n variables. The synonymous notions of higher degree forms and symmetric multilinear spaces are used interchangeably.

We start with computing $Z(f)$ via equation (2.2). It is well known that the matrix equations therein can be transformed to standard linear equations in the following way. Given an $n \times n$ matrix X, let X_{v} be the n^2-dimensional column vector

$$\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$$

where X_i is the i-th column of X. Let X_{v}^T denote the corresponding vector of X^T and let P be the permutation matrix such that $X_{\mathrm{v}}^T = PX_{\mathrm{v}}$. Then the matrix equation $X^T A^{(i_3 \cdots i_d)} = A^{(i_3 \cdots i_d)} X$ is equivalent to the following linear equations

$$[I_n \otimes A^{(i_3 \cdots i_d)} - (A^{(i_3 \cdots i_d)} \otimes I_n)P]X_{\mathrm{v}} = 0.$$

Let B denote the $n^d \times n^2$ matrix of coefficients of the previous linear equations, that is

$$B = \begin{pmatrix} I_n \otimes A^{(1 \cdots 11)} - (A^{(1 \cdots 11)} \otimes I_n)P \\ I_n \otimes A^{(1 \cdots 12)} - (A^{(1 \cdots 12)} \otimes I_n)P \\ \vdots \\ I_n \otimes A^{(n \cdots n)} - (A^{(n \cdots n)} \otimes I_n)P \end{pmatrix}. \tag{3.1}$$

Then clearly $Z(f)$ is obtained by putting the solution space of the linear equations

$$BX_{\mathrm{v}} = 0 \tag{3.2}$$

back to the form of $n \times n$ matrices.

Recall that a higher degree form f is called central if $Z(f) = \mathbb{k}$. A form is called absolutely indecomposable if it remains indecomposable under every field extension of the ground field $\mathbb{k}$. Thanks to items (2) and (4) of Proposition 2.1, a central form is absolutely indecomposable. It was showed in [6] that a general higher degree form is central. In the following we give another interpretation of this fact in terms of elementary algebraic geometry [5].

Proposition 3.1. *The set of central forms is an open subset of* $\mathrm{V}_{n,d}$.

Proof. According to (3.2), a form f is central if and only the associated matrix B has rank $n^2 - 1$. Therefore, the set of central forms in $\mathrm{V}_{n,d}$ is a union of all the principal open sets defined by the $n^2 - 1$ minors of B. □

In the following we give an explicit example of central forms in $\mathrm{V}_{n,d}$ for arbitrary $n,\ d$.

Example 3.2 (A central form). *For any integers $n \geq 2,\ d \geq 3$, let $f = x_1 x_2^{d-1} + \cdots + x_{n-1} x_n^{d-1} + x_n x_1^{d-1}$. Then $\frac{\partial^2 f}{\partial x_i \partial x_j} = 0$ for all $1 \leq i, j \leq n$ unless*

$$\frac{\partial^2 f}{\partial x_{j-1} \partial x_j} = (d-1)x_j^{d-2}, \quad \frac{\partial^2 f}{\partial x_j^2} = (d-1)(d-2)x_{j-1}x_j^{d-3}, \quad \frac{\partial^2 f}{\partial x_{j+1} \partial x_j} = (d-1)x_{j+1}^{d-2}$$

(by abuse of notation we consider $n+1 = 1$ here). Let $P = (p_{ij}) \in Z(f)$. Then HP is symmetric, and consequently we have

$$p_{i-1,j}x_i^{d-2} + (d-2)p_{ij}x_{i-1}x_i^{d-3} + p_{i+1,j}x_{i+1}^{d-2} = p_{j-1,i}x_j^{d-2} + (d-2)p_{ji}x_{j-1}x_j^{d-3} + p_{j+1,i}x_{j+1}^{d-2}. \tag{3.3}$$

When $d \geq 4$, the term $x_{i-1}x_i^{d-3}$ can not appear in the right hand side of the above equality unless $i = j$. Hence $p_{ij} = 0$ for $i \neq j$. When $i = j+1$, the above equality also shows that $p_{jj} = p_{j+1,j+1}$. So the matrix P is a scalar matrix and f is a central form.

When $d = 3$, the equality (3.3) becomes

$$p_{i-1,j}x_i + p_{ij}x_{i-1} + p_{i+1,j}x_{i+1} = p_{j-1,i}x_j + p_{ji}x_{j-1} + p_{j+1,i}x_{j+1}.$$

By comparing the coefficients of the x_i's, we have

- *if $|i-j| \geq 3$, then $p_{ij} = 0$;*
- *if $i = j-2$, then $p_{j-1,j-2} = p_{j-2,j} = 0$;*
- *if $i = j-1$, then $p_{j-1,j} = p_{j+1,j-1} = 0$, $p_{jj} = p_{j-1,j-1}$.*

Therefore, the matrix P is a scalar matrix and f is a central form.

