GENERALIZED KNÖRRER'S PERIODICITY THEOREM

JI-WEI HE, XIN-CHAO MA AND YU YE

ABSTRACT. Let A be a noetherian Koszul Artin-Schelter regular algebra, and let $f \in A_2$ be a central regular element of A. The quotient algebra A/(f) is usually called a (noncommutative) quadric hypersurface. In this paper, we use the Clifford deformation to study the quadric hypersurfaces obtained from the tensor products. We introduce a notion of simple graded isolated singularity and proved that, if B/(g) is a simple graded isolated singularity and proved that, if B/(g) is a simple graded isolated singularity of 0-type, then there is an equivalence of triangulated categories $\underline{\mathrm{mcm}}A/(f) \cong \underline{\mathrm{mcm}}(A \otimes B)/(f+g)$ of the stable categories of maximal Cohen-Macaulay modules. This result may be viewed as a generalization of Knörrer's periodicity theorem. As an application, we study the double branch cover $(A/(f))^{\#} = A[x]/(f+x^2)$ of a noncommutative conic A/(f).

0. INTRODUCTION

In noncommutative projective geometry, Artin-Schelter regular algebras are usually regarded as the coordinate rings of noncommutative projective spaces. Let A be a noetherian Koszul Artin-Schelter regular algebra, and let $f \in A_2$ be a central regular element of A. The quotient algebra A/(f) is usually called a *noncommutative quadric hypersurface*. Noncommutative quadric hypersurfaces have got lots of attentions in recent years (see [SVdB, CKMW, MU1, MU2, HU, HMM, Ue1, Ue2], etc). To study the graded Cohen-Macaulay modules of A/(f), Smith and Van den Bergh introduced in [SVdB] a finite dimensional algebra C(A/(f)), and proved that there is an equivalence of triangulated categories

$$\underline{\mathrm{mcm}}A/(f) \cong D^{b}(\mathrm{mod}C(A/(f))),$$

where $\underline{\mathrm{mcm}}A/(f)$ is the stable category of graded maximal Cohen-Macaulay modules over A/(f) and $D^b(\mathrm{mod}C(A/(f)))$ is the bounded derived category of finite dimensional modules over C(A/(f)). It is proved in [HMM] that C(A/(f)) is a Morita invariant. Hence, in some sense, the Cohen-Macaulay representations of A/(f) are determined by C(A/(f)).

²⁰¹⁰ Mathematics Subject Classification. 16S37, 16E65, 16G50.

Key words and phrases. Noncommutative quadric hypersurface, Knörrer's Periodicity Theorem, double branched cover.

However, one cannot recover the properties of A/(f) completely from C(A/(f)). For instance, let $A = \mathbb{C}[x, y], A' = \mathbb{C}_{-1}[x, y]$ and $f = x^2 + y^2$. Then $C(A/(f)) \cong C(A'/(f))$ and $\underline{\mathrm{mcm}}A/(f) \cong \underline{\mathrm{mcm}}A'/(f) \cong D^b(\mathrm{mod}\mathbb{C}^2)$. But if we consider the double branched covers of A/(f) and A'/(f) respectively, there will be a different situation. Indeed, $\underline{\mathrm{mcm}}(A/(f))^{\#} \cong D^b(\mathrm{mod}\mathbb{C}^4)$.

In [HY], we introduced the notion of *Clifford deformation* of a Koszul Frobenius algebra. Associated to every noncommutative quadric hypersurface A/(f), there is a Clifford deformation $C_{A^{!}}(\theta_{f})$, which is a strongly \mathbb{Z}_{2} -graded algebra. It turns out that the degree zero part of $C_{A^{!}}(\theta_{f})$ is isomorphic to the finite dimensional algebra C(A/(f)) as introduced in [SVdB]. The \mathbb{Z}_{2} -graded algebra $C_{A^{!}}(\theta_{f})$ may recover enough information of A/(f). For instance, the reason that $\underline{\mathrm{mcm}}(A/(f))^{\#}$ differs from $\underline{\mathrm{mcm}}(A'/(f))^{\#}$ in the previous paragraph, is because the \mathbb{Z}_{2} -graded algebra $C_{A^{!}}(\theta_{f})$ associated to A/(f) is a simple graded algebra, while $C_{A'^{!}}(\theta_{f})$ is not a simple graded algebra (see Examples 2.7, 2.9).

In this paper, we use the Clifford deformation to study the quadric hypersurfaces obtained from the tensor products of Koszul Artin-Schelter regular algebras. Let A and B be Koszul Artin-Schelter regular algebras, and let $f \in A_2, g \in B_2$ be central regular elements. Suppose that $A \otimes B$ is noetherian. We prove that $(A \otimes B)/(f + g)$ is a graded isolated singularity provided both A/(f) and B/(g) are graded isolated singularities (see Theorem 2.5). We introduce a notion of a *simple graded isolated singularity* in Section 2 and then prove a generalized version of Knörrer's Periodicity Theorem (see Theorem 2.15). In particular, it will recover the classical Knörrer's Periodicity Theorem of quadric singularities (see Remark 2.16). As an application, we study the double branch cover $(A/(f))^{\#}$ of a noncommutative conic A/(f) classified in [HMM] and proved that, in noncommutative case, $\underline{\mathrm{mcm}}(A/(f))^{\#} \cong \underline{\mathrm{mcm}}(A/(f)) \times \underline{\mathrm{mcm}}(A/(f))$ (see Corollary 3.6).

We assume the ground field $\mathbb{k} = \mathbb{C}$, and all the vector spaces and algebras are over \mathbb{C} .

1. Morita theory of \mathbb{Z}_2 -graded algebras revisited

In this section, we will recall some Morita type properties of \mathbb{Z}_2 -graded algebras.

Let $E = E_0 \oplus E_1$ be a finite dimensional \mathbb{Z}_2 -graded algebras. Denote $\operatorname{gr}_{\mathbb{Z}_2} E$ to be the category whose objects are finite dimensional graded right *E*-modules, and whose homsets are denoted by $\operatorname{Hom}_{\operatorname{gr}_{\mathbb{Z}_2} E}(M, N)$ consisting of right *E*-module homomorphisms which preserve the gradings. We use mod*E* to denote the category whose objects are all the finite dimensional right *E*-modules (ignoring the grading of *E*), and whose hom-sets are denoted by $\operatorname{Hom}_E(M, N)$ consisting of all the right *E*-module homomorphisms. Note that if *M* is finite dimensional, then $\operatorname{Hom}_E(M, N)$ is a \mathbb{Z}_2 -graded vector space.

We say that E is graded semisimple if E is a direct sum of simple objects in $\operatorname{gr}_{\mathbb{Z}_2} E$. Let $J^g(M)$ be the graded Jacobson radical of M, which is the intersection of all graded maximal submodules of M, it is clear that $J^g(E) = J^g(E_E)$ is a two-sided nilpotent graded ideal and E is graded semisimple if and only if $J^g(E) = 0$.

A \mathbb{Z}_2 -graded algebra E is called a *graded division algebra* if each non-zero homogeneous element of E is invertible.