Now it follows easily that

Corollary 3.3. *The set of central forms is dense in $\mathrm{V}_{n,d}$.*

In other words, almost all forms in $\mathrm{V}_{n,d}$ are central, a priori indecomposable.

Remark 3.4. *A similar result was obtained in [18]. It was shown that the set of indecomposable forms is an open algebraic subset in the projective space $\mathbb{P}(\mathrm{V}_{n,d})$.*

Next we consider the semisimplicity of the center algebras of higher degree forms. This turns out to be related to the geometric property of forms. Recall that a form $f(x_1, \cdots, x_n)$ is called smooth, or nonsingular, if the simultaneous equations

$$\frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0$$

have no nonzero solutions. There is also a notion of regularity of forms (see [4]) which generalizes and unifies the nondegeneracy and smoothness. That is, in terms of symmetric spaces, (V, Θ) is called l-regular if $\Theta(u, \cdots, u, v_{l+1}, \cdots, v_d) = 0$ for all $v_{l+1}, \cdots, v_d \in V$ implies $u = 0$. Suppose f is associated to (V, Θ) under the basis $e_1, \cdots, e_n$. If $u = a_1e_1 + a_2e_2 + \cdots + a_ne_n$, then it can be checked (see [3]) that

$$\Theta\left(u, \sum_{1\leq i\leq n} x_ie_i, \cdots, \sum_{1\leq i\leq n} x_ie_i\right) = \frac{1}{d}\sum_{1\leq i\leq n} a_i\frac{\partial f}{\partial x_i}. \tag{3.4}$$

With this it is clear that for corresponding multilinear symmetric spaces and higher degree forms 1-regular = nondegenerate, $(d-1)$-regular = smooth and l-regular implies $(l-1)$-regular.

It was noticed in [3, 4] that the condition of 2-regularity for forms imposes very strong restriction, namely the semisimpleness, on their centers. For completeness, we include a proof here.

Lemma 3.5. *Suppose f is 2-regular. Then $Z(f)$ is semisimple.*

Proof. It is enough to prove that if $\phi \in Z(f)$ is nilpotent, then $\phi = 0$. Otherwise, suppose there was a nonzero nilpotent element $\phi \in Z(f)$ with $\phi^{m+1} = 0$ while $\phi^m \neq 0$ for some $m \geq 1$. Then there is some $v \in V$ such that $\phi^m(v) \neq 0$. Hence

$$\Theta(\phi^m(v), \phi^m(v), v_3, \cdots, v_d) = \Theta(\phi^{2m}(v), v, v_3, \cdots, v_d) = \Theta(0, v, v_3, \cdots, v_d) = 0$$

for all $v_3, \cdots, v_d \in V$ as $2m \geq m+1$. Now the 2-regularity of f forces $\phi^m(v) = 0$. This leads to a desired contradiction. □

Remark 3.6. *There are non-2-regular higher degree forms whose center algebras are semisimple. For example, the determinant of a generic square matrix is non-2-regular as any rank 1 matrix is a common zero of its all degree $n-2$ differentials. However the generic determinant has trivial center, namely $\Bbbk$, see [1, 2, 17] and Example 4.3 for more details.*

For a complete description of the semisimplicity of center algebras, we need the notion of limit of direct sums forms introduced in [1].

Definition 3.7. *A higher degree form f is said to be a limit of direct sums (LDS) form if after a reversible linear change of variables,*

$$f(x_1, \cdots, x_n) = \sum_{i=1}^{l} x_i \frac{\partial h(x_{l+1}, \cdots, x_{2l})}{\partial x_{l+i}} + g(x_{l+1}, \cdots, x_n), \tag{3.5}$$

where h and g are forms of the same degree as f, in l and $n-l$ variables respectively.

The terminology is justified by the following

$$\begin{aligned} f(x_1, \cdots, x_n) &= \lim_{t \to 0} \frac{1}{t}[h(tx_1 + x_{l+1}, \cdots, tx_l + x_{2l}) - h(x_{l+1}, \cdots, x_{2l}) \\ &\quad + tg(tx_1 + x_{l+1}, \cdots, tx_l + x_{2l}, x_{2l+1}, \cdots, x_n)]. \end{aligned}$$

Theorem 3.8. *Suppose f is a nondegenerate higher degree form. Then the center $Z(f)$ is nonsemisimple if and only if f is an LDS form.*

Proof. Suppose $f(x_1, \cdots, x_n) = \sum_{i=1}^{l} x_i \frac{\partial h(x_{l+1}, \cdots, x_{2l})}{\partial x_{l+i}} + g(x_{l+1}, \cdots, x_n)$ is an LDS form. Then the Hessian matrix H of f is $\begin{pmatrix} 0 & H_h & 0 \\ H_h & & \\ 0 & & H_g \end{pmatrix}$ where H_g and H_h are the Hessian matrices of g and h respectively. Let N denote the block matrix $\begin{pmatrix} 0 & \mathrm{I}_l & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ which is clearly nilpotent. It is easy to verify that $(HN)^T = HN$. This implies that $Z(f)$ is nonsemisimple.