Let F be another finite dimensional \mathbb{Z}_2 -graded algebra. Denote $E \otimes F$ to be the twisted tensor product of E and F. The multiplications of elements in $E \otimes F$ is defined by

$$(x \hat{\otimes} y)(x' \hat{\otimes} y') = (-1)^{|y||x'|} x x' \hat{\otimes} y y'_{\pm}$$

where $x, x' \in E$ and $y, y' \in F$ are homogeneous elements, and |x'| denotes the degree of x'. Note that $E \otimes F$ is also a \mathbb{Z}_2 -graded algebra.

We say a \mathbb{Z}_2 -graded algebra E is graded Morita equivalent to F if there is a finitely generated \mathbb{Z}_2 -graded bimodule ${}_FP_E$ such that $\operatorname{Hom}_E(P, -) : \operatorname{gr}_{\mathbb{Z}_2}E \longrightarrow \operatorname{gr}_{\mathbb{Z}_2}F$ is an equivalence of abelian categories.

The following results are well known (see [Z, Lemmas 3.4, 3.10] and [NVO, Theorem 2.10.10]).

Lemma 1.1. Let E and F be \mathbb{Z}_2 -graded algebras.

- (i) Assume that E and F are graded Morita equivalent to E' and F' respectively. Then E ⊗F is graded Morita equivalent to E' ⊗F'.
- (ii) (Z₂-graded version of Wedderburn-Artin Theorem) If E is Z₂-graded semisimple, then E is isomorphic, as a graded algebra, to a direct product of finitely many matrix algebras over some division algebras.

Since the ground field is \mathbb{C} , there are only two classes of finite dimensional graded division algebras. Let $\mathbb{G} = \{1, \sigma\}$ be a group of order 2, and let \mathbb{CG} be the group algebra. Then \mathbb{CG} is a \mathbb{Z}_2 -graded algebra by setting $|\sigma| = 1$ and |1| = 0.

Lemma 1.2. Let E be a finite dimensional \mathbb{Z}_2 -graded division algebra over \mathbb{C} . Then E is isomorphic to either $\mathbb{C}\mathbb{G}$ or \mathbb{C} , where \mathbb{C} is viewed as a \mathbb{Z}_2 -graded algebra concentrated in degree 0.

Proof. Since E_0 is a finite dimensional division algebra over \mathbb{C} , hence $E_0 = \mathbb{C}$. If $E_1 \neq 0$, for any non-zero elements $x, y \in E_1$, we have $x^{-1}y \in \mathbb{C}$ and $x = \lambda y$ for some $\lambda \in \mathbb{C}$. Therefore, dim $E_1 = 1$ and E is isomorphic to the skew group algebra $\mathbb{C}\#\mathbb{G}$. The action of $\sigma \in \mathbb{G}$ is ± 1 , and both of them are isomorphic to the group algebra $\mathbb{C}\mathbb{G}$. \Box For a \mathbb{Z}_2 -graded algebra E, we write $\operatorname{gldim}_{\mathbb{Z}_2} E$ for the graded right global dimension of E.

Lemma 1.3. [HY, Corollary 7.4] Let E be a \mathbb{Z}_2 -graded algebra. Then

 $\operatorname{gldim}_{\mathbb{Z}_2} E = \operatorname{gldim}_{\mathbb{Z}_2} E \hat{\otimes} \mathbb{CG}.$

Proposition 1.4. Let E be a \mathbb{Z}_2 -graded algebra over \mathbb{C} , and let F be a \mathbb{Z}_2 -graded semisimple algebra over \mathbb{C} . Then $\operatorname{gldim}_{\mathbb{Z}_2} E \otimes F = \operatorname{gldim}_{\mathbb{Z}_2} E$. In particular, $E \otimes F$ is graded semisimple if and only if E is graded semisimple.

Proof. By the \mathbb{Z}_2 -graded version of Wedderburn-Artin Theorem, F is isomorphic to a direct product of finitely many matrix algebras over some graded division algebras. Note that a matrix algebra over a graded division algebra (with possible degree shifts) is graded Morita equivalent to a graded division algebra, and hence a matrix algebra over a graded division algebra is graded Morita equivalent to either \mathbb{CG} or \mathbb{C} by Lemma 1.2. Then the result follows from Lemma 1.1(i) and Lemma 1.3.

We have the following Morita cancellation type property (see [LWZ]).

Proposition 1.5. Let E and F be \mathbb{Z}_2 -graded algebras. Then $E \otimes \mathbb{CG}$ is graded Morita equivalent to $F \otimes \mathbb{CG}$ if and only if E is graded Morita equivalent to F.

Proof. Suppose that $E \otimes \mathbb{C}\mathbb{G}$ is graded Morita equivalent to $F \otimes \mathbb{C}\mathbb{G}$. By Lemma 1.1(i), $E \otimes \mathbb{C}\mathbb{G} \otimes \mathbb{C}\mathbb{G}$ is graded Morita equivalent to $F \otimes \mathbb{C}\mathbb{G} \otimes \mathbb{C}\mathbb{G}$. By [HY, Lemma 7.2], the \mathbb{Z}_2 graded algebra $E \otimes \mathbb{C}\mathbb{G} \otimes \mathbb{C}\mathbb{G}$ is graded Morita equivalent to E, and $F \otimes \mathbb{C}\mathbb{G} \otimes \mathbb{C}\mathbb{G}$ is graded Morita equivalent to F. Hence E is graded Morita equivalent to F. The other direction is a consequence of Lemma 1.1(i).

We next focus on strongly \mathbb{Z}_2 -graded algebras. Recall that a \mathbb{Z}_2 -graded algebra E is said to be *strongly graded* if $E_1E_1 = E_0$. Let E and F be strongly \mathbb{Z}_2 -graded algebras. Then $E \otimes F$ is also a strongly \mathbb{Z}_2 -graded algebra.

Proposition 1.6. Let *E* be a finite dimensional strongly \mathbb{Z}_2 -graded algebra. Then there is an equivalence of abelian categories

$$\operatorname{gr}_{\mathbb{Z}_2} E \hat{\otimes} \mathbb{C} \mathbb{G} \cong \operatorname{mod} E,$$

where modE is the category of finite dimensional right E-modules (ignoring the grading of E).

Proof. As a vector space, the degree 0 part of $E \otimes \mathbb{CG}$ is equal to

$$E_1 \otimes \mathbb{C}\sigma \oplus E_0 \otimes \mathbb{C} = E_1 \otimes \mathbb{C}\sigma \oplus E_0,$$

and the multiplication of $(E \otimes \mathbb{CG})_0$ is given as following (we temporarily write the multiplication of $E \otimes \mathbb{CG}$ by the symbol "*"): for $a, b \in E_1, c, d \in E_0$, we have

$$(a \otimes \sigma) * (b \otimes \sigma) = -ab, \quad (a \otimes \sigma) * c = ac \otimes \sigma$$

 $c * (a \otimes \sigma) = ca \otimes \sigma, \quad c * d = cd.$

We temporarily write E^{\natural} for the ungraded algebra obtained from E by ignoring the grading. Define a linear map

$$\Xi: (E\hat{\otimes}\mathbb{CG})_0 \longrightarrow E^{\natural},$$

by setting $\Xi(a \otimes \sigma) = \sqrt{-1}a$ if $a \in E_1$, and $\Xi(a) = a$ if $a \in E_0$. Then it is straightforward to check that Ξ is an isomorphism of algebras.