Conversely, suppose $Z(f)$ is nonsemisimple. Then take a nontrivial nilpotent element $\phi \in Z(f)$ satisfying $\phi^2 = 0$. This is possible as $Z(f)$ is a commutative algebra. Let (V, Θ) be the associated symmetric d-linear space of f. Assume $\operatorname{Rank} \phi = l$. Let $e_1, \cdots, e_l$ be a basis of $\operatorname{Im} \phi$ and choose $e_{l+i} \in V$ such that $\phi(e_{l+i}) = e_i$ for all $1 \le i \le l$. Then $e_1, \cdots, e_{2l}$ are linearly independent and we extend them to a basis $e_1, \cdots, e_n$ of V. Note that $\phi(e_j) = 0$ whenever $j \le l$ or $j \ge 2l+1$. It follows that $\Theta(e_{i_1}, \cdots, e_{i_d}) = 0$ whenever there are two indices $i_s, i_t \in [1, l]$ or $i_s \in [1, l]$ and $i_t \in [2l+1, n]$. Then, under this basis, the form f becomes

$$\begin{aligned} f(x_1, \cdots, x_n) &= \Theta\left(\sum_{i=1}^{n} x_i e_i, \cdots, \sum_{i=1}^{n} x_i e_i\right) \\ &= \sum_{i=1}^{l} x_i \sum_{1 \le i_2, \cdots, i_d \le l} \Theta(e_i, e_{l+i_2}, \cdots, e_{l+i_d}) x_{l+i_2} \cdots x_{l+i_d} \\ &\quad + \sum_{l+1 \le j_1, \cdots, j_d \le n} \Theta(e_{j_1}, \cdots, e_{j_d}) x_{j_1} \cdots x_{j_d}. \end{aligned}$$

Let $g(x_{l+1}, \cdots, x_n) = \sum_{l+1 \le j_1, \cdots, j_d \le n} \Theta(e_{j_1}, \cdots, e_{j_d}) x_{j_1} \cdots x_{j_d}$ and

$$h_i(x_{l+1}, \cdots, x_{2l}) = \sum_{1 \le i_2, \cdots, i_d \le l} \Theta(e_i, e_{l+i_2}, \cdots, e_{l+i_d}) x_{l+i_2} \cdots x_{l+i_d}$$

for all $1 \le i \le l$. Note that $h_i = \frac{\partial f}{\partial x_i}$ and thus by (3.4) one has

$$\begin{aligned} \frac{\partial h_i}{\partial x_{l+j}} = \frac{\partial^2 f}{\partial x_i \partial x_{l+j}} &= d(d-1)\Theta\left(e_i, e_{l+j}, \sum_{k=l+1}^{2l} x_k e_k, \cdots, \sum_{k=l+1}^{2l} x_k e_k\right) \\ &= d(d-1)\Theta\left(\phi(e_{l+i}), e_{l+j}, \sum_{k=l+1}^{2l} x_k e_k, \cdots, \sum_{k=l+1}^{2l} x_k e_k\right) \\ &= d(d-1)\Theta\left(e_{l+i}, \phi(e_{l+j}), \sum_{k=l+1}^{2l} x_k e_k, \cdots, \sum_{k=l+1}^{2l} x_k e_k\right) \\ &= d(d-1)\Theta\left(e_{l+i}, e_j, \sum_{k=l+1}^{2l} x_k e_k, \cdots, \sum_{k=l+1}^{2l} x_k e_k\right) \end{aligned}$$

$$
\begin{aligned}
&= d(d-1)\Theta\left(e_j, e_{l+i}, \sum_{k=l+1}^{2l} x_k e_k, \cdots, \sum_{k=l+1}^{2l} x_k e_k\right) \\
&= \frac{\partial^2 f}{\partial x_j \partial x_{l+i}} = \frac{\partial h_j}{\partial x_{l+i}}.
\end{aligned}
$$

Then by the well known Euler's identity, the following degree d form

$$
h(x_{l+1}, \cdots, x_{2l}) = \frac{1}{d} \sum_{1 \le i \le l} x_{l+i} h_i(x_{l+1}, \cdots, x_{2l})
$$

satisfies $\frac{\partial h}{\partial x_{l+i}} = h_i$ for all $1 \le i \le l$. Now we have shown that

$$
f(x_1, \cdots, x_n) = \sum_{i=1}^{l} x_i \frac{\partial h(x_{l+1}, \cdots, x_{2l})}{\partial x_{l+i}} + g(x_{l+1}, \cdots, x_n).
$$

That is to say, f is an LDS form. □

Corollary 3.9. *Suppose f is not an LDS form. Then $Z(f) \cong \Bbbk \times \cdots \times \Bbbk$ and the number of indecomposable direct summands of f is exactly $\dim Z(f)$. In particular, f is indecomposable if and only if $Z(f) \cong \Bbbk$.*

Proof. By the previous theorem, $Z(f)$ is semisimple since f is not an LDS form. Then according to the well known Wedderburn-Artin Theorem, the center algebra $Z(f) \cong \Bbbk \times \cdots \times \Bbbk$. The rest statements are direct consequences of Proposition 2.1. □

Remarks 3.10. *Keep the assumption that f is a nondegenerate higher degree form.*

(1) *The previous corollary gives an elementary criterion for direct sum decomposability of higher degree forms with semisimple center algebras. Suppose $Z(f)$ is semisimple. Then f is indecomposable (i.e. not a direct sum) if and only if $\dim Z(f) = 1$, if and only if $\operatorname{Rank} B = n^2 - 1$ where B is as in (3.1). Thus this is equivalent to computing the rank of a finite set of vectors. Note that smooth forms have semisimple centers by Lemma 3.5, so our criterion contains the cases treated in [1, 2, 18].*

(2) *If f is not an LDS form, then $\dim Z(f) \le n$ as the number of direct summands is not greater than the number of variables. In the specific case of $\dim Z(f) = n$, the form f is equivalent to the sum of d-th power of n linear forms. These are the so-called diagonalizable forms and have been investigated in our previous paper [6] via the theory of Harrison's centers.*

(3) *There is also a very simple algorithm for direct sum decompositions of higher degree forms with semisimple centers. Given a nondegenerate form f, the first step is to solve the linear equations (3.2) and take a basis of solution space. The second step is to diagonalize simultaneously the chosen basis. Note that $Z(f)$ is semisimple if and only if each $X \in Z(f)$ is diagonalizable. This can be detected via the minimal polynomial of any basis element, using Euclid algorithm to see whether the polynomial has multiple roots. This seems computationally cheaper than the Jacobian criterion of detecting the smoothness. The third step is to find out diagonal idempotent matrices from linear combinations of the obtained set of diagonal matrices. Finally determine a set of primitive orthogonal idempotents and decompose the form f accordingly.*

(4) *It was observed in [2] that if f is an LDS form, then f is a nullform, or GIT unstable. The converse is not true. For example, the binary form $f = x^2y^3 + xy^4 + y^5$ is a nullform (see e.g. [9]), however $Z(f) \cong \Bbbk$. It would be interesting to investigate the intrinsic difference between the nilpotency of $Z(f)$ and the unstableness of f.*

(5) *In the literature, it is considered very difficult to understand the nature of central higher degree forms, see for example [13]. Now by Theorem 3.8 and Corollary 3.9, we may interpret a central form as an indecomposable non LDS form. The previous item says that an indecomposable but not central form is extremely singular. It would be very interesting to unravel the connection between the nilpotency of $Z(f)$ and the singularity of f.*

(6) *It was conjectured in [10] that $\dim Z(f) \le n$ for all nondegenerate $f \in \mathrm{V}_{n,d}$. As LDS forms are extremely singular, this means that the dimension conjecture of O'Ryan-Shapiro holds for almost all nondegenerate forms. There do exist forms whose center algebras have dimension greater than n, for instance the Keet-Saxena cubic forms have dimension of magnitude $O(n^{\frac{3}{2}})$, see [7, 15]. In [15] Saxena*

also tried to amend the O'Ryan-Shapiro conjecture as: $\dim Z(f) \leq (d-1)n$ *for all nondegenerate* $f \in \mathrm{V}_{n,d}$*. As a matter of fact, this is also disproved by the same Keet-Saxena cubic forms.*

So far the exact upper bound of the dimensions of center algebras is not known yet. By the preceding items (2) and (6), we should focus on LDS forms for such bound. One may also approach this via the inverse problem proposed in [10]: which commutative subalgebras of $\Bbbk^{n\times n}$ are expressible as the center of some nondegenerate form in n variables? Since the upper bound is our primary goal, we start with a try on Schur's well known maximal commutative subalgebras in $\Bbbk^{n\times n}$.

In 1905 Schur [16] proved that the dimension of a commutative subalgebra of the full matrix algebra $\Bbbk^{n\times n}$ does not exceed $[\frac{n^2}{4}]+1 = m^2+1$ if $n = 2m$ and $= m(m-1)+1$ if $n = 2m-1$. The bound is attained by the following local algebra

$$\mathcal{S}_n := \lambda I_n + \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix},$$

where $A = k^{m\times m}$ if $n = 2m$ and $= k^{m\times(m-1)}$ if $n = 2m-1$.