Since E is strongly graded, $E \otimes \mathbb{C}\mathbb{G}$ is also strongly graded and there is an equivalence of abelian categories $\operatorname{gr}_{\mathbb{Z}_2} E \otimes \mathbb{C}\mathbb{G} \cong \operatorname{mod}(E \otimes \mathbb{C}\mathbb{G})_0$, which is in turn equivalent to $\operatorname{mod} E^{\natural}$. \Box

2. PRODUCTS OF QUADRIC HYPERSURFACES

Let us recall some notations and terminologies. An N-graded algebra $A = \bigoplus_{n \in \mathbb{N}} A_n$ is called a *connected graded algebra* if $A_0 = \mathbb{C}$. A connected graded algebra is said to be *locally finite*, if dim $A_n < \infty$ for all n. Let Gr A denote the category whose objects are graded right A-modules, and whose morphisms are right A-module morphisms which preserve the gradings of modules. For a graded right A-module X and an integer l, we write X(l) for the graded right A-module whose *i*th component is $X(l)_i = X_{i+l}$.

A locally finite connected graded algebra A is called a *Koszul algebra* (see [P]) if the trivial module \mathbb{k}_A has a *linear free resolution*; i.e.,

$$\cdots \longrightarrow P^n \longrightarrow \cdots \longrightarrow P^1 \longrightarrow P^0 \longrightarrow \mathbb{C} \longrightarrow 0,$$

where P^n is a graded free module generated in degree n for each $n \ge 0$. Note that a Koszul algebra is a *quadratic algebra*, that is, $A \cong T(V)/(R)$, where V is a finite dimensional vector space and $R \subseteq V \otimes V$. If A is a Koszul algebra, the *quadratic dual* of A is the quadratic algebra $A^! = T(V^*)/(R^{\perp})$, where V^* is the dual vector space and $R^{\perp} \subseteq V^* \otimes V^*$ is the orthogonal complement of R.

For graded right A-modules X and Y, denote $\underline{\operatorname{Hom}}_A(X,Y) = \bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}_{\operatorname{Gr} A}(X,Y(i))$. Then $\underline{\operatorname{Hom}}_A(X,Y)$ is a \mathbb{Z} -graded vector space. We write $\underline{\operatorname{Ext}}_A^i$ for the *i*th derived functor of $\underline{\operatorname{Hom}}_A$. Note that $\underline{\operatorname{Ext}}_A^i(X,Y)$ is also a \mathbb{Z} -graded vector space for each $i \geq 0$.

Definition 2.1. [AS] A connected graded algebra A is called an *Artin-Schelter Gorenstein* algebra of injective dimension d if

(i) A has finite injective dimension injdim $_AA = \text{injdim } A_A = d < \infty$,

JI-WEI HE, XIN-CHAO MA AND YU YE

- (ii) $\underline{\operatorname{Ext}}_{A}^{i}(\mathbb{C}_{A}, A_{A}) = 0$ for $i \neq d$, and $\underline{\operatorname{Ext}}_{A}^{d}(\mathbb{C}_{A}, A_{A}) \cong {}_{A}\mathbb{C}(l)$,
- (iii) the left version of (ii) holds.

If further, A has finite global dimension, then A is called an Artin-Schelter regular algebra.

An Artin-Schelter regular algebra A is called a *quantum polynomial algebra* if (i) A is a noetherian domain, (ii) A is a Koszul algebra with Hilbert series

$$H_t(A) := \sum_{i \ge 0} t^n \dim(A_n) = \frac{1}{(1-t)^d}$$

for some d > 0.

Let A be a noetherian Artin-Schelter Gorenstein algebra. Let X be a graded right A-module, and let $\Gamma(X) = \{x \in X | xA \text{ if finite dimensional}\}$. We obtain a functor Γ : Gr $A \longrightarrow$ Gr A. We write $R^i\Gamma$ for the *i*th right derived functor of Γ . Assume that injdim $_AA = \text{ injdim } A_A = d$. A finitely generated graded right A-module X is called a maximal Cohen-Macaulay module if $R^i\Gamma(X) = 0$ for all $i \neq d$. Let mcm A be the subcategory of Gr A consisting of all the maximal Cohen-Macaulay modules. Let mcm A be the stable category of mcm A. Then mcm A is a triangulated category.

Let V and U be finite dimensional vector spaces, and let $A = T(V)/(R_A)$ and $B = T(U)/(R_B)$ be Koszul algebras. Let us consider the tensor algebra $A \otimes B$. We view A and B as graded subalgebras of $A \otimes B$ through the injective maps $A \hookrightarrow A \otimes B, a \mapsto a \otimes 1$ and $B \hookrightarrow A \otimes B, b \mapsto 1 \otimes b$ respectively. Also, we identify A_1 with V and B_1 with U. So, the generating space of $A \otimes B$ is $W = V \oplus U$, and $A \otimes B \cong T(W)/(R_{A \otimes B})$, where

$$(2.1.1) R_{A\otimes B} = R_A \oplus [V, U] \oplus R_B,$$

in which (see [M, Chapter 3])

$$[V, U] = \{ v \otimes u - u \otimes v | v \in V, u \in U \}.$$

Moreover, $A \otimes B$ is also a Koszul algebra. Let $A^!$ and $B^!$ be the quadratic dual algebras of A and B respectively. The quadratic dual algebra of $A \otimes B$ is the following graded algebra [M, 3.5]

$$(A \otimes B)^! = A^! \hat{\otimes} B^!$$

Now we assume both A and B are Koszul Artin-Schelter regular algebras. Then $A^!$ and $B^!$ are graded Frobenius (see [Sm, Proposition 5.1]). It is easy to see that the graded tensor product $A^! \hat{\otimes} B^!$ is also a graded Frobenius algebra. Hence by [Sm, Proposition 5.1] again, $A \otimes B$ is an Artin-Schelter regular algebra.

In this section, we are interested in the quadric hypersurfaces obtained from $A \otimes B$.

Now assume both A and B are noetherian Koszul Artin-Schelter regular algebras, further more, $A \otimes B$ is also noetherian. The following lemma is clear.

Lemma 2.2. Let $f \in A_2$ and $g \in B_2$ be central regular elements of A and B respectively. View f and g as elements in $A \otimes B$. Then h = f + g is a central regular element of $A \otimes B$.

By the Rees Lemma [Le, Proposition 3.4(b)], $(A \otimes B)/(h)$ is a noetherian Koszul Artin-Schelter Gorenstein algebra. We are going to consider the Cohen-Macaulay modules over $(A \otimes B)/(h)$. Let us recall some notations from [SVdB, HY].