Proposition 3.11. *The algebra* $\mathcal{S}_n$ *is not the center of any nondegenerate forms in* $\Bbbk[x_1, \cdots, x_n]$*.*

Proof. Otherwise, suppose $Z(f) = \mathcal{S}_n$ for some nondegenerate form $f \in \Bbbk[x_1, \cdots, x_n]$. Let H be the Hessian matrix of f and partition it into the shape $\begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix}$ of those in $\mathcal{S}_n$. By (2.3), the matrix

$$\begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix}\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & H_1A \\ 0 & H_3A \end{pmatrix}$$

is symmetric. This forces $H_1 = 0$ and $H_3 = 0 = H_2$, as A is arbitrary and H is symmetric. It follows that the variables $x_1, \cdots, x_m$ do not appear in f and so it is degenerate. This is a contradiction. □

Finally we provide an algorithm, purely in terms of linear algebra, for the direct sum decompositions of any higher degree forms.

Algorithm 3.12. *Take an arbitrary* $f \in \mathrm{V}_{n,d}$*. Denote the associated symmetric tensor by* A*.*

Step 1. *Compute* $\mathrm{Rank}\{A_1, \cdots, A_n\}$*. If it is* n*, then* f *is nondegenerate and continue; otherwise, take a linearly independent set of maximal size, reduce variables and make* f *nondegenerate in lower dimension situation, then continue.*

Step 2. *Solve the linear equations* (3.2) *and get a basis* $(P_i)_{1\leq i\leq \dim Z(f)}$ *of the center* $Z(f)$*.*

Step 3. *Upper-triangularize* $(P_i)_{1\leq i\leq \dim Z(f)}$ *simultaneously, and get a set of uppertriangular matrices* $(Q_i)_{1\leq i\leq \dim Z(f)}$*. Let* $Z' = \oplus_{1\leq i\leq \dim Z(f)} \Bbbk Q_i$ *denote the conjugate algebra of* $Z(f)$*.*

Step 4. *Take the diagonal* α_i *of each* Q_i*. By the well known theorem of Jordan decomposition,* α_i *is a polynomial of* Q_i *and so an element of* Z'*. Determine a complete set of primitive orthogonal idempotents of* Z' *which are linearly spanned by the* α_i*'s. Write each* α_i *as a row vector and put them into a matrix* C*. Then a set of primitive orthogonal idempotents are obtained by a row echelon reduction of* C*. By the reverse conjugation of Step 3, get a complete set of primitive orthogonal idempotents, denoted by* $(\epsilon_j)_{1\leq j\leq \dim Z(f)}$*, of* $Z(f)$*.*

Step 5. *Decompose the form* f *according to the complete set* $(\epsilon_j)_{1\leq j\leq \dim Z(f)}$ *of primitive orthogonal idempotents.*

Remark 3.13. *The previous algorithm shows that the direct sum decompositions of higher degree forms can be boiled down to some standard tasks of linear algebra, specifically the computations of eigenvalues and eigenvectors for which there are efficient algorithms and well-established softwares. Our algorithm seems more elementary than those in some previous works* [1, 2, 17] *which involve sophisticated tools of Gröbner bases and associated forms.*

4. Examples

In the following we provide some examples to elucidate the proposed criteria and algorithms. Examples 4.1 and 4.2 are taken form [2]. It turns out they can be easily worked out by hand without involving computers. Example 4.3 was considered in [1, 2, 17] by apolarity. Example 4.4 is about Cayley's hyperdeterminant and seems new. Example 4.5 is the Keet-Saxena cubic form which provides the best upper bound of the dimension of centers so far. These seemingly very complicated forms can all be handled by the approach of center without difficulty. Note that the examples are over not necessarily algebraically closed ground fields. Thus our approach works as well for direct sum decompositions of forms over arbitrary fields.

Example 4.1. [2, Example 6.4] *Consider the rational form* $f(x_1, x_2, x_3) = x_1^3 + 3x_1^2x_2 + 3x_1x_2^2 + 2x_2^3 + 3x_1^2x_3 + 6x_1x_2x_3 + 4x_2^2x_3 + 3x_1x_3^2 + 4x_2x_3^2 + 2x_3^3 \in \mathbb{Q}[x_1,\ x_2,\ x_3]$. *Let*

$$A^{(1)} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & \frac{4}{3} \\ 1 & \frac{4}{3} & \frac{4}{3} \end{pmatrix}, \quad A^{(3)} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \frac{4}{3} & \frac{4}{3} \\ 1 & \frac{4}{3} & 2 \end{pmatrix}.$$

Then by direct calculation, $Z(f) = \{X \in \mathbb{Q}^{3\times 3} \mid A^{(i)}X = X^TA^{(i)},\ 1 \le i \le 3\} = \bigoplus_{1\le i\le 3} \mathbb{Q}X_i$, *where*

$$X_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & -1 & -4 \end{pmatrix}.$$

It is clear that X_1 *and* X_2 *are a pair of orthogonal idempotents. According to the proof of Proposition 2.1, take the change of variables*

$$y_1 = x_1 + x_2 + x_3, \quad y_2 = x_2, \quad y_3 = x_3$$

and decompose the form as

$$y_1^3 + (y_2^3 + y_2^2y_3 + y_2y_3^2 + y_3^3).$$

Let $g(y_2, y_3) = y_2^3 + y_2^2y_3 + y_2y_3^2 + y_3^3$. *Then one can read from the* X_i*'s that*

$$Z(g) = \mathbb{Q}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \mathbb{Q}\begin{pmatrix} 0 & 1 \\ -1 & -4 \end{pmatrix} \cong \mathbb{Q}[\sqrt{3}].$$

It follows by Proposition 2.1 that g *is indecomposable over* $\mathbb{Q}$ *as* $\mathbb{Q}[\sqrt{3}]$ *is a field. However it is not absolutely indecomposable. Over any field extension* $K/\mathbb{Q}$ *with* $\sqrt{3} \in K$, *one has easily* $Z(g) \otimes_{\mathbb{Q}} K \cong K \times K$ *and* g *can be further decomposed as*

$$2\left(\frac{3+\sqrt{3}}{6}y_2 + \frac{3-\sqrt{3}}{6}y_3\right)^3 + 2\left(\frac{3-\sqrt{3}}{6}y_2 + \frac{3+\sqrt{3}}{6}y_3\right)^3.$$

To summarize, f *can be diagonalized over* K *as*

$$f(x_1, x_2, x_3) = (x_1 + x_2 + x_3)^3 + 2\left(\frac{3+\sqrt{3}}{6}x_2 + \frac{3-\sqrt{3}}{6}x_3\right)^3 + 2\left(\frac{3-\sqrt{3}}{6}x_2 + \frac{3+\sqrt{3}}{6}x_3\right)^3.$$

Example 4.2. [2, Example 6.6] *Consider the following rational quaternary quartic*

$$\begin{aligned} f =& x_1^4 + 4x_1^3x_2 + 6x_1^2x_2^2 + 4x_1x_2^3 + 2x_2^4 + 8x_1^3x_3 + 24x_1^2x_2x_3 + 24x_1x_2^2x_3 + 8x_2^3x_3 + 24x_1^2x_3^2 \\ &+ 48x_1x_2x_3^2 + 24x_2^2x_3^2 + 32x_1x_3^3 + 32x_2x_3^3 + 17x_3^4 - 12x_1^3x_4 - 36x_1^2x_2x_4 - 36x_1x_2^2x_4 \\ &- 12x_2^3x_4 - 72x_1^2x_3x_4 - 144x_1x_2x_3x_4 - 72x_2^2x_3x_4 - 144x_1x_3^2x_4 - 144x_2x_3^2x_4 - 96x_3^3x_4 \\ &+ 54x_1^2x_4^2 + 108x_1x_2x_4^2 + 54x_2^2x_4^2 + 216x_1x_3x_4^2 + 217x_2x_3x_4^2 + 216x_3^2x_4^2 - 108x_1x_4^3 \\ &- 108x_2x_4^3 - 216x_3x_4^3 + 82x_4^4. \end{aligned}$$

Let A *be the associated symmetric tensor of* f *and for all* $1 \le i_3, i_4 \le 4$ *let* $A^{(i_3i_4)} = (a_{i_1i_2i_3i_4})_{1\le i_1,i_2\le 4}$. *Note that*

$$A^{(22)} - A^{(11)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad 12A^{(23)} - 24A^{(11)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$12A^{(24)}+36A^{(11)}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix},\qquad A^{(33)}-4A^{(11)}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix},$$

$$12A^{(34)}+72A^{(11)}=\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&1&0&0\end{pmatrix},\qquad 12A^{(44)}-144A^{(23)}+180A^{(11)}=\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&0\end{pmatrix}.$$

With the preceding observation it is easy to compute that $Z(f)=\{X\in\mathbb{Q}^{4\times 4}\mid A^{(ij)}X=X^TA^{(ij)},\ 1\le i,j\le 4\}=\mathbb{Q}X_1\oplus\mathbb{Q}X_2$ *where*

$$X_1=\begin{pmatrix}1&1&2&-3\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad X_2=\begin{pmatrix}0&-1&-2&3\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}.$$

Clearly X_1 *and* X_2 *are a pair of primitive orthogonal idempotents and* $Z(f)\cong\mathbb{Q}\times\mathbb{Q}$. *As the previous example, we have the following direct sum decomposition*

$$f=(x_1+x_2+2x_3-3x_4)^4+(x_2^4+x_3^4+x_2x_3x_4^2+x_4^4).$$

In conclusion, f *is the direct sum of two absolutely indecomposable forms over* $\mathbb{Q}$.