Let $E = T(X)/(R_E)$ be a Koszul Frobenius algebra. A linear map $\theta : R_E \to \mathbb{C}$ is called a *Clifford map* (see [HY, Definition 2.1]) if

$$(\theta \otimes 1 - 1 \otimes \theta)(X \otimes R_E \cap R_E \otimes X) = 0.$$

Given a Clifford map $\theta: R_E \to \mathbb{C}$, define an associative algebra

(2.2.1)
$$C_E(\theta) = T(X)/(r - \theta(r)) : r \in R).$$

The algebra $C_E(\theta)$ is called the *Clifford deformation* of E associated to θ . We may view T(X) as a \mathbb{Z}_2 -graded algebra by taking $T(X)_0 = \mathbb{C} \oplus (\bigoplus_{n \ge 1} X^{\otimes 2n})$ and $T(X)_1 = \bigoplus_{n \ge 1} X^{\otimes 2n-1}$. Since the definition relations of $C_E(\theta)$ lie in degree 0 component of T(X), we may view $C_E(\theta)$ as a \mathbb{Z}_2 -graded algebra.

Now let $S = T(Y)/(R_S)$ be a noetherian Koszul Artin-Schelter regular algebra. Denote by $\pi : T(Y) \to S$ the natural projection map. Let $z \in S_2$ be a central regular element of S. Pick an element $r_0 \in Y \otimes Y$ such that $\pi(r_0) = z$. Let $S' = T(Y^*)/(R_S^{\perp})$ be the quadratic dual algebra of S. Define a linear map

(2.2.2)
$$\theta_z \colon R_S^{\perp} \to \mathbb{C}$$
, by setting $\theta_z(\alpha) = \alpha(r_0), \ \forall \alpha \in R_S^{\perp}$.

Note that the map θ_z is independent of the choice of r_0 . The following results were proved in [HY, SVdB].

Theorem 2.3. Retain the notions as above.

- (i) θ_z is a Clifford map of $S^!$;
- (ii) The Clifford deformation $C_{S^1}(\theta_z)$ is a finite dimensional strongly \mathbb{Z}_2 -graded Frobenius algebra;
- (iii) There is an equivalence of triangulated categories

$$\underline{\mathrm{mcm}}S/(z) \cong D^{b}(\mathrm{gr}_{\mathbb{Z}_{2}}C_{S^{!}}(\theta_{z})) \cong D^{b}(\mathrm{mod}C_{S^{!}}(\theta_{z})_{0}),$$

where $\underline{\mathrm{mcm}}S/(z)$ is the stable category of graded maximal Cohen-Macaulay modules of S/(z), and $\mathrm{gr}_{\mathbb{Z}_2} C_{S^!}(\theta_z)$ is the category of finite dimensional right \mathbb{Z}_2 -graded modules of $C_{S^!}(\theta_z)$. Now let us go back to the quadric hypersurface $(A \otimes B)/(h)$. Recall $f \in A_2$, $g \in B_2$ and h = f + g, and $A = T(V)/(R_A)$ and $B = T(U)/(R_B)$. Then $A^! = T(V^*)/(R_A^{\perp})$, $B^! = T(U^*)/(R_B^{\perp})$ and $(A \otimes B)^! = T(V^* \oplus U^*)/(R_{A \otimes B}^{\perp})$. By Theorem 2.3, we have Clifford maps

(2.3.1)
$$\theta_f: R_A^{\perp} \to \mathbb{C}$$

(2.3.2)
$$\theta_g: R_B^\perp \to \mathbb{C},$$

(2.3.3)
$$\theta_h: R_{A\otimes B}^{\perp} \to \mathbb{C}.$$

Lemma 2.4. Retain the notations as above. we have an isomorphism of \mathbb{Z}_2 -graded algebras

$$C_{(A\otimes B)!}(\theta_h) \cong C_{A!}(\theta_f) \hat{\otimes} C_{B!}(\theta_g).$$

Proof. Let $\pi_A : T(V) \to A$, $\pi_B : T(U) \to B$ and $\pi_{A \otimes B} : T(V \oplus U) \to A \otimes B$ be the natural projection maps. Pick elements $r_1 \in V \otimes V$ and $r_2 \in U \otimes U$ such that $\pi_A(r_1) = f$ and $\pi_B(r_2) = g$. Then $\pi_{A \otimes B}(r_1 + r_2) = f + g = h$. Let us compute the generating relations of $C_{(A \otimes B)!}(\theta_h)$. By (2.2.1), $C_{(A \otimes B)!}(\theta_h) = T(V^* \oplus U^*)/I$, where I is generated by the space

 $\mathcal{R} := \{ \alpha - \theta_h(\alpha) | \alpha \in R_{A \otimes B}^{\perp} \} \subseteq (V^* \oplus U^*) \otimes (V^* \oplus U^*).$

Note that $R_{A\otimes B} = R_A \oplus [V, U] \oplus R_B$. It follows

$$R_{A\otimes B}^{\perp} = R_A^{\perp} \oplus [V^*, U^*]_+ \oplus R_B^{\perp},$$

where $R_A^{\perp} \subseteq V^* \otimes V^*$, $R_B^{\perp} \subseteq U^* \otimes U^*$ and $[V^*, U^*]_+ = \{\alpha \otimes \beta + \beta \otimes \alpha | \alpha \in V^*, \beta \in U^*\}$. Hence

(2.4.1)
$$\mathcal{R} = \{ \alpha - \alpha(f) | \alpha \in R_A^{\perp} \} + \{ \beta - \beta(g) | \beta \in R_B^{\perp} \} + [V^*, U^*]_+.$$

On the other hand, $C_{A^{!}}(\theta_{f}) \hat{\otimes} C_{B^{!}}(\theta_{g}) = T(V^{*} \oplus U^{*})/J$, where J is generated by the space

$$\hat{\mathcal{R}} := \{ \alpha - \theta_f(\alpha) | \alpha \in R_A^\perp \} + \{ \beta - \theta_g(\beta) | \beta \in R_B^\perp \} + [V^*, U^*]_+$$

Since $\theta_f(\alpha) = \alpha(f)$ and $\theta_g(\beta) = \theta(g)$ by (2.2.2), it follows that $\mathcal{R} = \hat{\mathcal{R}}$. Hence

$$C_{(A\otimes B)!}(\theta_h) \cong C_{A!}(\theta_f) \hat{\otimes} C_{B!}(\theta_g).$$

Let A be a noetherian graded algebra. Denote gr A for the category of finitely generated graded right A-modules, and tors A for the category of finite dimensional graded right A-modules. Let qgr A = gr A/tors A. Recall that A is called a *graded isolated singularity* if qgr A has finite global dimension. Lemma 2.4 implies the following result.

Theorem 2.5. Let A and B be noetherian Koszul Artin-Schelter regular algebras. Suppose that $A \otimes B$ is also noetherian. Assume that $f \in A$ and $g \in B$ are central regular homogeneous elements of degree 2. Suppose that B is a graded isolated singularity. Then $(A \otimes B)/(f+g)$ is a graded isolated singularity if and only if A/(f) is.

Proof. Let h = f + g. By Lemma 2.4, we have an isomorphism of \mathbb{Z}_2 -graded algebras

$$C_{(A\otimes B)!}(\theta_h) \cong C_{A!}(\theta_f) \hat{\otimes} C_{B!}(\theta_q).$$

By assumption, B/(f) is a graded isolated singularity. Hence, by [HY, Theorem 6.3], $C_{B^{!}}(\theta_{q})$ is a \mathbb{Z}_{2} -graded semisimple algebra.

By Proposition 1.4, $C_{(A \otimes B)^!}(\theta_h)$, which is isomorphic to $C_{A^!}(\theta_f) \otimes C_{B^!}(\theta_g)$, is a \mathbb{Z}_2 -graded semisimple algebra if and only if $C_{A^!}(\theta_f)$ is a \mathbb{Z}_2 -graded semisimple algebra. Applying [HY, Theorem 6.3] again, we obtain that $(A \otimes B)/(f)$ is a graded isolated singularity if and only if A/(f) is.

Definition 2.6. Let A be a noetherian Koszul Artin-Schelter regular algebra, and let $f \in A_2$ be a central regular element. If the \mathbb{Z}_2 -graded algebra $C_{A^{!}}(\theta_f)$ is a simple \mathbb{Z}_2 -graded algebra, then we call A/(f) is a simple graded isolated singularity.

Since the ground field is \mathbb{C} , there are two classes of simple graded algebras: matrix algebras over the \mathbb{Z}_2 -graded algebra $\mathbb{C}\mathbb{G}$, and matrix algebras over the \mathbb{Z}_2 -graded algebra \mathbb{C} . In the above definition, if $C_{A^{!}}(\theta_f)$ is a matrix algebra over the \mathbb{Z}_2 -graded \mathbb{C} , then we further call A/(f) is a simple graded isolated singularity of 0-type; if $C_{A^{!}}(\theta_f)$ is a matrix algebra over the \mathbb{Z}_2 -graded $\mathbb{C}\mathbb{G}$, then we further call A/(f) is a simple graded isolated singularity of 1-type. It is not hard to see that, if both A/(f) and B/(g) are simple graded isolated singularities of 1-type and $A \otimes B$ is also noetherian, then $(A \otimes B)/(f+g)$ is a simple graded isolated singularity of 0-type.

Example 2.7. Let $A = \mathbb{C}[x, y]$, and $f = x^2 + y^2$. Then

$$C_{A^{!}}(\theta_{f}) \cong \mathbb{C}_{-1}[u, v]/(u^{2} - 1, v^{2} - 1) \cong \mathbb{C}G \hat{\otimes} \mathbb{C}G \cong \mathbb{M}_{2}(\mathbb{C}),$$

where $\mathbb{M}_2(\mathbb{C})$ is the 2×2-matrix algebra which is viewed as a \mathbb{Z}_2 -graded algebra by setting

$$\mathbb{M}_2(\mathbb{C})_0 = \begin{bmatrix} \mathbb{C} & 0 \\ 0 & \mathbb{C} \end{bmatrix}, \quad \mathbb{M}_2(\mathbb{C})_1 = \begin{bmatrix} 0 & \mathbb{C} \\ \mathbb{C} & 0 \end{bmatrix}$$

Hence $\mathbb{C}[x, y]/(x^2 + y^2)$ is a simple graded isolated singularity of 0-type.

Example 2.8. Let $A = \mathbb{C}[x]$, and $f = x^2$. Then

$$C_{A'}(\theta_f) \cong \mathbb{C}[x]/(x^2 - 1) \cong \mathbb{C}G.$$

Therefore A/(f) is a simple graded isolated singularity of 1-type.

Example 2.9. Let $A = \mathbb{C}_{-1}[x, y]$, and $f = x^2 + y^2$. Then

$$C_{A^{!}}(\theta_{f}) \cong \mathbb{C}[u, v]/(u^{2} - 1, v^{2} - 1) \cong \mathbb{C}\mathbb{G} \times \mathbb{C}\mathbb{G}.$$

Hence A/(f) is not a simple graded isolated singularity.

Lemma 2.10. If $C_{A'}(\theta_f)_0$ is a simple algebra, then A/(f) is a simple graded isolated singularity of 1-type.

Proof. Since $C_{A^{!}}(\theta_{f})_{0}$ is simple and $C_{A^{!}}(\theta_{f})$ is strongly graded, we obtain that $C_{A^{!}}(\theta_{f})$ is simple graded. If $C_{A^{!}}(\theta_{f})$ is a matrix over \mathbb{C} , it must concentrate in degree 0 since $C_{A^{!}}(\theta_{f})_{0}$ is simple. But in this case, $C_{A^{!}}(\theta_{f})$ can not be strongly graded. Therfore, $C_{A^{!}}(\theta_{f})$ must be a matrix over \mathbb{CG} .

The next is an example of noncommutative simple graded isolated singularities.

Example 2.11. Let $A = \mathbb{C}\langle x_1, \ldots, x_5 \rangle / (r_1, \ldots, r_{10})$, where the generating relations are as follows:

$$r_1 = x_1 x_2 - x_2 x_1, r_2 = x_1 x_3 + x_3 x_1, r_3 = x_1 x_4 + x_4 x_1,$$

$$r_4 = x_1 x_5 + x_5 x_1, r_5 = x_2 x_3 - x_3 x_2, r_6 = x_2 x_4 + x_4 x_1,$$

$$r_7 = x_2 x_5 + x_5 x_2, r_8 = x_3 x_4 - x_4 x_3, r_9 = x_3 x_5 + x_5 x_3, r_{10} = x_4 x_5 + x_5 x_4.$$

Let $f = x_1^2 + \cdots + x_5^2$. By [MU2, Section 5.4.2], $\underline{\mathrm{mcm}}A/(f) \cong D^b(\mathbb{C})$. Hence $C_{A^!}(\theta_f)_0$ is simple and A/(f) is a simple graded isolated singularity of 1-type by Lemma 2.10.

Lemma 2.12. Let A be a quantum polynomial algebra of global dimension n, and let $f \in A_2$ be a central regular element, then dim $C_{A^{!}}(\theta_f) = 2^n$. Moreover

- (i) If A/(f) is a simple graded isolated singularity of 0-type, then n is even.
- (ii) If A/(f) is a simple graded isolated singularity of 1-type, then n is odd.

Proof. Since A has Hilbert series $1/(1-t)^n$, the Hilbert series of $A^!$ is $(1+t)^n$. Therefore, dim $A^! = 2^n$. Note that a Clifford deformation dose not change the dimension, we have dim $C_{A^!}(\theta_f) = 2^n$. If A/(f) is a simple graded isolated singularity of 0-type, then $C_{A^!}(\theta_f)$ is a matrix over \mathbb{C} and dim $C_{A^!}(\theta_f)$ is a square of an integer, so n must be even. If A/(f)is a simple graded isolated singularity of 1-type, then $C_{A^!}(\theta_f)$ is a matrix over $\mathbb{C}\mathbb{G}$, so n must be odd.