Example 4.3 (Determinant-like polynomials)**.** *Let* $n\ge 3$. *Consider the determinant-like polynomial*

$$D:=\sum_{\sigma\in S_n}c_\sigma x_{1,\sigma(1)}x_{2,\sigma(2)}\cdots x_{n,\sigma(n)},$$

where $c_\sigma\in\mathbb{k}^*$. *Denote* $D_{ij,kl}:=\frac{\partial^2 D}{\partial x_{ij}\partial x_{kl}}$. *Arrange the indeterminates* x_{ij} *lexicographically via their indices. Assume* $A=(a_{st,uv})\in Z(D)$. *Take an arbitrary pair of indices* ij *and* kl *with* $i\neq k,\ j\neq l$. *Then by* (2.3) *we have*

$$\sum_{st}D_{ij,st}a_{st,kl}=\sum_{uv}D_{kl,uv}a_{uv,ij}.$$

Note that $D_{ij,st}=D_{kl,uv}\neq 0$ *if and only if* $s=k,\ t=l,\ u=i,\ v=j$. *It follows that* $a_{ij,ij}=a_{kl,kl}$ *and* $a_{st,kl}=0$ *for any other* st *with* $s\neq i,\ t\neq j$. *By varying* ij *and* kl, *it can be checked that* A *is a scalar matrix. That is,* $Z(D)\cong\mathbb{k}$ *and thus* D *is absolutely indecomposable. One can also consider pfaffian-like and hafnian-like polynomials in the same manner.*

Example 4.4 (Cayley's hyperdetermiant)**.** *The well known Cayley's hyperdetermiant (of* $2\times2\times2$ *matrix) is the following* 8*-variate degree* 4 *form:*

$$\begin{aligned}f(x_1,\cdots,x_8)\ =\ & x_1^2x_8^2+x_2^2x_7^2+x_3^2x_6^2+x_4^2x_5^2\\ &-2(x_1x_2x_7x_8+x_1x_3x_6x_8+x_1x_4x_5x_8+x_2x_3x_6x_7+x_2x_4x_5x_7+x_3x_4x_5x_6)\\ &+4(x_1x_4x_6x_7+x_2x_3x_5x_8).\end{aligned}$$

Let $A=(a_{i_1i_2i_3i_4})$ *be the associated symmetric* 4*-tensor of* f *and let* $A^{(i,j)}$ *be the* 8×8 *matrix* $(a_{i_1i_2ij})_{1\le i_1,i_2\le 8}$. *By* $E_{i,j}$ *we denote the* 8×8 *matrix with* (i,j)*-entry* 1 *and other entries* 0. *Suppose* $T=(T_{ij})\in Z(f)$. *We have the following observations.*

- $A^{(i,i)}=E_{9-i,9-i}$ *for all* $1\le i\le 8$. *By* (2.3)*, it follows easily that the matrix* T *is diagonal, written simply as* $\operatorname{diag}(t_1,t_2,\dots,t_8)$*;*
- $A^{(1,j)}=-\frac{1}{6}(E_{9-j,8}+E_{8,9-j})$ *for* $j=2,3,4$. *It follows that* $t_5=t_6=t_7=t_8$*;*
- $A^{(k,8)}=-\frac{1}{6}(E_{9-k,1}+E_{1,9-k})$ *for* $j=4,6,7$. *It follows that* $t_1=t_2=t_3=t_5$*;*
- $A^{(5,8)}=-\frac{1}{6}(E_{4,1}+E_{1,4})+\frac{1}{3}(E_{3,2}+E_{2,3})$. *It follows that* $t_1=t_4$.

Thus the matrix T *must be a scalar matrix. Hence Cayley's hyperdetermiant is a central form and so absolutely indecomposable.*

Example 4.5 (The Keet-Saxena cubic forms). *Let* $f=\sum_{i=1}^{n} a_{ii}X_i^2+\sum_{1\le i<j\le n} 2a_{ij}X_iX_j$ *be a polynomial in the indeterminates* a_{ij}, X_i *with* $1\le i\le j\le n$. *Then* f *is a cubic form in* $\frac{n(n+3)}{2}$ *variables. Compute the second order partial derivatives of* f *to get*

$$\frac{\partial^2 f}{\partial a_{ij}\partial a_{i'j'}}=0,\quad \frac{\partial^2 f}{\partial X_i\partial X_j}=2a_{ij},\quad \frac{\partial^2 f}{\partial a_{ij}\partial X_k}=\begin{cases} 0, & k\ne i,j;\\ 2X_j, & k=i;\\ 2X_i, & k=j.\end{cases}$$

Arrange the a_{ij}*'s by lexicographic order. Then the Hessian matrix of* f *looks like*

$$H=\begin{pmatrix} 0 & X\\ X^T & A\end{pmatrix}$$

where X *is an* $\frac{n(n+1)}{2}\times n$ *matrix,* X^T *is the transpose of* X *and* A *is the generic* $n\times n$ *symmetric matrix. For any* $P\in Z(f)$, *partition it* $\begin{pmatrix} P_1 & P_2\\ P_3 & P_4\end{pmatrix}$ *as* H. *As* $HP=\begin{pmatrix} XP_3 & XP_4\\ X^TP_1+AP_3 & X^TP_2+AP_4\end{pmatrix}$ *is symmetric and* A *is generic, it follows easily that* $P_3=0$ *and* $\begin{pmatrix} P_1 & 0\\ 0 & P_4\end{pmatrix}$ *is a scalar matrix. Note that the* (k_1,k_2)*-entry of* X^TP_2 *is*