Remark 2.13. We don't know by now when A/(f) is a simple isolated singularity. It seems that the *rank* of f introduced by Mori-Ueyama in [MU2] is a candidate tool. Let A be a quantum polynomial algebra of global dimension n, and let $f \in A_2$ be a central

regular element. The rank of f (see [MU2, Definition 5.5]) is defined by

$$\operatorname{rank} f = \min \Big\{ r \in \mathbb{N}^+ \mid f = \sum_{i=1}^r u_i v_i, \ u_i, v_i \in A_1 \Big\}.$$

It has been proved that $C_{A'}(\theta_f)_0$ has no non-zero modules of dimension less than rank f (see [MU2, Lemma 5.10]). Since dim $C_{A'}(\theta_f)_0 = 2^{n-1}$, if n is odd and rank $f \ge 2^{\frac{n-1}{2}}$, then $C_{A'}(\theta_f)_0$ is a matrix over \mathbb{C} and A/(f) is a simple isolated singularity of 1-type by Lemma 2.10. For example, we can choose $A = \mathbb{C}[x, y, z]$ and $f = x^2 + y^2 + z^2$, then n = 3 and rank f = 2. In this case, $C_{A'}(\theta_f)_0 = M_2(\mathbb{C})$ and $C_{A'}(\theta_f) = M_2(\mathbb{C}\mathbb{G})$.

Proposition 2.14. Let A be a noetherian Koszul Artin-Schelter regular algebra, and let $f \in A_2$ be a central regular element.

- (i) If A/(f) is a simple graded isolated singularity of 0-type, then A/(f) has two indecomposable nonprojective graded Cohen-Macaulay modules (up to isomorphisms and degree shifts);
- (ii) If A/(f) is a simple graded isolated singularity of 1-type, then A/(f) has one indecomposable nonprojective graded Cohen-Macaulay module (up to isomorphisms and degree shifts).

Proof. (i) Since A/(f) is a simple graded isolated singularity of 0-type, then $C_{A^{!}}(\theta_{f})$ is a matrix algebra over the \mathbb{Z}_{2} -graded algebra \mathbb{C} . Then $C_{A^{!}}(\theta_{f}) \cong \operatorname{End}_{\mathbb{C}}(\mathbb{C}^{s} \oplus (\mathbb{C}(1))^{t})$ for some $s, t \geq 1$, where $\mathbb{C}(1)$ is the graded \mathbb{Z}_{2} -graded \mathbb{C} -module by putting \mathbb{C} in degree 1. By Theorem 2.3(ii), $C_{A^{!}}(\theta_{f})$ is a strongly \mathbb{Z}_{2} -graded algebra. Hence $t \neq 0$. Therefore $C_{A^{!}}(\theta_{f})_{0} \cong M_{s}(\mathbb{C}) \times M_{t}(\mathbb{C})$, where $M_{s}(\mathbb{C})$ (resp. $M_{t}(\mathbb{C})$) is the $s \times s$ (resp. $t \times t$) matrix algebra over the field \mathbb{C} . By Theorem 2.3(iii), $\underline{\operatorname{mcm}} A/(f) \cong D^{b}(C_{A^{!}}(\theta_{f})_{0})$ since $C_{A^{!}}(\theta_{f})$ is strongly graded. Since $C_{A^{!}}(\theta_{f})_{0}$ has two nonisomorphic simple modules, A/(f) has two nonprojective indecomposable graded Cohen-Macaulay modules (up to isomorphisms and degree shifts).

(ii) If A/(f) is a simple graded isolated singularity of 1-type, then $C_{A^{!}}(\theta_{f})$ is a matrix algebra over the \mathbb{Z}_{2} -graded algebra \mathbb{CG} . Then $C_{A^{!}}(\theta_{f})_{0}$ is a matrix algebra over \mathbb{C} . Therefore $C_{A^{!}}(\theta_{f})_{0}$ has one nonisomorphic simple module. By Theorem 2.3(iii) again, A/(f) has one nonprojective indecomposable graded Cohen-Macaulay module (up to isomorphisms and degree shifts).

Theorem 2.15. Let A and B be noetherian Koszul Artin-Schelter regular algebras, and let $f \in A_2$ and $g \in B_2$ be central regular elements. Suppose that $A \otimes B$ is noetherian.

(i) If B/(g) is a simple graded isolated singularity of 0-tpye, then there are equivalences of triangulated categories

 $\underline{\mathrm{mcm}}(A \otimes B)/(f+g) \cong D^b(\mathrm{mod}C_{A^!}(\theta_f)_0) \cong \underline{\mathrm{mcm}}A/(f);$

(ii) If B/(g) is a simple graded isolated singularity of 1-type, there is an equivalence of triangulated categories

$$\underline{\mathrm{mcm}}(A \otimes B)/(f+g) \cong D^{b}(\mathrm{mod}C_{A^{!}}(\theta_{f})).$$

Proof. (i) Since B/(g) is a simple graded isolated singularity of 0-type, $C_{B'}(\theta_g)$ is a matrix algebra over the \mathbb{Z}_2 -graded algebra \mathbb{C} , and hence is graded Morita equivalent to \mathbb{C} . By Lemma 1.1, $C_{A'}(\theta_f) \otimes C_{B'}(\theta_g)$ is graded Morita equivalent to $C_{A'}(\theta_f) \otimes \mathbb{C} \cong C_{A'}(\theta_f)$, and the later one is isomorphic to $C_{A'}(\theta_f)$. Hence by Theorem 2.3(iii) and Lemma 2.4, we have equivalences of triangulated categories

$$\underline{\mathrm{mcm}}(A \otimes B)/(f+g) \cong D^{b}(\mathrm{gr}_{\mathbb{Z}_{2}} C_{(A \otimes B)^{!}}(\theta_{f+g}))$$
$$\cong D^{b}\left(\mathrm{gr}_{\mathbb{Z}_{2}} \left(C_{A^{!}}(\theta_{f})\hat{\otimes}C_{B^{!}}(\theta_{g})\right)\right)$$
$$\cong D^{b}(\mathrm{gr}_{\mathbb{Z}_{2}} C_{A^{!}}(\theta_{f}))$$
$$\cong \mathrm{mcm}A/(f).$$

Since $C_{A^{!}}(\theta_{f})$ is a strongly \mathbb{Z}_{2} -graded, $D^{b}(\operatorname{gr}_{\mathbb{Z}_{2}} C_{A^{!}}(\theta_{f})) \cong D^{b}(\operatorname{mod} C_{A^{!}}(\theta_{f})_{0})$ as triangulated categories. Hence the statement (i) follows.

(ii) As in the proof of (i), we have an equivalence of triangulated categories

$$\underline{\mathrm{mcm}}(A \otimes B)/(f+g) \cong D^b\left(\mathrm{gr}_{\mathbb{Z}_2}\left(C_{A^!}(\theta_f)\hat{\otimes}C_{B^!}(\theta_g)\right)\right)$$

Since B/(g) is a simple graded isolated singularity of 1-type, $C_{B'}(\theta_g)$ is a matrix algebra over the \mathbb{Z}_2 -graded algebra \mathbb{CG} , and hence is graded Morita equivalent to \mathbb{CG} . By Lemma 2.4 again, $D^b\left(\operatorname{gr}_{\mathbb{Z}_2}\left(C_{A'}(\theta_f)\hat{\otimes}C_{B'}(\theta_g)\right)\right) \cong D^b\left(\operatorname{gr}_{\mathbb{Z}_2}\left(C_{A'}(\theta_f)\hat{\otimes}\mathbb{CG}\right)\right)$. By Proposition 1.6, $D^b\left(\operatorname{gr}_{\mathbb{Z}_2}\left(C_{A'}(\theta_f)\hat{\otimes}\mathbb{CG}\right)\right) \cong D^b(\operatorname{mod} C_{A'}(\theta_f))$. Hence the statement (ii) follows. \Box

Remark 2.16. Theorem 2.15(i) may be viewed as a generalization of Knörrer's periodicity theorem (see [K, Theorem 3.1], which has been generalized to noncommutative algebras in [CKMW, HY, MU2]). Indeed, let $B = \mathbb{C}[x, y]$ and $g = x^2 + y^2$. By Example 2.7, B/(g) is a simple graded isolated singularity of 0-type. Let A be a noetherian Koszul Artin-Schelter regular algebra, and let $f \in A_2$ be a central regular element. The second double branch cover of A/(f) is defined to be the quotient algebra [K, CKMW]

$$(A/(f))^{\#\#} = A[x,y]/(f+x^2+y^2).$$

Since $A[x, y] \cong A \otimes B$ is notherian and $A[x, y]/(f + x^2 + y^2) \cong (A \otimes B)/(f + g)$, by Theorem 2.15(i), $\underline{\mathrm{mcm}}(A/(f))^{\#\#} \cong \underline{\mathrm{mcm}}A/(f)$.

3. Double branch covers of noncommutative conics

Noncommutative conics in Calabi-Yau quantum planes were recently classified by Hu-Matsuno-Mori in [HMM]. In this section, we will study the double branch covers of the noncommutative conics obtained in [HMM, Corollary 3.8].

We say that A/(f) is a noncommutative conic (see [HMM, Definition 1.3]) if A is a 3-dimensional Calabi-Yau quantum polynomial algebra, and $f \in A_2$ is a central regular element. The double branch cover of A/(f) is defined to be

$$(A/(f))^{\#} = A[x]/(f + x^2).$$

By Theorem 2.15(ii), we also have

$$\underline{\mathrm{mcm}}(A/(f))^{\#} \cong D^{b}(\mathrm{mod}C_{A^{!}}(\theta_{f}))$$

If A/(f) is commutative, by [HMM, Corollary 3.8(i)], A/(f) is isomorphic to one of the following algebras:

$$\mathbb{C}[x, y, z]/(x^2), \quad \mathbb{C}[x, y, z]/(x^2 + y^2), \quad \mathbb{C}[x, y, z]/(x^2 + y^2 + z^2),$$

and we have the following table:

TABLE 1. Commutative case.

A/(f)	$C_{A^!}(heta_f)$	$C_{A^!}(heta_f)_0$
$\mathbb{C}[x,y,z]/(x^2)$	$\mathbb{C}_{-1}[x, y, z]/(x^2 - 1, y^2, z^2)$	$\bigwedge(u,v)$
$\mathbb{C}[x,y,z]/(x^2+y^2)$	$\mathbb{C}_{-1}[x, y, z]/(x^2 - 1, y^2 - 1, z^2)$	$\mathbb{C}_{-1}[u,v]/(u^2-1,v^2)$
$\mathbb{C}[x,y,z]/(x^2+y^2+z^2)$	$M_2(\mathbb{CG})$	$M_2(\mathbb{C})$

If A/(f) is noncommutative, by [HMM, Corollary 3.8(ii)], A/(f) is isomorphic to

$$S^{(\alpha,\beta,\gamma)}/(ax^2+by^2+cz^2)$$

for some $\alpha, \beta, \gamma \in \mathbb{C}$ and $(a, b, c) \in \mathbb{P}^2$, where

$$S^{(\alpha,\beta,\gamma)} = \mathbb{C}\langle x, y, z \rangle / (yz + zy + \alpha x^2, zx + xz + \beta y^2, xy + yx + \gamma z^2).$$

In this case, $C_{A'}(\theta_f)$ is commutative.

Next, we focus on the noncommutative case. We need some preparations.

Let G be a finite group with identity e, and let E be a finited dimensional G-graded algebra. A G-element x is an invertible homogeneous element of E such that $x^i \mapsto |x|^i$ is an injective group homomorphism from $\langle x \rangle$ to G. Here, we use |x| to denote the degree of x and $\langle x \rangle$ is a cyclic group generated by x via multiplication in E. **Example 3.1.** $\mathbb{C}[x]/(x^n)$ graded by \mathbb{Z}_n has no \mathbb{Z}_n -element for positive degrees and $\mathbb{C}G$ graded by G has G-elements for every degree.

Example 3.2. Let $E = M_3(\mathbb{C})$ graded by \mathbb{Z}_2 , where

$$E_0 = \begin{bmatrix} \mathbb{C} & \mathbb{C} & 0 \\ \mathbb{C} & \mathbb{C} & 0 \\ 0 & 0 & \mathbb{C} \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0 & 0 & \mathbb{C} \\ 0 & 0 & \mathbb{C} \\ \mathbb{C} & \mathbb{C} & 0 \end{bmatrix}.$$

Then E is strongly graded $(E_1E_1 = E_0)$, but E has no \mathbb{Z}_2 -element in degree 1.

Remark 3.3. Let $x \in E$ be a *G*-element. Consider the skew group algebra $E_e \# \langle x \rangle$: as a vector space $E_e \# \langle x \rangle = E_e \otimes \mathbb{C} \langle x \rangle$, and the multiplication of $E_e \# \langle x \rangle$ is defined by

$$(a_e \# g)(b_e \# h) = a_e(gb_e g^{-1}) \# gh.$$

One sees that $E_e \# \langle x \rangle$ may be regarded as a graded subalgebra of E via $(a_e, x^i) \mapsto a_e x^i$. The Moreover, if $G \cong \langle x \rangle$, then $E \cong E_e \# \langle x \rangle$.

Lemma 3.4. Assume that I is a nilpotent homogeneous ideal of E, then any G-element \overline{x} of E/I can be lifted into E.

Proof. Assume $x \in E$ is a homogeneous preimage of \overline{x} . Since $\overline{x}^n = \overline{1}$ for some $n \geq 0$, there is an element $r \in I$ such that $x^n = 1 + r$. Note that r is a nilpotent element of degree e, so 1 + r is invertible.

More precisely, since $|x^n| = e$, $r = x^n - 1 \in I_e$ and commutative with x. If $I^2 = 0$, then x' := x(1 - r/n) is a lifting of \overline{x} . In fact, x' is homogeneous,

$$x'^{n} = x^{n}(1 - r/n)^{n} = (1 + r)(1 - r) = 1$$

and $x - x' = (x/n)r \in I$.