$$2(X_1P_{1k_1,k_2}+\cdots+X_{k_1}P_{k_1k_1,k_2}+X_{k_1+1}P_{k_1\ k_1+1,k_2}+\cdots+X_nP_{k_1n,k_2}),$$

where $P_2=(P_{ij,k})_{1\le i\le j\le n,\ 1\le k\le n}$. *Then* X^TP_2 *is symmetric if and only if*

$$\sum_{i\le k_1} P_{ik_1,k_2}X_i+\sum_{i>k_1} P_{k_1i,k_2}X_i=\sum_{i\le k_2} P_{ik_2,k_1}X_i+\sum_{i>k_2} P_{k_2i,k_1}X_i$$

for all $1\le k_1\ne k_2\le n$. *By comparing the coefficients of* $X_i's$, *we have that*

- *if* $i\le\min\{k_1,k_2\}$, *then* $P_{ik_1,k_2}=P_{ik_2,k_1}$;
- *if* $k_1<i<k_2$, *then* $P_{k_1i,k_2}=P_{ik_2,k_1}$;
- *if* $k_2<i<k_1$, *then* $P_{ik_1,k_2}=P_{k_2i,k_1}$;
- *if* $i\ge\max\{k_1,k_2\}$, *then* $P_{k_1i,k_2}=P_{k_2i,k_1}$.

Therefore, we have $P_{i_1j_1,k_1}=P_{i_2j_2,k_2}$ *if and only if the triples* $\{i_1,j_1,k_1\}=\{i_2,j_2,k_2\}$ *counting with the multiplicities, and if and only if* $X_{i_1}X_{j_1}X_{k_1}=X_{i_2}X_{j_2}X_{k_2}$ *as monomials. It is well known that the number of degree* 3 *monomials in* n *variables is* $\binom{n+2}{3}$. *Hence we have* $\dim Z(f)=1+C_{n+2}^3=1+\frac{(n+2)(n+1)n}{6}$. *Moreover,* $Z(f)$ *is a local algebra with square zero radical. Therefore, the Keet-Saxena cubic* f *is an indecomposable LDS form.*

References

[1] W. Buczyńska, J. Buczyński, J. Kleppe and Z. Teitler, Apolarity and direct sum decomposability of polynomials, Michigan Math. J. 64 (2015) 675–719.

[2] M. Fedorchuk, Direct sum decomposability of polynomials and factorization of associated forms, Proc. London Math. Soc. (3) 120 (2020) 305–327.

[3] D. Harrison, A Grothendieck ring of higher degree forms, J. Algebra 35 (1975) 123–138.

[4] D. Harrison, B. Pareigis, Witt rings of higher degree forms, Comm. Algebra 16 (6) (1988) 1275–1313.

[5] R. Hartshorne, Algebraic geometry. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977. xvi+496 pp.

[6] H.-L. Huang, H. Lu, Y. Ye, C. Zhang, Diagonalizable higher degree forms and symmetric tensors, arXiv:2003.08041.

[7] A. Keet, Higher degree hyperbolic forms, Quaestiones Math. 16 (1993) 413–442.

[8] J. Kleppe, Additive splittings of homogeneous polynomials, Ph.D. thesis, U. Oslo, 2005, arXiv:1307.3532.

[9] S. Mukai, An introduction to invariants and moduli. Translated from the 1998 and 2000 Japanese editions by W. M. Oxbury. Cambridge Studies in Advanced Mathematics, 81. Cambridge University Press, Cambridge, 2003. xx+503 pp.

[10] M. O'Ryan, D.B. Shapiro, Centers of higher degree forms, Linear Alg. Appl. 371 (2003) 301–314.

[11] M. O'Ryan, D.B. Shapiro, Centers of higher degree trace forms, Journal of Pure and Applied Algebra 217 (2013) 2263–2273.

[12] A. Prószyński, On orthogonal decomposition of homogeneous polynomials, Fund. Math. 98 (1978) 201–217.

[13] S. Pumplün, Indecomposable forms of higher degree, Math. Z. 253 (2006) 347–360.

[14] B. Reichstein, On symmetric operators of higher degree and their applications, Linear Algebra Appl. 75 (1986)155–172.

[15] N. Saxena, On the centers of higher degree forms, unpublished article (2005),
posted at: www.math.uni-bonn.de/people/saxena/papers/laa05.pdf.

[16] I. Schur, Zur Theorie vertauschbaren Matrizen, J. Reine Angew. Math. vol. 130 (1905) 66–76.

[17] S. M. Shafiei, Apolarity for determinants and permanents of generic matrices, J. Commut. Algebra 7 (2015) 89–123.
[18] Z. Wang, On homogeneous polynomials determined by their Jacobian ideal, Manuscripta Math. 146 (2015) 559–574.

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