For general case, the result follows from the induction on the exact sequence

$$0 \to I^{2^{k-1}}/I^{2^k} \to E/I^{2^k} \to E/I^{2^{k-1}} \to 0.$$

Proposition 3.5. Let *E* be a finite dimensional commutative algebra graded by \mathbb{Z}_2 , if *A* is strongly graded, then $A \cong A_0 \times A_0$ as ungraded algebras.

Proof. Since E is finite dimensional, the graded radical $J^g(E)$ is nilpotent and $E/J^g(E)$ is graded semisimple. The condition that E is strongly graded implies that $E/J^g(E)$ is also strongly graded, hence $E/J^g(E)$ is a product of some copies of \mathbb{CG} by Lemma 1.1(ii)

and has a \mathbb{Z}_2 -element in degree 1. By Lemma 3.4, this \mathbb{Z}_2 -element can be lifted into E, denoted by g. Now, $E_0 \# \langle g \rangle$ is a subalgebra of E, and as ungraded algebras

$$E_0 \# \langle g \rangle \cong E_0[x]/(x^2 - 1) \cong E_0 \times E_0$$

For surjective, note that if $a_1 \in E_1$, then $a_1 = (a_1g)g \in E_0 \# \langle g \rangle$.

Corollary 3.6. Let A/(f) be a noncommutative conic. If A/(f) is noncommutative, then

$$\underline{\mathrm{mcm}}(A/(f))^{\#} \cong \underline{\mathrm{mcm}}(A/(f)) \times \underline{\mathrm{mcm}}(A/(f))$$

Example 3.7. Let $A = \mathbb{C}\langle x, y, z \rangle / (yz + zy + x^2, zx + xz + y^2, xy + yx)$ and $f = 3x^2 + 3y^2 + 4z^2$. Then

$$A^{!} = \mathbb{C}[x, y, z]/(yz - x^{2}, zx - y^{2}, z^{2}),$$

$$C_{A^{!}}(\theta_{f}) = \mathbb{C}[x, y, z]/(yz - x^{2} + 3, zx - y^{2} + 3, z^{2} - 4).$$

Since $(yz - x^2 + 3) - (zx - y^2 + 3) = 0$, we have (x - y)(x + y + z) = 0. One can compute that $\text{Spec}(C_{A'}(\theta_f))$ has 4 points and $C_{A'}(\theta_f)$ is a product of 4 commutative local rings.

Let $R = \mathbb{C}[z]/(z^2 - 4)$, then

$$C_{A^{!}}(\theta_{f}) \cong R[x, y]/(yz - x^{2} + 3, zx - y^{2} + 3) \cong A' \times A''$$

where $A' = \mathbb{C}[x, y]/(2y - x^2 + 3, 2x - y^2 + 3)$ and $A'' = \mathbb{C}[x, y]/(-2y - x^2 + 3, -2x - y^2 + 3)$.

In A', we have

$$y^{4} = (2x+3)^{2} = 4x^{2} + 12x + 9 = 4(2y+3) + 6(y^{2}-3) + 9 = 6y^{2} + 8y + 3.$$

Hence, $y^4 - 6y^2 - 8y - 3 = (y+1)^3(y-3) = 0$, which implies $A' \cong \mathbb{C}[u]/(u^3) \times \mathbb{C}$. A similar computation shows that $A' \cong A''$. Therefore, $C_{A'}(\theta_f)_0$ is also isomorphic to $\mathbb{C}[u]/(u^3) \times \mathbb{C}$ by Propsition 3.5.

By Remark 2.13, rank f must be 1. In fact, $f = 3x^2 + 3y^2 + 4z^2 = (-x - y + 2z)^2$.

Acknowledgments. J.-W. He was supported by NSFC (No. 11971141). Y. Ye was supported by NSFC (No. 11971449).

References

- [AS] M. Artin, W. Schelter, Graded algebras of global dimension 3, Adv. Math. 66 (1987), 171–216.
- [CKMW] A. Conner, E. Kirkman, W. F. Moore, C. Walton, Noncommutative Knörrer periodicity and noncommutative Kleinian singularities, arXiv:1809.06524.
- [HU] A. Higashitani, K. Ueyama, Combinatorial study of stable categories of graded Cohen-Macaulay modules over skew quadric hypersurfaces, Collectanea Math. 135 (2021), doi.org/10.1007/s13348-020-00306-1.

- [HY] J.-W. He, Y. Ye, Clifford deformations of Koszul Frobenius algebras and noncommutative quadrics, arXiv:1905.04699.
- [HMM] H. Hu, M. Matsuno, I. Mori, Noncommutative conics in Calabi-Yau quantum planes, arXiv:2104.00221.
- [K] H. Knörrer, Cohen-Macaulay modules on hypersurface singularities I, Invent. Math. 88 (1987), 153–164.
- [Le] T. Levasseur, Some properties of noncommutative regular graded rings, Glasgow Math. J. 34 (1992), 277–300.
- [LWZ] D.-M. Lu, Q.-S. Wu, J. J. Zhang, A Morita cancellation problem, Canad. J. Math. 72 (2019), 708–731.
- [M] Yu. I. Manin, *Quantum Groups and Non-Commutative Geometry*, Centre de Recherches Mathématiques, 1988.
- [MU1] I. Mori, K. Ueyama, Noncommutative matrix factoriztions with an application to skew exterior algebras, arXiv:1806.07577.
- [MU2] I. Mori, K. Ueyama, Noncommutative Knörrer's Periodicity Theorem and noncommutative quadric surfaces, arXiv:1905.12266.
- [NVO] C. Năstăsescu, F. Van Oystaeyen, Methods of Graded Rings, Lect. Note Math. 1836, Springer, 2004.
- [P] S. B. Priddy, Koszul resolutions, Trans. Amer. Math. Soc. 152 (1970), 39–60.
- [Sm] S. P. Smith, Some finite dimensional algebras related to elliptic curves, Representation Theory of Algebras and Related Topics, CMS Conf. Proc., American Mathematical Society, Providence, RI (1996), 315–348.
- [SVdB] S. P. Smith, M. Van den Bergh, Noncommutative quadric surfaces, J. Noncommut. Geom. 7 (2013), 817–856.
- [Ue1] K. Ueyama, Graded maximal Cohen-Macaulay modules over noncommutative graded Gorenstein isolated singularities, J. Algebra 383 (2013), 85–103.
- [Ue2] K. Ueyama, Derived categories of skew quadic hypersurfaces, arXiv:2008.02255.
- [Z] D. Zhao, Graded Morita equivalence of Clifford superalgebras, Adv. Appl. Clifford Algebras 23 (2013), 269–281.

HE: DEPARTMENT OF MATHEMATICS, HANGZHOU NORMAL UNIVERSITY, HANGZHOU ZHEJIANG 311121, CHINA

Email address: jwhe@hznu.edu.cn

MA: School of Mathematical Sciences, University of Science and Technology of China, Hefei Anhui 230026, China

Email address: mic@mail.ustc.edu.cn

YE: SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA, HEFEI ANHUI 230026, CHINA

CAS WU WEN-TSUN KEY LABORATORY OF MATHEMATICS, UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA, HEFEI, ANHUI, 230026, PR CHINA

Email address: yeyu@ustc.edu.cn